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A modified generalized projective Riccati equation method

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ABSTRACT

A modification of the generalized projective Riccati equation method is proposed to treat some nonlinear evolution equations and obtain their exact solutions. Some known methods are obtained as special cases of the proposed method. In addition, the method is implemented to find new exact solutions for the well-known Drinfel'd–Sokolov–Wilson system of nonlinear partial differential equations.

Keywords

Exact solutions; Projective Riccati equation method;Drinfel'd–Sokolov–Wilson equations; sech ξ – tanh ξ method; Extended tanh method.

1. Introduction

The investigation of exact travelling wave solutions to nonlinear evolution equations (NLEE's) plays an important role in the study of nonlinear phenomena in many fields such as physics, biology, chemistry and mechanics, etc. As the mathematical model for these phenomena helps us understand them better. Due to the increasing interest in obtaining exact solutions of nonlinear partial differential equations (NLPDE's), many powerful methods are now available for treating and obtaining solitary wave solutions. They have been developed since the establishment of the inverse scattering technique [1], Backland and Darboux transform [2–4], Hirota method [5], Jacobi elliptic function method [6], Hyperbolic functions expansion method [7], tanh method [8], cosine-function method [9], the auxiliary equation method [10], Mapping method [11], and so on. Implementing these methods yield large nonlinear systems, which are usually difficult to solve. But with the help of Maple or Mathematica programs, that perform tedious algebraic calculations, one can solve such systems easily.

In this paper an improvement on the work done in [12–18] is achieved. A series of new and more exact solutions of Dreinfeld's–Sokolov–Wilson system are obtained including solitary wave solutions and triangular periodic solutions.

This paper is organized as follows: in section 2, we describe the modified generalized projective Riccati equation method. In section 3, we derive two well-known methods, namely, the sech ξ –tanh ξ and the extended tanh methods as special cases. In section 4, we apply the method to treat theDrinfel'd–Sokolov–Wilson(DSW) equations. Finally, some concluding remarks are given in section 5.

2. Description of the modified generalized projective Riccati equation method

In this section, we will describe the modified generalized Projective Riccati equation (MGPRE) method based on [18].

Consider the nonlinear PDE

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{1}$$

Using the wave transformation

$$u(x,t) = u(\xi), \xi = x - ct, \tag{2}$$

where c is a constant, Eq. (1) converts to an ordinary differential equation (ODE) of the function $u(\xi)$

$$O(u, u', u'', \ldots) = 0. \tag{3}$$

To apply the method, we use the following steps:

Step 1. We assume that the solution of Eq. (3) is in the form:

$$u(\xi) = H(\sigma(\xi), \tau(\xi)) = a_0 + \sum_{i=1}^{n} \sigma^{i-1}\left[a_i \sigma(\xi) + b_i \tau(\xi)\right], \tag{4}$$

where a_i and b_i are constants to be determined later, $H(\sigma(\xi), \tau(\xi))$ is a rational function in the new variables $\sigma(\xi)$, $\tau(\xi)$ which satisfy the system

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi), \tag{5}$$

$$\tau'(\xi) = -R + \epsilon\tau^2(\xi) + \mu\sigma(\xi), \epsilon = \pm 1, \tag{6}$$

$$\tau^2(\xi) = \frac{1}{\epsilon}[R - 2\mu\sigma(\xi) + \frac{(\mu^2 + \delta)}{R}\sigma^2(\xi)], \ R \neq 0 \ , \tag{7}$$

where ϵ, R, μ are real constants and $\epsilon \neq 0$. It is noticeable that $\sigma(\xi)\tau(\xi) \neq 0.$

Step 2. The parameter n can be determined by balancing the highest order derivative term with the nonlinear term in Eq. (3), n is usually a positive integer. If n is a fraction or a negative integer, we make the following transformation

1. When $n = p / q$ is a fraction, we let

$$u(\xi) = v^{p/q}(\xi),$$

and then substitute into (3) to determine n.

2. When n is a negative integer, we let

$$u(\xi) = v^n(\xi),$$

and then substitute into (3) to determine n .

Now, we will solve Projective Riccati equations (5-7) based on the idea of Zuntao Fu, see [18] for details, by making appropriate generalizations.

We will assume

$$\sigma(\xi) = \frac{1}{\phi(\xi)}, \tag{8}$$

then using (5), we obtain

$$\tau(\xi) = \frac{1}{\epsilon}\frac{\sigma'(\xi)}{\sigma(\xi)}. \tag{9}$$

Substituting (8) and (9) in (6) and after simplifying, we obtain:

$$\phi''(\xi) - R\epsilon\phi(\xi) + \mu\epsilon = 0, \tag{10}$$

Solutions of Eq. (10) are obtained as follows:

(i) If we assume $R \neq 0$ and $R\epsilon > 0$. If we let $R\epsilon = \lambda^2$, then Eq. (10) can be rewritten as

$$\phi''(\xi) - \lambda^2\phi(\xi) + \mu\epsilon = 0. \tag{11}$$

and the general solution of Eq. (11) is given by

$$\phi(\xi) = \phi_c(\xi) + \phi_p(\xi) = a_0 + a_1 \sinh(\lambda\xi) + a_2 \cosh(\lambda\xi),$$

where $\lambda = \sqrt{R\epsilon}$ and $a_0 = \frac{\mu\epsilon}{\lambda^2} = \frac{\mu}{R}$. Explicitly, we can write the solution as follows

$$\phi_1(\xi) = \frac{\mu}{R} + a_1 \sinh(\sqrt{R\epsilon}\xi) + a_2 \cosh(\sqrt{R\epsilon}\xi), \tag{12}$$

so

$$\sigma_1(\xi) = \frac{R}{\mu + \alpha_1 \sinh(\sqrt{R\epsilon}\xi) + \beta_1 \cosh(\sqrt{R\epsilon}\xi)},$$

and

$$\tau_1(\xi) = \frac{-\sqrt{R\epsilon}}{\epsilon}\frac{\alpha_1 \cosh(\sqrt{R\epsilon}\xi) + \beta_1 \sinh(\sqrt{R\epsilon}\xi)}{\mu + \alpha_1 \sinh(\sqrt{R\epsilon}\xi) + \beta_1 \cosh(\sqrt{R\epsilon}\xi)}, \quad (13)$$

where α_1 and β_1 are arbitrary constants, and $\delta = \alpha_1^2 - \beta_1^2$.

(ii) If we assume $\epsilon R < 0$ and $R\epsilon = -\lambda^2$, then Eq. (10) can be rewritten as

$$\phi''(\xi) + \lambda^2\phi(\xi) + \mu\epsilon = 0, \quad (14)$$

the general solution of Eq. (14) is

$$\phi_2(\xi) = b_0 + b_1 \sin(\lambda\xi) + b_2 \cos(\lambda\xi), \lambda = \sqrt{-R\epsilon},$$

where $b_0 = \frac{\mu}{R}$. Therefore, we can write the solution as follows

$$\phi_2(\xi) = \frac{\mu}{R} + b_1 \sin(\sqrt{-R\epsilon}\xi) + b_2 \cos(\sqrt{-R\epsilon}\xi), \quad (15)$$

$$\sigma_2(\xi) = \frac{R}{\mu + \alpha_2 \sin(\sqrt{-R\epsilon}\xi) + \beta_2 \cos(\sqrt{-R\epsilon}\xi)},$$

and

$$\tau_2(\xi) = \frac{-\sqrt{-R\epsilon}}{\epsilon}\frac{\alpha_2 \cos(\sqrt{-R\epsilon}\xi) - \beta_2 \sin(\sqrt{-R\epsilon}\xi)}{\mu + \alpha_2 \sin(\sqrt{-R\epsilon}\xi) + \beta_2 \cos(\sqrt{-R\epsilon}\xi)}, \quad (16)$$

where α_2 and β_2 are arbitrary constants, and $\delta = -\alpha_2^2 - \beta_2^2$.

(iii) Assume $R = 0$, then Eq. (10) becomes

$$\phi'' + \epsilon\mu = 0, \quad (17)$$

The solution of Eq. (17) is given by

$$\phi_3(\xi) = a_0 + a_1\xi - \frac{\mu\epsilon}{2}\xi^2, \quad (18)$$

where a_0 and a_1 are two arbitrary constants. Using (18), we have

$$\sigma_3(\xi) = \frac{1}{a_0 + a_1\xi - \frac{\mu\epsilon}{2}\xi^2},$$

and

$$\tau_3(\xi) = \frac{-1}{\epsilon}\left(\frac{a_1 - \epsilon\mu\xi}{a_0 + a_1\xi - \frac{\mu\epsilon}{2}\xi^2}\right).$$

The following derivations are needed for the results in Section 3.

Assume

$$\sigma = \sigma(\xi) = \psi^{\frac{1}{m}}(\xi) = \psi^{\frac{1}{m}}, \quad (19)$$

then (7) can be written as

$$\tau^2(\xi) = \frac{1}{\epsilon}[R - 2\mu\psi^{\frac{1}{m}}(\xi) + \frac{(\mu^2+\delta)}{R}\psi^{\frac{2}{m}}(\xi)], R \neq 0. \tag{20}$$

From (19), we get

$$\sigma'(\xi) = \frac{1}{m}\psi^{\frac{1}{m}-1}\psi' \tag{21}$$

Solving for ψ', get

$$\psi' = m\sigma'\psi^{-\frac{1}{m}}\psi = m(\epsilon\sigma\tau)\psi^{-\frac{1}{m}}\psi = \epsilon m\tau\psi^{\frac{1}{m}}\psi^{-\frac{1}{m}}\psi = \epsilon m\tau\psi \tag{22a}$$

where $\psi = \psi(\xi), \sigma = \sigma(\xi)$ and $\tau = \tau(\xi)$.

Squaring both sides of Eq.(22a), we obtain

$$\psi'^2 = \epsilon^2 m^2\psi^2\tau^2, \tag{22b}$$

substituting (20) in (22b), we get

$$\psi'^2 = \epsilon m^2\psi^2(\xi)[R - 2\mu\psi^{\frac{1}{m}}(\xi) + \frac{(\mu^2+\delta)}{R}\psi^{\frac{2}{m}}(\xi)]. \tag{23}$$

If we take $m = -1$ in (23), we obtain

$$\psi'^2 = \epsilon\left(R\psi^2 - 2\mu\psi + \frac{(\mu^2+\delta)}{R}\right). \tag{24}$$

Equations (22b) - (24) are used to drive the special cases below.

Special cases

3.1. The sech ξ – tanh ξ method

In this section, we will obtain the $\operatorname{sech}\xi - \tanh\xi$ method as special case of our proposed method.

The main steps of the $\operatorname{sech}\xi - \tanh\xi$ method are as follow:

We assume the solution of the ODE (3) is in the form:

$$u(\xi) = a_0 + \sum_{i=1}^{n}\sigma^{i-1}(\xi)\left[a_i\sigma(\xi) + b_i\tau(\xi)\right], \tag{25}$$

with

$$\sigma(\xi) = \operatorname{sech}\xi, \quad \tau(\xi) = \tanh\xi, \tag{26}$$

The relation (25) becomes

$$u(\xi) = a_0 + \sum_{i=1}^{n}\operatorname{sech}^{i-1}\xi[a_i\operatorname{sech}\xi + b_i\tanh\xi], \tag{27}$$

where a_i, b_i are constants to be determined later, n can be determined by balancing the highest order derivative term with the high degree nonlinear term in ODE (3).

If we assume

$$\epsilon = -1, \mu = 0, R = -1, \delta = -1, m = -1,$$

Then Eq.'s (5) – (7) become

$$\begin{aligned} \sigma'(\xi) &= -\operatorname{sech}\xi\tanh\xi = -\sigma(\xi)\tau(\xi) \\ \tau'(\xi) &= 1-\tanh^2\xi = 1-\tau^2(\xi) \\ \tau^2(\xi) &= 1-\operatorname{sech}^2\xi = 1-\sigma^2(\xi), \end{aligned} \quad (28)$$

If we take $n = 1,2$ in (25), obtain

$$u(\xi) = H(\sigma(\xi),\tau(\xi)) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi)$$

and

$$u(\xi) = a_0 + a_1\sigma(\xi) + a_2\sigma^2(\xi) + b_1\tau(\xi) + b_2\sigma(\xi)\tau(\xi),$$

respectively, where $\sigma(\xi)$ and $\tau(\xi)$ are obtained as follows: from Eq. (22b)

$$(\psi')^2 = \psi^2(\xi)\tau^2(\xi).$$

and

$$\psi' = \psi\sqrt{1-\psi^2},$$

or

$$\sigma'(\xi) = \sigma(\xi)\sqrt{1-\sigma(\xi)^2}, \quad (29)$$

the solution to (29) is $\sigma(\xi) = \operatorname{sech}\xi,$ and using the relation $\tau(\xi) = -\frac{\sigma'(\xi)}{\sigma(\xi)},$ we obtain that $\tau(\xi) = \tanh\xi$.

3.2 The extended tanhmethod

The main steps of the extended tanh method are as follow:

If we take $\tau(\xi) = R\psi^{-1} + \psi$ and $m = \epsilon = -1$ in Eq. (22)

$$(\psi')^2 = \psi^2(R\psi^{-1}+\psi)^2 = (R+\psi^2)^2. \quad (30)$$

The last equation gives

$$\psi' = R + \psi^2 \quad (31)$$

The general solution of Riccati equation (31) is given by:

If $R < 0,$

$$\psi = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \quad (32)$$

and

$$\psi = -\sqrt{-R}\coth(\sqrt{-R}\xi). \quad (33)$$

If $R = 0,$

$$\psi = \frac{-1}{\xi}, \quad (34)$$

If $R > 0,$

$$\psi = \sqrt{R}\tan(\sqrt{R}\xi) \quad (35a)$$

and

$$\psi = -\sqrt{R}\cot(\sqrt{R}\xi). \quad (35b)$$

Now projective Riccati equations are

$$\begin{cases} \sigma'(\xi) = -\sigma(\xi)\tau(\xi) \\ \tau^2(\xi) = (R\sigma(\xi) - \frac{1}{\sigma(\xi)})^2 \end{cases} . \quad (36)$$

According to (32)-(35) the solutions to system (36) are given by

$$\sigma_1(\xi) = \frac{-1}{\sqrt{-R}}\coth(\sqrt{-R}\xi),$$

$$\tau_1(\xi) = -\frac{\sigma_1'(\xi)}{\sigma_1(\xi)} = \sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi).$$

$$\sigma_2(\xi) = \frac{-1}{\sqrt{-R}}\tanh(\sqrt{-R}\xi),$$

$$\tau_2(\xi) = -\frac{\sigma_2'(\xi)}{\sigma_2(\xi)} = -\sqrt{-R}\,\mathrm{sech}(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi).$$

$$\sigma_3(\xi) = -\xi \quad \text{and} \quad \tau_3(\xi) = -\frac{\sigma'(\xi)}{\sigma(\xi)} = \frac{1}{\xi}.$$

$$\sigma_4(\xi) = \frac{1}{\sqrt{R}}\cot(\sqrt{R}\xi),$$

$$\tau_4(\xi) = -\frac{\sigma_5'(\xi)}{\sigma_5(\xi)} = \sqrt{R}\sec(\sqrt{R}\xi)\csc(\sqrt{R}\xi).$$

$$\sigma_5(\xi) = \frac{-1}{\sqrt{R}}\tan(\sqrt{R}\xi),$$

$$\tau_5(\xi) = -\frac{\sigma'(\xi)}{\sigma(\xi)} = -\sqrt{R}\sec(\sqrt{R}\xi)\csc(\sqrt{R}\xi).$$

To solve Eq. (3), with $R < 0,$ we can try to find solutions using expansions,

$$v(\xi) = a_0 + \sum_{j=1}^{n}\tanh^{j-1}(\sqrt{-R}\xi)[a_j\tanh(\sqrt{-R}\xi) + b_j\,\mathrm{sech}(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi)], \quad (37)$$

or the expansion

$$v(\xi) = a_0 + \sum_{j=1}^{n} a_j(-\xi)^j + a_{-j}(-\xi)^{-j}, R = 0. \quad (38a)$$

If $R > 0,$ to find solutions using the expansion

$$v(\xi) = a_0 + \sum_{j=1}^{n}\tan^{j-1}(\sqrt{R}\xi)[a_j\tan(\sqrt{R}\xi) + b_j\sec(\sqrt{R}\xi)\csc(\sqrt{R}\xi)]. \quad (38b)$$

The value of n is calculated by the balancing procedure. Usually, $n = 1,2$. For these particular choices ansatzes (36), (37) and (38) take the following forms:

$$v(\xi) = a_0 + a_1\tanh(\sqrt{-R}\xi) + b_1\,\mathrm{sech}(\sqrt{-R}\xi)\,\mathrm{csch}(\sqrt{-R}\xi),$$

$$v(\xi) = a_0 + a_1 \tanh(\sqrt{-R}\xi) + a_2 \tanh^2 \sqrt{-R}\xi$$
$$+b_1 \operatorname{sech} \sqrt{-R}\xi \operatorname{csch} \sqrt{-R}\xi + b_2 \operatorname{sech}^2 \sqrt{-R}\xi$$

$$v(\xi) = a_0 - \ a_1\xi + a_{-1}\xi^{-1}$$

$$v(\xi) = a_0 - \ a_1\xi + a_{-1}\xi^{-1} \ + \ a_2\xi^2 + a_{-2}\xi^{-2}$$

$$v(\xi) = a_0 + a_1 \tan(\sqrt{R}\xi) + b_1 \sec(\sqrt{R}\xi)\csc(\sqrt{R}\xi)$$

$$v(\xi) = a_0 + a_1 \tan(\sqrt{R}\xi) + a_2 \tan^2(\sqrt{R}\xi)$$
$$+b_1 \sec(\sqrt{R}\xi)\csc(\sqrt{R}\xi) + b_2 \sec^2(\sqrt{R}\xi).$$

4. Drinfel'd–Sokolov–Wilson equations

In this section, we are going to apply the MGPRE method to solve the DSW equations [19–20]. The DSW equations are a nonlinear partial differential equation with (1+1) dimension

$$u_t + pvv_x = 0, \tag{39}$$

$$v_t + qv_{xxx} + ruv_x + su_xv = 0, \tag{40}$$

where p, q, r, s are nonzero parameters. Eq. (39) and (40) are proposed by Drinfel'd–Sokolov in [19] and Wilson [20].

Using the traveling wave transformation

$$u(x,t) = u(\xi), \ \ v \ \ x,t \ \ = v(\xi), \ \ \xi = x - ct,$$

Eqs. (39) and (40) can be transformed to the ODE's

$$-cu' + pvv' = 0, \tag{41}$$

$$-cv'+ qv'''+ ruv'+ su'v = 0. \tag{42}$$

Integrating (41) once and neglecting the constant of integration, we get

$$u = \frac{p}{2c}v^2, \tag{43}$$

substituting Eq. (43) into (42) and integrating, we get,

$$-cv(\xi) + qv''(\xi) + \frac{p(r+2s)}{6c}(v(\xi))^3 = 0, \tag{44}$$

Balancing the terms v'' with v^3 in (44), we find $n + 2 = 3n, \ \ n = 1,$

Using the MGPRE method (4) we assume the solution of Eq. (44) is

$$v(\xi) = a_0 + a_1\sigma(\xi) + b_1\tau(\xi), \tag{45}$$

where $\sigma(\xi), \tau(\xi)$ satisfy the system

$$\sigma'(\xi) = \epsilon\sigma(\xi)\tau(\xi),$$

$$\tau'(\xi) = -R + \epsilon\tau^2(\xi) + \mu\sigma(\xi), \epsilon = \pm 1, \tag{46}$$

$$\tau^2(\xi) = \frac{1}{\epsilon}[R - 2\mu\sigma(\xi) + \frac{(\mu^2 + \delta)}{R}\sigma^2(\xi)], \ \ R \neq 0 \ \ , \tag{47}$$

and $a_0, a_1\ b_1$ constants to be determined later. Substituting (45) – (47) into (44), yields a system with coefficients of $\sigma^i(\xi)\tau^j(\xi)\ \ (i = 0,1,2,\dots\ ,\ \ j = 0,1)$. Setting the coefficients of $\sigma^i\tau^j$ to zero, we obtain algebraic equations with the respect to unknowns $a_0, a_1, b_1, R, \mu, \delta$ and c. To get a nontrivial solution of this algebraic equations, we need to assume $a_1^2 + b_1^2 \neq 0$. With the aid of the computer program Maple, we obtain the following sets of solutions:

Case 1. When $\delta = -1,$ we obtain the following sets of solutions:

First set

$$a_0 = 0,\ \ a_1 = \pm 2\sqrt{\frac{3}{p(r+2s)}}\epsilon q,\ \ b_1 = 0,\ \ R\epsilon = \frac{c}{q},\ \ \mu = 0,$$

Second set

$$a_0 = 0,\ \ a_1 = \pm\sqrt{\frac{3(\mu^2-1)}{2p(r+2s)}}\epsilon q,\ \ b_1 = \mp\sqrt{-\frac{3cq}{p(r+2s)}}\epsilon,\ \ R\epsilon = -\frac{2c}{q},\ \ \mu = \mu,$$

Third set

$$a_0 = 0,\ \ a_1 = 0,\ \ b_1 = \pm 2\sqrt{\frac{3cq}{p(r+2s)}}\epsilon q,\ \ R\epsilon = -\frac{1}{2}\frac{c}{q},\ \ \mu = 0,$$

Using to the first set and assuming $R\epsilon = \frac{c}{q} > 0,$ the solutions for Eq. (44) read:

$$v_1 = \pm\frac{2c\sqrt{\frac{3}{p(r+2s)}}}{\sqrt{\beta_1^2-1}\sinh\left[\sqrt{\frac{c}{q}}(x-ct)\right] + \beta_1\cosh\left[\sqrt{\frac{c}{q}}(x-ct)\right]},$$

and

$$u_1 = \pm\frac{6c}{(r+2s)\ \ \sqrt{\beta_1^2-1}\sinh\left[\sqrt{\frac{c}{q}}(x-ct)\right] + \beta_1\cosh\left[\sqrt{\frac{c}{q}}(x-ct)\right]^2}.$$

In particular, if $\beta_1 = \pm 1,$ we get

$$v_2 = \pm\frac{2c\sqrt{\frac{3}{p(r+2s)}}}{\cosh[\sqrt{\frac{c}{q}}(x-ct)]},$$

and

$$u_2 = \frac{6c}{(r+2s)\cosh^2[\sqrt{\frac{c}{q}}(x-ct)]}.$$

However, for $R\epsilon\ \ = -\frac{c}{q} < 0,$ the solutions are

$$v_3 = \pm\frac{2c\sqrt{\frac{3}{p(r+2s)}}}{\sqrt{-\beta_2^2+1}\sin[\sqrt{-\frac{c}{q}}(x-ct)] + \beta_2\cos[\sqrt{-\frac{c}{q}}(x-ct)]},$$

and

$$u_3 = \frac{6c}{(r+2s)(\sqrt{-\beta_2^2+1}\sin[\sqrt{-\frac{c}{q}}(x-ct)] + \beta_2\cos[\sqrt{-\frac{c}{q}}(x-ct)])^2}.$$

Note if $\beta_2 = \pm 1,$ we get

$$v_4 = \pm \frac{2c\sqrt{\frac{3}{p(r+2s)}}}{\cos[\sqrt{-\frac{c}{q}}(x-ct)]},$$

and

$$u_4 = \frac{6c}{(r+2s)\cos^2[\sqrt{-\frac{c}{q}}(x-ct)]},$$

Using to the second set and assuming $R\epsilon = -\frac{2c}{q} > 0,$ the solutions for Eq. (44) read:

$$v_5 = \pm \frac{2c\sqrt{\frac{3(\mu^2-1)}{2p(r+2s)}}}{\mu + \sqrt{\beta_1^2 - 1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1 \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}$$
$$\mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\ \ \beta_1 \sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\mu + \sqrt{\beta_1^2 - 1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1 \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}$$
$$\frac{+\sqrt{\beta_1^2 - 1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\mu + \sqrt{\beta_1^2 - 1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1 \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]},$$

and

$$u_5 = \frac{p}{2c}\left(\pm \frac{2c\sqrt{\frac{3(\mu^2-1)}{2p(r+2s)}}}{\mu+\sqrt{\beta_1^2-1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p\ r+2s}}\ \ \beta_1\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\mu+\sqrt{\beta_1^2-1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \frac{+\sqrt{\beta_1^2-1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\mu+\sqrt{\beta_1^2-1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \right)^2$$

Note if we set $\mu = 0,$ in the second set and assuming $R\epsilon > 0,$ we get

$$v_6 = \pm \frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\sqrt{\beta_1^2 - 1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1 \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}$$
$$\mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p\ r+2s}}\ \ \beta_1 \sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2 - 1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1 \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}$$
$$\frac{+\sqrt{\beta_1^2 - 1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2 - 1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1 \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]},$$

and

$$u_6 = \frac{p}{2c}\left(\pm\frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\sqrt{\beta_1^2-1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\ \ \beta_1\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2-1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \frac{+\sqrt{\beta_1^2-1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2-1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\right)^2$$

In particular, if $\beta_1 = \pm 1$, we get

$$v_7 = \pm\frac{2c\sqrt{\frac{3(\mu^2-1)}{2p(r+2s)}}}{\mu \pm \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\mu \pm \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]},$$

and

$$u_7 = \frac{p}{2c}\left(\pm\frac{2c\sqrt{\frac{3(\mu^2-1)}{2p(r+2s)}}}{\mu \pm \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\mu \pm \cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\right)^2,$$

Note if we set $\mu = 0$ in the second set and assume $R\epsilon \ < 0,$ we get

$$v_8 = \pm\frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]},$$

and

$$u_8 = \frac{p}{2c}\left(\pm\frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\right)^2,$$

However, for $R\epsilon \ < 0,$ the solutions are

$$v_9 = \pm\frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\sqrt{1-\beta_2^2}\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} - \frac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{2p(r+2s)}}\ \ -\beta_2\sin[\sqrt{\frac{2c}{q}}(x-ct)]}{\sqrt{1-\beta_2^2}\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \frac{+\sqrt{1-\beta_2^2}\cos\left[\sqrt{\frac{2c}{q}}\ \ x-ct\ \ \right]}{\sqrt{1-\beta_2^2}\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]},$$

and

$$u_9 = \frac{p}{2c}\left(\begin{array}{c} \pm \dfrac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\sqrt{1-\beta_2^2}\,\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \\ \mp \dfrac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{2p(r+2s)}}\ \ -\beta_2\sin[\sqrt{\frac{2c}{q}}(x-ct)]}{\sqrt{1-\beta_2^2}\,\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \\ \dfrac{+\sqrt{1-\beta_2^2}\,\cos[\sqrt{\frac{2c}{q}}(x-ct)]}{\sqrt{1-\beta_2^2}\,\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \end{array}\right)^2,$$

Note if $\alpha = 0, \beta = \pm 1,$ we get

$$v_{10} = \pm\frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\cos[\sqrt{\frac{2c}{q}}(x-ct)]} - \frac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{2p(r+2s)}}\sin[\sqrt{\frac{2c}{q}}(x-ct)]}{\cos[\sqrt{\frac{2c}{q}}(x-ct)]},$$

and

$$u_{10} = \frac{p}{2c}\left(\pm\frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\cos[\sqrt{\frac{2c}{q}}(x-ct)]} - \frac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{2p(r+2s)}}\sin[\sqrt{\frac{2c}{q}}(x-ct)]}{\cos[\sqrt{\frac{2c}{q}}(x-ct)]}\right)^2,$$

Using to the Third Set and assuming $R\epsilon = -\frac{1}{2}\frac{c}{q} > 0,$ the solutions for Eq. (44) read:

$$v_{11} = \pm\frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\ \ \beta_1\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2-1}\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]}$$
$$\frac{+\sqrt{\beta_1^2-1}\cosh\left[\frac{1}{2}\sqrt{-\frac{2c}{q}}\ \ x-ct\ \right]}{\sqrt{\beta_1^2-1}\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]},$$

and

$$u_{11} = \pm\frac{3c(\beta_1\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]+\sqrt{\beta_1^2-1}\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)])^2}{(r+2s)(\sqrt{\beta_1^2-1}\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)])^2},$$

However, for $R\epsilon\ < 0,$ the solutions are

$$v_{12} = \pm\frac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{2p(r+2s)}}(-\beta_2\sin[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]}{\sqrt{1-\beta_2^2}\sin[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]}$$
$$\frac{+\sqrt{1-\beta_2^2}\cos[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)])}{\sqrt{1-\beta_2^2}\sin[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]},$$

and

$$u_{12} = -\frac{3c(-\beta_2 \sin[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)] + \sqrt{1-\beta_2^2}\cos[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)])^2}{(r+2s)(\sqrt{1-\beta_2^2}\sin[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)] + \beta_2 \cos[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)])^2}$$

Note if we let $\beta = \pm 1$, we get

$$v_{13} = \pm\frac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{2p(r+2s)}}\sin[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]}{\cos[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]}$$

and

$$u_{13} = -\frac{3c\sin^2[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]}{(r+2s)\cos^2[\frac{1}{2}\sqrt{\frac{2c}{q}}(x-ct)]}$$

Case 2. When $\delta = 1$, we obtain the following sets of solutions:

First set

$$a_0 = 0,\ a_1 = \pm 2\sqrt{-\frac{3}{p(r+2s)}}\epsilon q,\ b_1 = 0,\ R\epsilon = \frac{c}{q},\ \mu = 0,$$

Second set

$$a_0 = 0,\ a_1 = \pm\sqrt{\frac{3\ \mu^2+1}{2p(r+2s)}}\epsilon q,\ b_1 = \mp\sqrt{-\frac{3cq}{p(r+2s)}}\epsilon,\ R\epsilon = -\frac{2c}{q},\ \mu = \mu,$$

Third set

$$a_0 = 0,\ a_1 = 0,\ b_1 = \pm 2\sqrt{-\frac{3cq}{p(r+2s)}}\epsilon q,\ R\epsilon = -\frac{1}{2}\frac{c}{q},\ \mu = 0,$$

Using the first set and assuming $R\epsilon = \frac{c}{q} > 0,$ the solutions for Eq. (44)

$$v_{14} = \pm\frac{2c\sqrt{-\frac{3}{p(r+2s)}}}{\sqrt{\beta_1^2+1}\sinh[\sqrt{\frac{c}{q}}(x-ct)] + \beta_1\cosh[\sqrt{\frac{c}{q}}(x-ct)]},$$

and

$$u_{14} = \mp\frac{6c}{(r+2s)(\sqrt{\beta_1^2+1}\sinh[\sqrt{\frac{c}{q}}(x-ct)] + \beta_1\cosh[\sqrt{\frac{c}{q}}(x-ct)])^2},$$

However, for $R\epsilon = \frac{c}{q} < 0,$ the solutions are

$$v_{15} = \pm\frac{2c\sqrt{-\frac{3}{p(r+2s)}}}{\sqrt{-(\beta_2^2+1)}\sin[\sqrt{-\frac{c}{q}}(x-ct)] + \beta_2\cos[\sqrt{-\frac{c}{q}}(x-ct)]},$$

$$u_{15} = -\frac{6c}{(r+2s)(\sqrt{-(\beta_2^2+1)}\sin[\sqrt{-\frac{c}{q}}(x-ct)] + \beta_2\cos[\sqrt{-\frac{c}{q}}(x-ct)])^2},$$

Using the second set and assuming $R\epsilon = -\frac{2c}{q} > 0, \mu = 0,$ the solutions for Eq. (44) read:

$$v_{16}=\pm\frac{2c\sqrt{\frac{3}{2p(r+2s)}}}{\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\mp\frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\ \beta_1\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\frac{+\sqrt{\beta_1^2+1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]},$$

$$u_{16}=\frac{p}{2c}\left(\pm\frac{2c\sqrt{-\frac{3}{2p(r+2s)}}}{\sqrt{\beta_1^2-1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\mp\frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}\ \beta_1\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\frac{+\sqrt{\beta_1^2-+1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\right)^2,$$

Using the second set and assuming $R\epsilon=-\frac{2c}{q}>0,\mu\neq 0,$ the solutions for Eq. (44) read:

$$v_{17}=\pm\frac{2c\sqrt{\frac{3\ \mu^2+1}{2p(r+2s)}}}{\mu+\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\mp\frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p\ \ r+2s}}(\beta_1\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{(r+2s)(\mu+\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)])^2}\frac{+\sqrt{\beta_1^2+1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)])}{(r+2s)(\mu+\sqrt{\beta_1^2+1}\sinh\left[\sqrt{-\frac{2c}{q}}(x-ct)\right]+\beta_1\cosh\left[\sqrt{-\frac{2c}{q}}(x-ct)\right])^2},$$

and

$$u_{17}=\frac{p}{2c}\left(\pm\frac{2c\sqrt{\frac{3\ \mu^2+1}{2p(r+2s)}}}{\mu+\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\mp\frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p\ \ r+2s}}\ \beta_1\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]}{(r+2s)(\mu+\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)])^2}\frac{+\sqrt{\beta_1^2+1}\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}{(r+2s)(\mu+\sqrt{\beta_1^2+1}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)]+\beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)])^2}\right)^2,$$

if $R\epsilon\ =-\frac{2c}{q}<0,\ \mu\neq 0,$ the solutions are

$$v_{18} = \pm \frac{2c\sqrt{\frac{3\ \mu^2+1}{2p(r+2s)}}}{\mu + \sqrt{-(\beta_2^2+1)}\sin[\sqrt{\frac{2c}{q}}(x-ct)] + \beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{p\ r+2s}}\ \ -\beta_2\sin[\sqrt{\frac{2c}{q}}(x-ct)]}{\mu + \sqrt{-(\beta_2^2+1)}\sin[\sqrt{\frac{2c}{q}}(x-ct)] + \beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \frac{+\sqrt{-\ \beta_2^2+1}\ \ \cos[\sqrt{\frac{2c}{q}}(x-ct)]}{\mu + \sqrt{-(\beta_2^2+1)}\sin[\sqrt{\frac{2c}{q}}(x-ct)] + \beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]},$$

and

$$u_{18} = \frac{p}{2c}\left(\pm\frac{2c\sqrt{\frac{3(\mu^2+1)}{2p(r+2s)}}}{\mu+\sqrt{-(\beta_2^2+1)}\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \mp \frac{\sqrt{\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}(-\beta_2\sin[\sqrt{\frac{2c}{q}}(x-ct)]}{\mu+\sqrt{-(\beta_2^2+1)}\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]} \frac{+\sqrt{-\ \beta_2^2+1}\ \ \cos[\sqrt{\frac{2c}{q}}(x-ct)])}{\mu+\sqrt{-(\beta_2^2+1)}\sin[\sqrt{\frac{2c}{q}}(x-ct)]+\beta_2\cos[\sqrt{\frac{2c}{q}}(x-ct)]}\right)^2$$

Using the third set and assuming $R\epsilon = -\frac{1}{2}\frac{c}{q} > 0,$ the solutions for Eq. (44) read:

$$v_{19} = \pm\frac{\sqrt{-\frac{2c}{q}}\sqrt{-\frac{3qc}{p(r+2s)}}(\beta_1\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]}{\sqrt{\beta_1^2+1}\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]} \frac{+\sqrt{\beta_1^2+1}\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)])}{\sqrt{\beta_1^2+1}\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)]},$$

and

$$u_{19} = \pm\frac{3c(\beta_1\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)] + \sqrt{\beta_1^2+1}\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)])^2}{(r+2s)(\sqrt{\beta_1^2+1}\sinh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1\cosh[\frac{1}{2}\sqrt{-\frac{2c}{q}}(x-ct)])^2},$$

Case 3. When $\delta \neq \pm 1,$ we obtain the following sets of solutions:

First set

$$a_0 = \pm\frac{c\sqrt{6}}{\sqrt{\frac{1}{p\ r+2s}}p(r+2s)},\ \ a_1 = \pm\sqrt{\frac{6}{p(r+2s)}}\epsilon\mu q,\ \ b_1 = 0,\ \ R\epsilon = -\frac{2c}{q},\ \ \delta = 0,$$

Second set

$$a_0 = 0,\ \ a_1 = 0,\ \ b_1 = \mp\sqrt{-\frac{6\epsilon}{p(r+2s)R}}c,\ \ \mu = 0,\ \ \delta = 0,$$

Third set

$$a_0 = \pm\sqrt{\frac{3}{2p(r+2s)}}c, \quad a_1 = 0, \quad b_1 = \pm\sqrt{-\frac{3\epsilon}{2p(r+2s)R}}c,$$
$$R = \frac{3}{2}\frac{\epsilon c^2}{pb_1^2(r+2s)}, \quad \mu = 0, \delta = 0,$$

Using the first set and assuming $R\epsilon = -\frac{2c}{q} > 0,$ the solutions for Eq. (44) read:

$$v_{20} = \mp\frac{c\sqrt{6}}{\sqrt{\frac{1}{p\ \ r+2s}}p(r+2s)} \pm \frac{2c\mu\sqrt{\frac{6}{p(r+2s)}}}{\mu + \sqrt{\beta_1^2}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]},$$

and

$$u_{20} = \frac{p}{2c}\left(\frac{c\sqrt{6}}{\mp\sqrt{\frac{1}{p\ \ r+2s}}p(r+2s)} \pm \frac{2c\mu\sqrt{\frac{6}{p\ \ r+2s}}}{\mu + \sqrt{\beta_1^2}\sinh[\sqrt{-\frac{2c}{q}}(x-ct)] + \beta_1\cosh[\sqrt{-\frac{2c}{q}}(x-ct)]}\right)^2.$$

Using the second set and assuming $R\epsilon > 0,$ the solutions for Eq. (44) read:

$$v_{21} = \pm\frac{c\sqrt{R\epsilon}\sqrt{\frac{6\epsilon}{pR(r+2s)}}(\beta_1\sinh[\sqrt{R\epsilon}(x-ct)] + \sqrt{\beta_1^2}\cosh[\sqrt{R\epsilon}(x-ct)])}{\epsilon(\sqrt{\beta_1^2}\sinh[\sqrt{R\epsilon}(x-ct)] + \beta_1\cosh[\sqrt{R\epsilon}(x-ct)])},$$

and

$$u_{21} = \frac{3c(\beta_1\sinh[\sqrt{R\epsilon}(x-ct)] + \sqrt{\beta_1^2}\cosh[\sqrt{R\epsilon}(x-ct)])^2}{(r+2s)(\sqrt{\beta_1^2}\sinh[\sqrt{R\epsilon}(x-ct)] + \beta_1\cosh[\sqrt{R\epsilon}(x-ct)])^2},$$

Using the third set and assuming $R\epsilon > 0,$ the solutions for Eq. (44) read:

$$v_{22} = \pm\frac{1}{2}c\sqrt{\frac{6}{p(r+2s)}} \mp\frac{1}{2}\frac{c\sqrt{R\epsilon}\sqrt{\frac{6\epsilon}{pR(r+2s)}}(\beta_1\sinh[\sqrt{R\epsilon}(x-ct)]}{\epsilon(\sqrt{\beta_1^2}\sinh[\sqrt{R\epsilon}(x-ct)] + \beta_1\cosh[\sqrt{R\epsilon}(x-ct)])} \frac{+\sqrt{\beta_1^2}\cosh[\sqrt{R\epsilon}(x-ct)])}{\epsilon(\sqrt{\beta_1^2}\sinh[\sqrt{R\epsilon}(x-ct)] + \beta_1\cosh[\sqrt{R\epsilon}(x-ct)])},$$

and

$$u_{22} = \frac{p}{2c}\left(\pm\frac{1}{2}c\sqrt{\frac{6}{p(r+2s)}} \mp \frac{\frac{c\sqrt{R\epsilon}}{2}\sqrt{\frac{6\epsilon}{pR(r+2s)}}(\beta_1\sinh[\sqrt{R\epsilon}(x-ct)]+\sqrt{\beta_1^2}\cosh[\sqrt{R\epsilon}(x-ct)])}{\epsilon(\sqrt{\beta_1^2}\sinh[\sqrt{R\epsilon}(x-ct)]+\beta_1\cosh[\sqrt{R\epsilon}(x-ct)])}\right)^2.$$

The solution obtained in this section are new up to our knowledge.

3. Conclusion

The modified generalized projective Riccati equation method is an efficient method for finding exact solutions of the nonlinear partial differential equations. The method is a powerful method to search for exact solutions to NLPDE´s, but is more complicated than other methods, in the sense that it demands more computer resources since the algebraic system may require a lot of time to be solved. In some cases, this system is so complicated that no computer algorithm may solve it, especially if the value of m is greater than four. Using modified generalized projective Riccati equation method, a series of new and more general exact solutions of the DSW equations are obtained including solitary wave solutions and triangular periodic solutions. A wide range of solutions are obtained which will be helpful for the understanding various physical phenomena described by these equations.

References

[1] Ablowitz MJ, Clarkson PA. Nonlinear evolution and inverse scattering. Cambridge, New York: Cambridge University Press;1991. http://www.dtic.mil/dtic/tr/fulltext/u2/a246164.pdf

[2] Lu D. C., Hong B. J. Backlund transformation and n-soliton-like solutions to the combined KdV-Burgers equation with variable coefficients, Int. J .Nonlinear Sci. 1(2), (2006) 3-10. http://www.worldacademicunion.com/journal/1749-3889-3897IJNS/IJNSVol2No1Paper1.pdf

[3] Matveev VA, Salle MA. Darboux transformation and solitons. Berlin, Heidelberg: Springer-Verlag; 1991. https://www.researchgate.net/publication/265505481_Darboux_Transformation_and_Solitons

[4] H.Q. Zhang et al. Darboux transformation and soliton solutions for the (2+1)-dimensional nonlinear Schrödinger hierarchy with symbolic computation. Physica A: Statistical Mechanics and its Applications, Volume 388, Issue 1, 1 January 2009, Pages 9-20. doi:10.1016/j.physa.2008.09.032

[5] Hirota R, Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, J Math Phys 14 (810) 1973. http://dx.doi.org/10.1063/1.1666400

[6] Liu S. K., Fu Z. T., Liu S. D. and Zhao Q., Jacobi Elliptic function Expansion method and Periodic Wave Solutions of Nonlinear Wave Equations, Phys. Lett. A 289 (2001) 69-74. doi:10.1016/S0375-9601(01)00580-1

[7] Malfliet W., Travelling Wave Solutions of Coupled Nonlinear Evolution equations, Am. J. Phys. 60 (1992) 650-654. doi:10.1016/j.mcm.2012.11.026

[8] Wazwaz A. M., Tanh method: Exact solutions of Sine-Gordon and Sinh-Gordon Equations, Appl. Maths and Computation 167 (2005) 1196-1210. doi:10.1016/j.amc.2004.08.005

[9] A.H.A. Ali et al. Soliton solution for nonlinear partial differential equations by cosine-function method, Physics Letters A 368 (2007) 299-304. doi:10.1016/j.physleta.2007.04.017

[10] Sirendaoreji. New exact travelling wave solutions for the Kawahara and modified Kawahara equations, Chaos Solitons Fractals 19 (2004) 147-150. doi:10.1016/S0960-0779(03)00102-4

[11] Xin Zeng, Xuelin Yong. A new mapping method and its applications to nonlinear partial differential equations. Physics Letters A 372 (2008) 6602--6607. doi:10.1016/j.physleta.2008.09.025

[12] R. Conte, M. Musette, Link between solitary waves and projective Riccati equations, J. Phys. A: Math. Gen. 25 (1992) 5609--5623. http://dx.doi.org/10.1088/0305-4470/25/21/019

[13] Yan Z. Chaos. Generalized method and its application in the higher-order nonlinear Schrodinger equation in nonlinear optical fibers. Solitons & Fractals 16 (2003) 759. doi:10.1016/S0960-0779(02)00435-6

[14] Chen Y, Li Biao. General projective Riccati equation method and exact solutions for generalized KdV-type and KdV--Burgers-type equations with nonlinear terms of any order. Chaos, Solitons & Fractals 19 (2004) 977. doi:10.1016/S0960-0779(03)00250-9

[15] X. Yong, Y. Chen. On two approaches of finding exact solutions to nonlinear evolution equations. Applied Mathematics and Computation 194 (2007) 74--84. doi:10.1016/j.amc.2007.04.013

[16] C.A. Gomez S, A.H. Salas. The Cole--Hopf transformation and improved tanh--coth method applied to new integrable system (KdV6). Applied Mathematics and Computation 204 (2008) 957—962. doi:10.1016/j.amc.2008.08.006

[17] Ding- Jiang Huang , Hong --Qing Zhang. Exact travelling wave solutions for the Boiti--Leon--Pempinelli equation Chaos, Solitons and Fractals 22 (2004) 243-247. doi:10.1016/j.chaos.2004.01.004

[18] Zuntao Fu, Shida Liu, Shikuo Liu. New solutions to mKdV equation. Phys Lett A 2004;364:374. doi:10.1016/j.physleta.2004.04.059

[19] V. G. Drinfel'd,V. V. Sokolov, Equations of Korteweg de Vries type, and simple Lie algebras, (Russian) Dokl. Akad. Nauk SSSR 258(1981)11-16.

[20] G. Wilson, The affine Lie algebra C(1) 2 and an equation of Hirota and Satsuma, Phys. Lett. A 89(1982)332-334. doi:10.1016/0375-9601(82)90186-4