



Leibniz’s rule and Fubini’s theorem associated with Hahn difference operators

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ABSTRACT

In 1945, Wolfgang Hahn introduced his difference operator $D_{q,\omega}$, which is defined by

$$D_{q,\omega}f(t) = \frac{f(qt + \omega) - f(t)}{(qt + \omega) - t}, \quad t \neq \omega_0,$$

where $\omega_0 = \frac{\omega}{1-q}$ with $0 < q < 1, \omega > 0$. In this paper, we establish Leibniz’s rule and Fubini’s theorem associated with this Hahn difference operator.

Keywords. q, ω -difference operator; q, ω -Integral; q, ω -Leibniz Rule; q, ω -Fubini’s Theorem.

1 Introduction

The Hahn difference operator is defined by

$$D_{q,\omega}f(t) = \frac{f(qt + \omega) - f(t)}{(qt + \omega) - t}, \quad t \neq \omega_0, \tag{1}$$

where $q \in (0,1)$ and $\omega > 0$ are fixed, see [2]. This operator unifies and generalizes two well known difference operators. The first is the Jackson q -difference operator which is defined by

$$D_qf(t) = \frac{f(qt) - f(t)}{qt - t}, \quad t \neq 0, \tag{2}$$

see [3, 4, 5, 6]. The second difference operator which Hahn’s operator generalizes is the forward difference operator

$$\Delta_\omega f(t) = \frac{f(t + \omega) - f(t)}{(t + \omega) - t}, \quad t \in \mathbf{R}, \tag{3}$$

where ω is a fixed positive number, see [9, 10, 13, 14]. The associated integral of (3) is the well known Nörlund sum

$$\int_\infty^x f(t) \Delta_\omega t = -\omega \sum_{k=0}^\infty f(x + k\omega), \tag{4}$$

see [12, 13, 15]. In some literature Nörlund sums are called the indefinite sums, cf. [14]. Then we can define

$$\int_a^b f(t) \Delta_\omega t = \omega \sum_{k=0}^\infty [f(a + k\omega) - f(b + k\omega)], \tag{5}$$

whenever the convergence of the series is guaranteed. Note that, under appropriate conditions,



$$\begin{aligned} \lim_{q \downarrow 1} D_{q,\omega} f(t) &= \Delta_\omega f(t), \\ \lim_{\omega \uparrow 0} D_{q,\omega} f(t) &= D_q f(t), \\ \lim_{q \downarrow 1, \omega \uparrow 0} D_{q,\omega} f(t) &= f'(t). \end{aligned} \tag{6}$$

In [1], A.Hamza et al. gave a rigorous analysis of the calculus associated with $D_{q,\omega}$. They stated and proved some basic properties of such a calculus. For instance, they defined the inverse of $D_{q,\omega}$ which contains the right inverse of D_q and the right inverse of Δ_ω . Then, they proved a fundamental lemma of Hahn calculus.

This paper is devoted to establishing Leibniz's rule and Fubini's theorem associated with the q, ω - difference operator. We organize this paper as follows. Section 2 gives an introduction to q, ω - difference calculus. In Section 3, we prove Leibniz's rule which is concerning with differentiating under the integral sign. Some related results are obtained. Also, we prove Fubini's theorem in Hahn difference operator setting, that is, we prove that the iterated integrals are equal.

2 Preliminaries

Let \mathbf{N} be the set of natural numbers and $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. For $k \in \mathbf{N}_0$ and $0 < q < 1$, we define the q -numbers

$$[k]_q := \frac{1-q^k}{1-q}.$$

Let I is an interval of \mathbf{R} containing ω_0 , where $\omega_0 := \omega/(1-q)$, and h denote the transformation

$$h(t) := qt + \omega, t \in I.$$

One can see that

$$h(t) \begin{cases} > t, & \text{for } t < \omega_0, \\ = t, & \text{for } t = \omega_0, \\ < t, & \text{for } t > \omega_0. \end{cases}$$

The transformation h has the inverse $h^{-1}(t) = (t - \omega)/q, t \in I$. The k -th order iteration of h is given by

$$h^k(t) := \underbrace{h \circ h \circ \dots \circ h}_{k\text{-times}}(t) = q^k t + \omega [k]_q, \quad t \in I, \tag{1}$$

$$(h^k(t))^{-1} := h^{-k}(t) := \underbrace{h^{-1} \circ h^{-1} \circ \dots \circ h^{-1}}_{k\text{-times}}(t) = \frac{t - \omega [k]_q}{q^k}, \quad t \in I. \tag{2}$$

Furthermore, $\{h^k(t)\}_{k=1}^\infty$ is a decreasing (an increasing) sequence in k when $t > \omega_0$ ($t < \omega_0$) with



$$\omega_0 = \begin{cases} \inf_{k \in \mathbb{N}} h^k(t), & t > \omega_0, \\ \sup_{k \in \mathbb{N}} h^k(t), & t < \omega_0. \end{cases} \quad (3)$$

The sequence $\{h^{-k}(t)\}_{k=1}^{\infty}$ is increasing (decreasing), $t > \omega_0$ ($t < \omega_0$) with

$$\infty = \begin{cases} \sup_{k \in \mathbb{N}} h^{-k}(t), & t > \omega_0, \\ -\inf_{k \in \mathbb{N}} h^{-k}(t), & t < \omega_0. \end{cases} \quad (4)$$

Let f be a function defined on I . The Hahn difference operator is defined in [2] by

$$D_{q,\omega}f(t) := \frac{f(qt + \omega) - f(t)}{(qt + \omega) - t}, \quad \text{if } t \neq \omega_0, \quad (5)$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$, provided that f is differentiable at ω_0 , where $q \in (0,1)$ and $\omega > 0$. In this case, we call $D_{q,\omega}f$, the q,ω -derivative of f . Finally, we say that f is q,ω -differentiable, i.e. throughout I , if $D_{q,\omega}f(\omega_0)$ exists.

One can easily check that if f, g are q,ω -differentiable at $t \in I$, then

$$D_{q,\omega}(\alpha f + \beta g)(t) = \alpha D_{q,\omega}f(t) + \beta D_{q,\omega}g(t), \quad \alpha, \beta \in \mathbb{C},$$

$$D_{q,\omega}(fg)(t) = D_{q,\omega}(f(t))g(t) + f(qt + \omega)D_{q,\omega}g(t),$$

$$D_{q,\omega}\left(\frac{f}{g}\right)(t) = \frac{D_{q,\omega}(f(t))g(t) - f(t)D_{q,\omega}g(t)}{g(t)g(qt + \omega)},$$

provided in the last identity $g(t)g(qt + \omega) \neq 0$, cf. [1]. The right inverse for $D_{q,\omega}$ is defined in [1] in terms of Jackson-Nörlund sums as follows. Let $a, b \in I$ the q,ω -integral of f from a to b is defined to be

$$\int_a^b f(t) d_{q,\omega}t := \int_{\omega_0}^b f(t) d_{q,\omega}t - \int_{\omega_0}^a f(t) d_{q,\omega}t, \quad (6)$$

$$\int_{\omega_0}^x f(t) d_{q,\omega}t := (x(1-q) - \omega) \sum_{k=0}^{\infty} q^k f(xq^k + \omega[k]_q), \quad x \in I, \quad (7)$$

provided that the series converges at $x = a$ and $x = b$. It is known that if f is continuous at ω_0 , then the series in (7) is uniformly convergent.

In the following we present some needed results from [1] concerning the calculus associated with $D_{q,\omega}$.

Corollary 1 The series



$$\sum_{k=0}^{\infty} |q^k (t(1-q) - \omega)|,$$

is uniformly convergent to $|t - \omega_0|$ on interval $I = [a, b]$ containing ω_0 .

Theorem 2 Let X be a Banach space endowed with a norm $\|\cdot\|$. Assume $f : I \rightarrow X$ is continuous at ω_0 . Then the following statements are true

- $\{f((sq^k) + \omega[k]_q)\}_{k \in \mathbb{N}}$ converges uniformly to $f(\omega_0)$ on I .
- $\sum_{k=0}^{\infty} q^k f(sq^k + \omega[k]_q)$ is uniformly convergent on I and consequently f is q, ω -integrable over I .
- The function

$$F(x) = \int_{\omega_0}^x f(t) d_{q,\omega} t, \quad x \in I,$$

is continuous at ω_0 . Furthermore, $D_{q,\omega} F(x)$ exists for every $x \in I$ and

$$D_{q,\omega} F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega} f(t) d_{q,\omega} t = f(b) - f(a) \quad \text{for all } a, b \in I.$$

Consequently, the q, ω -integration by parts for continuous function f, g is giving in by

$$\int_a^b f(t) D_{q,\omega} g(t) d_{q,\omega} t = f(t) g(t) \Big|_a^b - \int_a^b D_{q,\omega} (f(t)) g(qt + \omega) d_{q,\omega} t, \quad a, b \in I. \quad (8)$$

Theorem 3 Let $f : I \rightarrow \mathbb{R}$ be continuous function at ω_0 , then for $t \in I$

$$\int_{h(t)}^t f(s) d_{q,\omega} s = (t - h(t)) f(t).$$

We will apply the time scales calculus tools, see [7, 8], to obtain a q, ω -analog of the chain rule.

Theorem 4 Let $g : I \rightarrow \mathbb{R}$ be continuous and q, ω -differentiable and $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuously differentiable. Then, there exists c between $qt + \omega$ and t such that,

$$D_{q,\omega} (f \circ g)(t) = f'(g(c)) D_{q,\omega} g(t). \quad (9)$$

3 q, ω -Differentiation Under The Integral Sign

In this section we study the continuity and the q, ω -differentiation of the integral

$$\int_{\phi(t)}^{\psi(t)} f(t, s) d_{q,\omega} s.$$

We establish Libnitz's rule. Finally, we prove that the iterated integrals are equal (this theorem is known by Fubini's Theorem).

Let $f : I \times I \rightarrow \mathbb{R}$. We begin by the following definitions.



Definition 1

(i) We say that $f(t, s)$ is continuous at $t = t_0$ uniformly with respect to $s \in [a, b] \subseteq I$ if $\lim_{t \rightarrow t_0} f(t, s) = f(t_0, s)$ uniformly with respect to $s \in [a, b] \subseteq I$.

(ii) The q, ω -partial derivative of f with respect to t is defined by

$$D_{q, \omega, t} f(t, s) = \begin{cases} \frac{f(t, s) - f(h(t), s)}{t - h(t)} & t \in I \setminus \omega_0, \\ \lim_{t \rightarrow \omega_0} \frac{f(t, s) - f(\omega_0, s)}{t - \omega_0} & t = \omega_0, \end{cases}$$

whenever the limit exists.

(iii) We say that $f(t, s)$ is uniformly partially differentiable at $t = \omega_0 \in I$, with respect to $s \in [a, b] \subseteq I$ if

$$\lim_{t \rightarrow \omega_0} \frac{f(t, s) - f(\omega_0, s)}{t - \omega_0}$$

exists uniformly with respect to $s \in [a, b]$.

Definition 2 Let $\omega_0 \in [a, b] \subseteq I$. We define the q, ω -interval by

$$[a, b]_{q, \omega} = \{aq^k + \omega[k]_q : k \in \mathbf{N}_0\} \cup \{bq^k + \omega[k]_q : k \in \mathbf{N}_0\} \cup \{\omega_0\}.$$

For any point $c \in I$, we denote by

$$[c]_{q, \omega} = \{cq^k + \omega[k]_q : k \in \mathbf{N}_0\} \cup \{\omega_0\}.$$

From now on, we assume some appropriate conditions that imply the integrals of the form $\int_{\phi(t)}^{\psi(t)} g(t, s) d_{q, \omega} s := F(t)$ exist, where $g : I \times I \rightarrow \mathbf{R}$ and $\phi, \psi : I \rightarrow I$.

Lemma 3 The following statements are true

(i) Assume that $f(t, s)$ is continuous at $t = \omega_0$ uniformly with respect to $s \in [b]_{q, \omega}$. Then

$$F(t) = \int_{\omega_0}^b f(t, s) d_{q, \omega} s$$

is continuous at $t = \omega_0$.

(i) $D_{q, \omega} F(t) = \int_{\omega_0}^b D_{q, \omega, t} f(t, s) d_{q, \omega} s, \quad t \neq \omega_0.$

(ii) If $D_{q, \omega, t} f(t, s)$ exists uniformly at $t = \omega_0$ with respect to $s \in [b]_{q, \omega}$, then $D_{q, \omega} F(t)$ exists at $t = \omega_0$

and

$$D_{q, \omega_0} F(\omega_0) = \int_{\omega_0}^b D_{q, \omega, t} f(\omega_0, s) d_{q, \omega} s.$$

Proof.



(i) Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$\forall t \in I (|t - \omega_0| < \delta \Rightarrow |f(t, bq^k + \omega[k]_q) - f(\omega_0, bq^k + \omega[k]_q)| < \frac{\varepsilon}{|b - \omega_0|}, \forall k).$$

In view of corollary 1, for $t \in I \cap]\omega_0 - \delta, \delta + \omega_0[$, we have

$$|F(t) - F(\omega_0)| = \left| \sum_{k=0}^{\infty} q^k (b(1-q) - \omega) [f(t, bq^k + \omega[k]_q) - f(\omega_0, bq^k + \omega[k]_q)] \right| < \varepsilon.$$

(ii) For $t \neq \omega_0$, we have

$$\begin{aligned} D_{q,\omega} F(t) &= \sum_{k=0}^{\infty} \frac{q^k (b(1-q) - \omega)}{t - h(t)} [f(t, bq^k + \omega[k]_q) - f(h(t), bq^k + \omega[k]_q)] \\ &= \sum_{k=0}^{\infty} q^k (b(1-q) - \omega) D_{q,\omega,t} f(t, bq^k + \omega[k]_q) \\ &= \int_0^b D_{q,\omega,t} f(t, s) d_{q,\omega} s. \end{aligned}$$

(iii) Assume that $D_{q,\omega,t} f(t, s)$ exists uniformly at $t = \omega_0$ with respect to $s \in [b]_{q,\omega}$. We conclude that

$$\begin{aligned} & \left| \frac{F(t) - F(\omega_0)}{t - \omega_0} - \int_{\omega_0}^b D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right| \\ &= \left| \sum_{k=0}^{\infty} \frac{q^k (b(1-q) - \omega)}{t - \omega_0} f(t, bq^k + \omega[k]_q) \right. \\ & \quad \left. - \sum_{k=0}^{\infty} \frac{q^k (b(1-q) - \omega)}{t - \omega_0} f(\omega_0, bq^k + \omega[k]_q) \right. \\ & \quad \left. - \sum_{k=0}^{\infty} q^k (b(1-q) - \omega) D_{q,\omega,t} f(\omega_0, bq^k + \omega[k]_q) \right| \\ &= \left| \sum_{k=0}^{\infty} (q^k (b(1-q) - \omega) \left[\frac{f(t, bq^k + \omega[k]_q) - f(\omega_0, bq^k + \omega[k]_q)}{t - \omega_0} \right. \right. \right. \\ & \quad \left. \left. - D_{q,\omega,t} f(\omega_0, bq^k + \omega[k]_q) \right] \right|. \end{aligned}$$

For $\varepsilon > 0$, there exists $\delta > 0$ such that for all $t \in I$ we have $0 < |t - \omega_0| < \delta$

$$\Rightarrow \left| \frac{f(t, bq^k + \omega[k]_q) - f(\omega_0, bq^k + \omega[k]_q)}{t - \omega_0} - D_{q,\omega,t} f(\omega_0, bq^k + \omega[k]_q) \right| < \frac{\varepsilon}{|b - \omega_0|}.$$

In view of corollary 1, for $t \in I$ such that $0 < |t - \omega_0| < \delta$, we see that

$$\left| \frac{F(t) - F(\omega_0)}{t - \omega_0} - \int_{\omega_0}^b D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right| < \varepsilon.$$

Corollary 4 If $D_{q,\omega,t} f(t, s)$ exists uniformly at $t = \omega_0$ with respect to $s \in [a]_{q,\omega}$ and $s \in [b]_{q,\omega}$, then



$$D_{q,\omega,t} \int_a^b f(t,s) d_{q,\omega} s = \int_a^b D_{q,\omega,t} f(t,s) d_{q,\omega} s, \quad t \in I.$$

Proof. By lemma 3, we get the desired result from the following inequality

$$D_{q,\omega,t} \int_a^b f(t,s) d_{q,\omega} s = D_{q,\omega,t} \left[\int_{\omega_0}^b f(t,s) d_{q,\omega} s - \int_{\omega_0}^a f(t,s) d_{q,\omega} s \right].$$

Theorem 5 (Leibniz's integral rule) Define the function F by

$$F(t) := \int_{\omega_0}^t f(t,s) d_{q,\omega} s.$$

The following statements are true

(i) For $t \neq \omega_0$, we have

$$D_{q,\omega} F(t) = f(h(t),t) + \int_{\omega_0}^t D_{q,\omega,t} f(t,s) d_{q,\omega} s. \quad (1)$$

(ii) If $f(t,s)$ is continuous at (ω_0, ω_0) , then F is differentiable at $t = \omega_0$ and $F'(\omega_0) = f(\omega_0, \omega_0)$.

Proof.

(i) At $t \neq \omega_0$, we have

$$\begin{aligned} D_{q,\omega} F(t) &= \sum_{k=0}^{\infty} \frac{q^k (t(1-q) - \omega)}{t - h(t)} [f(t, tq^k + \omega[k]_q) \\ &+ f(h(t), tq^k + \omega[k]_q) - f(h(t), tq^k + \omega[k]_q)] \\ &- \sum_{k=0}^{\infty} \frac{q^{k+1} (t(1-q) - \omega)}{t - h(t)} f(h(t), tq^{k+1} + \omega[k+1]_q) \\ &= \sum_{k=0}^{\infty} \frac{q^k (t(1-q) - \omega)}{t - h(t)} [f(t, tq^k + \omega[k]_q) - f(h(t), tq^k + \omega[k]_q)] \\ &+ \sum_{k=0}^{\infty} \frac{q^k (t(1-q) - \omega)}{t - h(t)} f(h(t), tq^k + \omega[k]_q) \\ &- \sum_{k=0}^{\infty} \frac{q^{k+1} (t(1-q) - \omega)}{t - h(t)} f(h(t), tq^{k+1} + \omega[k+1]_q) \\ &= \frac{1}{t - h(t)} \left[\int_{\omega_0}^{h(t)} f(h(t), s) d_{q,\omega} s - \int_{\omega_0}^t f(h(t), s) d_{q,\omega} s \right] \\ &+ \int_{\omega_0}^t \frac{f(h(t), s) - f(t, s)}{t - h(t)} d_{q,\omega} s \\ &= \frac{1}{t - h(t)} \int_t^{h(t)} f(h(t), s) d_{q,\omega} s + \int_{\omega_0}^t D_{q,\omega,t} f(t, s) d_{q,\omega} s. \end{aligned}$$



By theorem 3, we get

$$D_{q,\omega} F(t) = f(h(t), t) + \int_{\omega_0}^t D_{q,\omega,t} f(t, s) d_{q,\omega} s.$$

(ii) Using corollary 1, we see that

$$\begin{aligned} & \left| \frac{F(s) - F(\omega_0)}{s - \omega_0} - f(\omega_0, \omega_0) \right| \\ &= \frac{1}{(s - \omega_0)} \left| \sum_{k=0}^{\infty} q^k (s(1-q) - \omega) [f(s, sq^k + \omega[k]_q) - f(\omega_0, \omega_0)] \right|. \end{aligned}$$

The continuity of $f(t, s)$ at (ω_0, ω_0) implies that given $\varepsilon > 0$, there exists $\delta > 0$ such that $|s - \omega_0| < \delta$, (and consequently $sq^k + \omega[k]_q < \delta$), implies $|f(s, sq^k + \omega[k]_q) - f(\omega_0, \omega_0)| < \varepsilon$. Thus

$$\left| \frac{F(s) - F(\omega_0)}{s - \omega_0} - f(\omega_0, \omega_0) \right| < \varepsilon,$$

whenever $0 < |s - \omega_0| < \delta$, which completes the proof.

Theorem 6 Let $\phi: I \rightarrow I$ be bounded. Define the function F by

$$F(t) := \int_{\omega_0}^{\phi(t)} f(t, s) d_{q,\omega} s.$$

The following statements are true

(i) The function F is q, ω -differentiable at $t \neq \omega_0$ and

$$D_{q,\omega} F(t) = \frac{1}{t - h(t)} \int_{\phi(t)}^{\phi(h(t))} f(h(t), s) d_{q,\omega} s + \int_{\omega_0}^{\phi(t)} D_{q,\omega,t} f(t, s) d_{q,\omega} s, \quad t \neq \omega_0. \quad (2)$$

(ii) Assume that the following conditions hold

- ϕ is q, ω -differentiable.

- $f(t, s)$ is uniformly partially differentiable at $t = \omega_0$ with respect to $s \in I$ such that $D_{q,\omega,t} f(\omega_0, s)$ is continuous at $s = \omega_0$.

- The function $H(t) = \int_{\omega_0}^t f(\omega_0, s) d_{q,\omega} s$ is differentiable at $\phi(\omega_0)$.

Then $D_{q,\omega} F(t)$ exists at $t = \omega_0$ and

$$D_{q,\omega} F(\omega_0) = f(\omega_0, \phi(\omega_0)) \phi'(\omega_0) + \int_{\omega_0}^{\phi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s.$$

Proof.

(i) For $t \neq \omega_0$ we have

$$\begin{aligned}
 D_{q,\omega}F(t) &= \sum_{k=0}^{\infty} \frac{q^k(\phi(t)(1-q)-\omega)}{t-h(t)} [f(t, \phi(t)q^k + \omega[k]_q) \\
 &- f(h(t), \phi(t)q^k + \omega[k]_q)] \\
 &+ \sum_{k=0}^{\infty} \frac{q^k(\phi(t)(1-q)-\omega)}{t-h(t)} f(h(t), \phi(t)q^k + \omega[k]_q) \\
 &- \sum_{k=0}^{\infty} \frac{q^{k+1}(\phi(t)(1-q)-\omega)}{t-h(t)} f(h(t), \phi(t)q^{k+1} + \omega[k+1]_q) \\
 &= \int_{\omega_0}^{\phi(t)} \frac{f(h(t), s) - f(t, s)}{t-h(t)} d_{q,\omega} s + \frac{1}{t-h(t)} \left[\int_{\omega_0}^{\phi(t)} f(t, s) d_{q,\omega} s \right. \\
 &- \left. \int_{\omega_0}^{\phi(h(t))} f(t, s) d_{q,\omega} s \right] \\
 &= \frac{1}{t-h(t)} \int_{\phi(h(t))}^{\phi(t)} f(t, s) d_{q,\omega} s + \int_{\omega_0}^{\phi(t)} D_{q,\omega,t} f(t, s) d_{q,\omega} s.
 \end{aligned}$$

(ii) To ensure the differentiability at $t = \omega_0$, we write F as follows

$$F(t) = P(t) + (H \circ \phi)(t),$$

where

$$P(t) = \int_{\omega_0}^{\phi(t)} \{f(t, s) - f(\omega_0, s)\} d_{q,\omega} s.$$

First, we show that $D_{q,\omega}P(\omega_0) = \int_{\omega_0}^{\phi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s$. Indeed, one can see that

$$\begin{aligned}
 & \left| \frac{P(t) - P(\omega_0)}{t - \omega_0} - \int_{\omega_0}^{\phi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right| \\
 &= \left| \int_{\omega_0}^{\phi(t)} \left[\frac{f(t, s) - f(\omega_0, s)}{t - \omega_0} - D_{q,\omega,t} f(\omega_0, s) \right] d_{q,\omega} s + \int_{\omega_0}^{\phi(t)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right. \\
 &\quad \left. - \int_{\omega_0}^{\phi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right| \\
 &\leq \int_{\omega_0}^{\phi(t)} \left| \left[\frac{f(t, s) - f(\omega_0, s)}{t - \omega_0} - D_{q,\omega,t} f(\omega_0, s) \right] \right| d_{q,\omega} s + \left| \int_{\omega_0}^{\phi(t)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right. \\
 &\quad \left. - \int_{\omega_0}^{\phi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right|.
 \end{aligned}$$

Since $f(t, s)$ is uniformly partially differentiable at $t = \omega_0$ with respect to $s \in I$, and ϕ is bounded, then

$$\int_{\omega_0}^{\phi(t)} \left| \frac{f(t, s) - f(\omega_0, s)}{t - \omega_0} - D_{q,\omega,t} f(\omega_0, s) \right| d_{q,\omega} s \rightarrow 0 \text{ as } t \rightarrow \omega_0.$$

The continuity of $D_{q,\omega,t} f(\omega_0, s)$ at $s = \omega_0$ implies that $K(t) = \int_{\omega_0}^t D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s$ is continuous at $t = \omega_0$ which in turn implies that $K(\phi(t))$ is continuous at $t = \omega_0$, that is

$$\left| \int_{\omega_0}^{\phi(t)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s - \int_{\omega_0}^{\phi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right| \rightarrow 0 \text{ as } t \rightarrow \omega_0.$$

Consequently, we conclude that

$$\left| \frac{P(t) - P(\omega_0)}{t - \omega_0} - \int_{\omega_0}^{\phi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s \right| \rightarrow 0 \text{ as } t \rightarrow \omega_0. \quad (3)$$

Since H is differentiable at $\phi(\omega_0)$ and ϕ is differentiable at ω_0 , then $H \circ \phi$ is differentiable at $t = \omega_0$ and $D_{q,\omega}(H \circ \phi)(\omega_0) = (H \circ \phi)'(\omega_0) = H'(\phi(\omega_0))\phi'(\omega_0) = f(\omega_0, \phi(\omega_0))\phi'(\omega_0)$. Therefore, we get the desired result.

Corollary 7 Let $\phi, \psi : I \rightarrow I$ be bounded functions. Define the function F by

$$F(t) := \int_{\phi(t)}^{\psi(t)} f(t, s) d_{q,\omega} s.$$

Then, the following statements are true

(i) For $t \neq \omega_0$, we have

$$\begin{aligned} D_{q,\omega} F(t) &= \frac{1}{t - h(t)} \left[\int_{\psi(h(t))}^{\psi(t)} f(h(t), s) d_{q,\omega} s - \int_{\phi(h(t))}^{\phi(t)} f(h(t), s) d_{q,\omega} s \right] \\ &\quad + \int_{\phi(t)}^{\psi(t)} D_{q,\omega,t} f(t, s) d_{q,\omega} s, \quad t \neq \omega_0. \end{aligned}$$

(ii) Assume that the following conditions hold

- ϕ and ψ are q, ω -differentiable.
- $f(t, s)$ is uniformly partially differentiable at $t = \omega_0$ with respect to $s \in I$ such

that $D_{q,\omega,t} f(\omega_0, s)$ is continuous at $s = \omega_0$.

- The function $H(t) = \int_{\omega_0}^t f(\omega_0, s) d_{q,\omega} s$ is differentiable at $\phi(\omega_0)$ and $\psi(\omega_0)$.

Then $D_{q,\omega} F(t)$ exists at $t = \omega_0$ and

$$D_{q,\omega} F(\omega_0) = f(\omega_0, \psi(\omega_0))\psi'(\omega_0) - f(\omega_0, \phi(\omega_0))\phi'(\omega_0) + \int_{\phi(\omega_0)}^{\psi(\omega_0)} D_{q,\omega,t} f(\omega_0, s) d_{q,\omega} s.$$

Proof. By theorem 6 and using the definition

$$D_{q,\omega,t} \int_{\phi(t)}^{\psi(t)} f(t, s) d_{q,\omega} s = D_{q,\omega,t} \left[\int_{\omega_0}^{\psi(t)} f(t, s) d_{q,\omega} s - \int_{\omega_0}^{\phi(t)} f(t, s) d_{q,\omega} s \right],$$

we get the desired result.

Theorem 8 (Fubini's theorem)

Let f be defined on the closed rectangle $R = [\omega_0, a] \times [\omega_0, b] \subset I \times I$. Assume that $f(t, s)$ is

continuous at $t = \omega_0$ uniformly with respect to $s \in [a]_{q,\omega}$ and continuous at $s = \omega_0$ uniformly with respect to $t \in [b]_{q,\omega}$. Then, the double q, ω -integrals

$$\int_{\omega_0}^b \int_{\omega_0}^a f(t, s) d_{q,\omega} t d_{q,\omega} s \quad \text{and} \quad \int_{\omega_0}^a \int_{\omega_0}^b f(t, s) d_{q,\omega} s d_{q,\omega} t$$

exist and they are equal, that is

$$\int_{\omega_0}^b \int_{\omega_0}^a f(t, s) d_{q,\omega} t d_{q,\omega} s = \int_{\omega_0}^a \int_{\omega_0}^b f(t, s) d_{q,\omega} s d_{q,\omega} t.$$

Proof. By assumptions, lemma 3 tells us that $\int_{\omega_0}^a f(t, s) d_{q,\omega} t$ and $\int_{\omega_0}^b f(t, s) d_{q,\omega} s$ are continuous at $s = \omega_0$ and at $t = \omega_0$ respectively. Therefore both double q, ω -integrals above exist and we have

$$\begin{aligned} \int_{\omega_0}^b \int_{\omega_0}^a f(t, s) d_{q,\omega} t d_{q,\omega} s &= \int_{\omega_0}^b \left[\sum_{j=0}^{\infty} q^j (a(1-q) - \omega) f(aq^j + \omega[j]_q, s) \right] d_{q,\omega} s \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} q^{k+j} (b(1-q) - \omega)(a(1-q) - \omega) f(aq^j + \omega[j]_q, bq^k + \omega[k]_q) \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q^{j+k} (a(1-q) - \omega)(b(1-q) - \omega) f(aq^j + \omega[j]_q, bq^k + \omega[k]_q) \\ &= \int_{\omega_0}^a \left[\sum_{k=0}^{\infty} q^k (b(1-q) - \omega) f(t, bq^k + \omega[k]_q) \right] d_{q,\omega} t = \int_{\omega_0}^a \int_{\omega_0}^b f(t, s) d_{q,\omega} s d_{q,\omega} t. \end{aligned}$$

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