Uniqueness of Solutionfor Nonlinear Implicit Fractional Differential Equation

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ABSTRACT

We study the uniquenessof solutionfor nonlinear implicit fractional differential equation with initial condition involving Caputo fractional derivative. The technique used in our analysis is based on an application of Bihari and Medved inequalities.

Keywords:

Caputo fractional derivative, fractional integral, Implicit fractional differential equation, BihariandMedved inequalities.

Subject Classification: 26D10, 26A33, 34A08, 34A09, 34A12, 34A34.

1. INTRODUCTION

Fractional calculus is generalization of ordinary differential equations and integrations to arbitrary non-integerorders.One can describe many phusical phenomena arising in engineering, physics, economics and science more accurately through the fractional derivative formulation, see [4, 5, 7, 9, 11, 18, 19].

Withinyears the fractional calculus became a very attractive subject to mathematicians,and many different forms of fractional (i.e., noninteger) differential operators wereintroduced: the Grunwald–Letnikow, Riemann–Liouville, Hadamard, Caputo, Riesz(Hilfer 2000; Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993) and the morerecent notions of Cresson (2007), and Katugampola (2011) or variable order fractional operators introduced by Samko and Ross(1993). Several authors have studied the existence, uniqueness and qualitative properties of the initial value problems of fractional order by different techniques, refer [5, 7, 10, 14, 18, 21].

Recently, Pachpatte D. B. and Chinchane V. L. [1] have discussed the uniqueness of solution of fractional differential equation with the Riemann-Liouville derivative.

By motivation of the recent works [1, 15], we extends the results presented by Pachpatte D. B. and Chinchane V. L. for nonlinear implicit fractional differential equation with Caputo fractional derivative. In this paper we consider the initial valve problem of the type

$$^cD^\alpha x(t) = f\big(t, x(t), \; ^cD^\alpha x(t)\big), \tag{1.1}$$

$$x(0) = x_0 \in \mathbb{R}, \quad t \in J = [0, b] \tag{1.2}$$

where $^cD^\alpha$ $(0 < \alpha < 1)$ denotes the Caputo fractional derivative operator and f is real conitnuous valued function on $J \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; $\mathbb{R}$ denotes the real space.

The paper is organized as follows. In Section 2, some definitions, lemmas and preliminary results are intoduced to be used in the sequel. Section 3 will deals the results and proofs of uniqueness problm of (1.1).

2. PRELIMINARIES

Let us recall the concepts of the fractional calculus[7, 8, 13, 15, 18, 19, 20] for more details.

Definition 2.1 A real valued function $f(t)$ $(t > 0)$ is said to be in space C_μ $(\mu \in \mathbb{R})$ if thereexists a real number $p(> \mu)$ such that $f(t) = t^p g(t)$, where $g \in C[0, \infty)$.

Definition 2.2 A real function $f(t)(t > 0)$ is said to be in the space C_μ^n, $n \in \mathbb{N} \cup \{0\}$, iff $(n) \in C_\mu$.

Definition 2.3. Let $f \in C_\mu$ $(\mu \geq -1)$, then the (left-sided) Riemann–Liouville fractionalintegral of order $\alpha > 0$ of the function f is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds, \; t > 0 \text{ and } I^0 f(t) = f(t) \tag{2.1}$$

where$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha - 1} dx$.

Definition 2.4 The (left-sided) Caputo fractional derivative of order $\alpha > 0$ of the function $f \in C_{-1}^n$ $(n \in \mathbb{N} \cup \{0\})$, is given by:

$$^cD^\alpha f(t) = f(x) = \begin{cases} -f^{(n)}(t), & \text{if } \alpha = n \\ I^{n-\alpha} f^{(n)}(t), & \text{if } n - 1 < \alpha < n, \; n \in \mathbb{N}. \end{cases} \tag{2.2}$$

Note that $I^\alpha \; ^cD^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$.

Definition 2.5. The (left-sided) Riemann–Liouville fractional derivative of order $\alpha > 0$ ofthe function $f \in C_{-1}^n$ ($n \in \mathbb{N} \cup \{0\}$), is given by:

$$D^\alpha f(t) = \frac{d^n}{dt^n} I^{n-\alpha} f(t), \quad n - 1 < \alpha \leq n, \quad n \in \mathbb{N}. \quad (2.3)$$

In [15], the author have been studied the existence and uniqueness of theinitial value problem (1.1)-(1.2), first, let us reduce the problem (1.1)-(1.2) into equivalent fractional integral equation, we obtain

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), \quad {}^cD^\alpha x(s)) ds. (2.4)$$

The results of our work in the present paper differ substantially from the work of [8, 15] in terms of assumptions and methods of proof.

Also, in [17] Medved defined a special class of nonlinear function and developeda method to estimate solution for nonlinear integral inequalities with singularkernel. The class of function defined as follows:

Definition 2.6.[17] Let $q > 0$ be a real number and $0 < b \leq \infty$. The function $w : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following condition

$$e^{-qt}[w(u)]^q \leq R(t) w(e^{-qt} u^q), (2.5)$$

for all $u \in \mathbb{R}_+$, $t \in [0, b]$, where $R(t)$ is a continuous, nonnegative function.

Remark 2.1. If $w(u) = u^m$, $m > 0$, then

$$e^{-qt}[w(u)]^q = e^{(m-1)qt} w(e^{-qt} u^q), \quad (2.6)$$

for any $q > 1$. i.e the condition (2.5) is satisfies with $R(t) = e^{(m-1)qt}$. For$w(u) = u + au^m$, where $0 \leq a \leq 1, m \geq 1$, the function w satisfies the condition(2.5) with $q > 1$ and $R(t) = 2^{q-1} e^{qmt}$, see [12].

In [17], Medved introduced the following inequalities which are the best widely and known in the study of many qualitative properties of solution of differential equations.

Lemma 2.1 Let $0 \leq T \leq \infty$, $u(t)$, $b(t), a(t)$, $a'(t) \in C([0, T), R+)$; $w \in C(\mathbb{R}^+, \mathbb{R})$ be nondecreasing function, $w(0) = 0, w(u) > 0$ on $(0, T)$, and

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s) w(u(s)) ds, (2.7)$$

for $t \in [0, T)$where $\beta > 0$ is constant. Then following hold:

(i). Suppose $\beta > \frac{1}{2}$, and if w satisfies the condition (2.5) with $q = 2$, then

$$u(t) \leq e^t \{\Omega^{-1}[\Omega(2a(t)^2) + g_1(t)]\}^{\frac{1}{2}}, \quad (2.8)$$

for $t \in [0, T_1]$, where

$$g_1(t) = \frac{\Gamma(2\beta-1)}{4^{\beta-1}} \int_0^t R(t) b(s)^2 ds, (2.9)$$

where Γ is gamma function, $\Omega(v) = \int_{v_0}^v \frac{ds}{w(s)}, v_0 > 0$, $\Omega^{-1}$is the inverse of Ω, and $t \in \mathbb{R}_+$ is such that $\Omega(2a(t)^2) + g_1(t) \in \text{Dom}(\Omega^{-1})$ for all $t \in [0, T_1]$.

(ii). Let $\beta = (0, \frac{1}{2}]$and w satisfies the condition (2.5) with $q = z = 2$, where $z = \frac{1-\beta}{\beta}$, i. e. $\beta = \frac{1}{z+1}$. LetΩ, Ω^{-1}be as in part (i). Then

$$u(t) \leq e^t \{\Omega^{-1}[\Omega(2^{q-1} a(t)^2) + g_2(t)]\}^{\frac{1}{q}}, \quad (2.10)$$

for $t \in [0, T_1]$, where

$$g_2(t) = 2^{q-1} K_z^q \int_0^t R(t) b(s)^2 ds, \quad (2.11)$$

$$K_z = \left[\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right]^{\frac{1}{p}}, \quad \alpha = \frac{z}{z+1}, \quad p = \frac{z+2}{z+1} \quad (2.12)$$

and$T_1 \in \mathbb{R}_+$is such that $\Omega(2^{q-1} a(t)^2) + g_2(t) \in \text{Dom}(\Omega^{-1})$ for all $t \in [0, T_1]$.

Lemma 2.2 Let $0 \leq T \leq \infty$, $u(t)$, $b(t), a(t)$, $a'(t) \in C([0, T), R+)$; $w \in C(\mathbb{R}^+, \mathbb{R})$ and

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} b(s) u(s) ds, \quad (2.13)$$

for $t \in [0, T)$where $\beta > 0$ is constant. Then following hold:

(i). Suppose $\beta > \frac{1}{2}$, then

$$u(t) \leq (\sqrt{2})a(t)\exp\left[\frac{2\Gamma(2\beta-1)}{4^\beta}\int_0^t b(s)^2 ds + t\right], \tag{2.14}$$

for $t \in [0, T]$.

(ii). If $\beta = \frac{1}{z+1}$, for some $z \geq 1$, then

$$u(t) \leq (2^{q-1})^{\frac{1}{q}} a(t)\exp\left[\frac{2^{q-1}}{q} K_z^q \int_0^t b(s)^q ds + t\right], \tag{2.15}$$

for $t \in [0, T]$, where K_z is defined as in (2.12), $q = z + 2$.

For detail proof of above two theorems see [16].

Lemma 2.3 ([2, 16], p. 152) (Bihari inequality) Let u and f be nonnegativedefined on $\mathbb{R}_+$, let $w(u)$ be continuous nondecreasing function defined on $\mathbb{R}^+$, and $w(u) > 0$, on $(0, \infty)$. If

$$u(t) \leq k + \int_0^t f(s)w\big(u(s)\big)ds, \tag{2.16}$$

for $t \in \mathbb{R}_+$ where k is nonnegative constant, for $0 \leq t \leq T$,

$$u(t) \leq G^{-1}\left[G(k) + \int_0^t f(s)ds\right], \tag{2.17}$$

where

$$G(r) = \int_0^t \frac{ds}{w(s)}, r > 0, r_0 > 0$$

and G^{-1} is the inverse fucntion of G and $t_1 \in \mathbb{R}_+$ is chosen so that $G(k) + \int_0^t f(s)ds \in \text{Dom}(G^{-1})$ for all $t \in \mathbb{R}_+$ laying in the interval $0 \leq t \leq t_1$.

3. RESULTS

In this section, we establish the uniqueness of solution of the following initialvalue problem.

Theorem 3.1 If $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfiescondition

$$|f(t, x, y) - f(t, \bar{x}, \bar{y})| \leq M\Phi(|x - \bar{x}|) + L|y - \bar{y}|, \tag{3.1}$$

whre M is positive constant, $L \in (0,1)$ and Φ is a continuous nondecreasing function on $0 < u \leq A$, with $\Phi(0) = 0$ and

$$\int_0^A \frac{du}{\Phi(u)}, \tag{3.2}$$

then the problem (1.1)-(1.2) has unique solution on J.

Proof.Assume that there exists two solutions$x(t)$ and $y(t)$ of (1.1)-(1.2). Then we have

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, x(s), \ ^cD^\alpha x(s))ds \tag{3.3}$$

and

$$y(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s, y(s), \ ^cD^\alpha y(s))ds. \tag{3.4}$$

Hence we have

$$|x(t) - y(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f\big(s, x(s), \ ^cD^\alpha x(s)\big) - f(s, y(s), \ ^cD^\alpha y(s))\big|ds$$

$$\leq \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[M\Phi(|x(s) - y(s)|) + L\,|^cD^\alpha(x(s) - y(s))|]ds. \tag{3.5}$$

But by hypothesis (3.1) for any $t \in [0, b]$ and any $x, y \in \mathbb{R}$,

$$|^cD^\alpha(x(t) - y(t))| = |\, f(t, x(t), \ ^cD^\alpha x(t)) - f(t, y(t), \ ^cD^\alpha y(t))|$$

$$\leq M\Phi(|x(t) - y(t)|) + L\,|^cD^\alpha(x(t) - y(t))|.$$

This implies

$$|^cD^\alpha(x(t) - y(t))| \leq \frac{M}{1-L}\Phi(|x(t) - y(t)|). \tag{3.6}$$

Using above estimation in (3.5), we get

$$|x(t) - y(t)| \leq \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M\Phi(|x(s) - y(s)|) + \frac{ML}{1-L}\Phi(|x(s) - y(s)|)\right]ds$$

$$\leq \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M + \frac{ML}{1-L}\right]\Phi(|x(s) - y(s)|)ds$$

$$\leq \ \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, M\left[1+\frac{L}{1-L}\right]\Phi(|x(s)-y(s)|)ds$$

$$\leq \ \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\frac{M}{1-L}\Phi(|x(s)-y(s)|)ds. \tag{3.7}$$

Now an application of Lemma 2.3 to (3.7) which yields

$$|x(t)-y(t)| < \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha)(1-L)}\int_0^b (t-s)^{\alpha-1}ds\right]$$

$$< \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha)(1-L)}\frac{(t-s)^{\alpha}}{-\alpha}\Big|_0^b\right]$$

$$< \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha)(1-L)}\left(\frac{b^{\alpha}}{\alpha}-\frac{(t-b)^{\alpha}}{\alpha}\right)\right]$$

$$< \Phi^{-1}\left[\Phi(\epsilon)+\frac{M}{\Gamma(\alpha+1)(1-L)}(b^{\alpha}-(t-b)^{\alpha})\right], \tag{3.8}$$

where $\Phi(x)$ is primitive for $\frac{1}{\Phi(x)}$. We shall prove that the right-hand side of(3.8) tends toward zero as $\epsilon \to 0$. As$|x(t)-y(t)|$ is independent of ϵ, it followsthat $x(t)=y(t)$, which we need. Let us remark that condition (3.2) implies$\Phi(\epsilon) \to -\infty$ for $\epsilon \to 0$, no matter how we choose the primitive of $\frac{1}{\Phi(x)}$. Thus$\Phi^{-1}(x) \to 0$ as $x \to -\infty$. Consequently,$\epsilon \to 0$in the inequality (3.8), theright-hand side tends toward zero. This completes the proof of the theorem.

Theorem 3.2 If the function f is continuous and satisfies the conditionz
$$|f(t,x,y)-f(t,\bar{x},\bar{y})| \leq M|x-\bar{x}|+L|y-\bar{y}|, \tag{3.9}$$

for some positive constant M and $L \in (0,1)$ then the initial value problem (3.1)-(3.2) has unique solution inthe interval J.
Proof.Assume that there exists two solutions$x(t)$ and $y(t)$ of (1.1)-(1.2). Then we have

$$x(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,x(s),\ \ ^cD^{\alpha}x(s))ds \tag{3.10}$$

and

$$y(t)=x_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s),\ \ ^cD^{\alpha}y(s))ds. \tag{3.11}$$

Therefore, using these (3.10), (3.11) and hypothesis (3.9), we have

$$|x(t)-y(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f(s,x(s),\ \ ^cD^{\alpha}x(s))-f(s,y(s),\ \ ^cD^{\alpha}y(s))\big|ds$$

$$\leq \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(|x(s)-y(s)|)+L\,|^cD^{\alpha}(x(s)-y(s))|\right]ds \tag{3.12}$$

But by hypothesis (3.9) for any $t \in [0,b]$ and any $x,y \in \mathbb{R}$,
$$|^cD^{\alpha}(x(t)-y(t))| = |f(t,x(t),\ \ ^cD^{\alpha}x(t))-f(t,y(t),\ \ ^cD^{\alpha}y(t))|$$
$$\leq M(|x(t)-y(t)|)+L\,|^cD^{\alpha}(x(t)-y(t))|.$$

This implies

$$|^cD^{\alpha}(x(t)-y(t))| \leq \frac{M}{1-L}(|x(t)-y(t)|). \tag{3.13}$$

Using (3.13) in (3.12), we obtain

$$|x(t)-y(t)| \leq \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M(|x(s)-y(s)|)+\frac{ML}{1-L}(|x(s)-y(s)|)\right]ds$$

$$\leq \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[M+\frac{ML}{1-L}\right](|x(s)-y(s)|)ds$$

$$\leq \epsilon + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}M\left[1+\frac{L}{1-L}\right](|x(s)-y(s)|)ds$$

$$\leq \epsilon + \int_0^t (t-s)^{\alpha-1}\frac{1}{\Gamma(\alpha)}\frac{M}{1-L}(|x(s)-y(s)|)ds. \tag{3.14}$$

Now, (a) suppose that $\alpha > \frac{1}{2}$, then applying Leema 2.2 (i) to (3.14), we have

$$|x(t)-y(t)| < \sqrt{2}\,\epsilon\, exp\left[\frac{2\sqrt{2\alpha-1}}{4^{\alpha}}\int_0^t \left(\frac{1}{\Gamma(\alpha)}\frac{M}{(1-L)}\right)^2 ds + t\right]$$

$$< \sqrt{2}\,\epsilon\, exp\left[\frac{2\sqrt{2\alpha-1}}{4^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\frac{M}{(1-L)}\right)^2\int_0^t ds + t\right]$$

$$< \sqrt{2}\,\epsilon\, exp\left[\frac{2\sqrt{2\alpha-1}}{4^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\frac{M}{(1-L)}\right)^2 t + t\right], \tag{3.15}$$

for $t \in J$. Since ε was arbitrary, as $\varepsilon \to 0$ the inequality (3.13) implies that$x(t)=y(t)$ on J.

(b) Let $\alpha > \frac{1}{z+1}$ for some $z \geq 1$. Then by Lemma 2.2(ii) to (3.14), again we have,

$$|x(t) - y(t)| < (2^{q-1})^{\frac{1}{q}} \epsilon \exp\left[\frac{2^{q-1}}{q} K_z^q \int_0^t (\frac{1}{\Gamma(\alpha)} \frac{M}{(1-L)})^2 ds + t\right]$$

$$< (2^{q-1})^{\frac{1}{q}} \epsilon \exp\left[\frac{2^{q-1}}{q} K_z^q (\frac{1}{\Gamma(\alpha)} \frac{M}{(1-L)})^2 t + t\right], \tag{3.16}$$

for $t \in [0, T)$ where K_z is defined by (2.12). Since ϵ was arbitrary in (3.16), implies that $x(t) = y(t)$ as $\epsilon \to 0$. This completes the proof of the theorem.

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