ISSN 2347-1921

Dynamics of certain anti-competitive systems of rational difference equations in the plane

Vu Van Khuong[1], Mai Nam Phong[2]
[1] Department of Mathematical Analysis,
University of Transport and Communications,
Hanoi City, Vietnam
vuvankhuong@gmail.com
[2] Department of Mathematical Analysis,
University of Transport and Communications,
Hanoi City, Vietnam
mnphong@gmail.com

ABSTRACT

In this paper, we consider a system of rational difference equations in the plane

$$\begin{cases} x_{n+1} = \dfrac{x_n}{y_n^b x_n - a} \\ y_{n+1} = \dfrac{y_n}{x_n^b y_n - a} \end{cases}, \qquad n = 0,1,2 \ldots$$

where $a \in (0, \infty)$, $b \in (0, \infty)$ and the initial values x_0, $y_0 \in [0, \infty)$. We will prove that the unique positive equilibrium point of this system is globally asymptotically stable. We also determine the rate of convergence of a solution that converges to the equilibrium point $(\bar{x}, \bar{y})$ of this system.

Keywords:

Equilibrium; asymptotic; positive solution; system of difference equation; strongly competitive systems.

Mathematics Subject Classification: 39A10

1. INTRODUCTION AND PRELIMINARY RESULTS

Let I and J be intervals of real numbers. Consider a first order system of difference equations of the form.

$$x_{n+1} = f(x_n, y_n), y_{n+1} = g(x_n, y_n), n = 0, 1, 2, \ldots \qquad (1.1)$$

where $f: I \times J \to I, g: I \times J \to J$ and $(x_0, y_0) \in I \times J$, when the function $f(x, y)$ is increasing in x and decreasing in y and the function $g(x, y)$ is decreasing in x and increasing in y, the systems (1.1) is called competitive. One can consider a map $T = (f(x, y), g(x, y))$ associated with the system (1.1) and define the notions of competitive map accordingly.

If $\boldsymbol{v} = (u, v) \in \boldsymbol{R}^2$, we denote with $Q_l(\boldsymbol{v}), l \in \{1, 2, 3, 4\}$, the four quadrants in $\boldsymbol{R}^2$ relative to $\boldsymbol{v}$, i.e., $Q_1(\boldsymbol{v}) = \{(x, y) \in \boldsymbol{R}^2 : x \geq u, y \geq v\}$, $Q_2(\boldsymbol{v}) = \{(x, y) \in \boldsymbol{R}^2 : x \leq u, y \geq v\}$, and so on. Define the *South-East* partial order $\leq_{se}$ on $\boldsymbol{R}^2$ by $(x, y) \leq_{se} (s, t)$ if and only if $x \leq s$ and y$\geq$ t. Similarly, we define the *North-East* partial order $\leq_{ne}$ on $\boldsymbol{R}^2$ by $(x, y) \leq_{ne} (s, t)$ if and only if $x \leq s$ and y$\leq$ t. For $A \subset \boldsymbol{R}^2$ and $x \in \boldsymbol{R}^2$, define the *distance from* x *to* A as *dist (x, A):=inf{||x-y||: y* $\in$ *A}*. By *intA* we denote the interior of a set *A*.

It is easy to show that a map F is competitive if it is non-decreasing with respect to the *South-East* partial order, that is if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \leq_{se} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \leq_{se} F\begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \qquad (1.2)$$

For standard definitions of attracting fixed point, saddle point, stable manifold, and related notions see [7, 9, 10] and [16].

When the function $f(x, y)$ is increasing in x and increasing in y and the function $g(x, y)$ is increasing in x and increasing in y, system (1.1) is called *cooperative*. *Strongly competitive* systems of difference equations or strongly competitive maps are those for which the function f and g are coordinate wise stricly monotone.

System (1.1) where the function f and g have monotonic character opposite of the monotonic character in competitive system will be called *anti-competitive*, while system (1.1) where the function f and g have monotonic character opposite of the monotonic character in cooperative system will be called *anti-cooperative*. Anti-competitive and anti-cooperative systems will be called *anti-monotone* systems.

Competitive and cooperative systems have been investigated by many authors, see [2, 3, 4, 9, 12, 17] and others. The study of anti-monotone systems started recently in [5]. The rational systems of difference equations play an important role in modelling in biology and economics, see [6] and [7].

The following result gives a convergence result for a system in R^2 when there exists an invariant rectangle and the map of the system satisfies certain monotonicity and algebraic conditions. See [8] and [6, 11].

Theorem 1.1

Let R $= [a, b] \times [c, d]$ and

f: R$\to [a, b]$, g: R$\to [c, d]$

be a continuous functions such that:

(a) $f(x, y)$ is decreasing in both variables and $g(x, y)$ is decreasing in both variables for each $(x, y) \in$ R;

(b) If $(m_1, M_1, m_2, M_2) \in$ R^2 is a solution of

$$\begin{cases} M_1 = f(m_1, m_2), \ m_1 = f(M_1, M_2) \\ M_2 = g(m_1, m_2), \ m_2 = g(M_1, M_2) \end{cases} \quad (1.3)$$

Then $m_1 = M_1$ and $m_2 = M_2$. Then the system (1.1) has a unique equilibrium $(\bar{x}, \bar{y})$ and every solution (x_n, y_n) of the system (1.1) with $(x_0, y_0) \in R$ converges to the unique equilibrium $(\bar{x}, \bar{y})$. In addition, the equilibrium $(\bar{x}, \bar{y})$ is globally asymptotically stable.

In this paper we want to give an example of anti-monotonic system with a unique equilibrium which is globally asymptotically stable.

In Section 2 we consider the following system of difference equations

$$\begin{cases} x_{n+1} = \frac{x_n}{y_n^b x_n - a} \\ y_{n+1} = \frac{y_n}{x_n^b y_n - a} \end{cases}, \qquad n = 0,1,2 \dots \quad (1.4)$$

where $a \in (0, \infty)$, $b \in (0, \infty)$ and the initial values x_0, $y_0 \in [0, \infty)$ and $x_n^b y_n - a > 0, y_n^b x_n - a > 0$. This system has exactly one positive equilibrium point $(\bar{x}, \bar{y}) = \left((1 + a)^{1/(1+b)}, (1 + a)^{1/(1+b)}\right)$ which is locally asymptotically stable. We use Theorem 1.1 to show that the positive equilibrium point $(\bar{x}, \bar{y})$ is locally asymptotically stable.

Finally, in Section 3 we give the rate of convergence of a solution that converges to the equilibrium $(\bar{x}, \bar{y})$ of the systems (1.4) for all values of parameters. The rate of convergence of solutions that converge to an equilibrium has been obtained for some two-dimensional systems in [13] and [14].

The following results give the rate of convergence of solutions of a system of difference equations

$$\boldsymbol{x}_{n+1} = [A + B(n)]\boldsymbol{x}_n \quad (1.5)$$

where $\boldsymbol{x}_n$ is a k-dimensional vector, $A \in \boldsymbol{C}^{k\times k}$ is a constant matrix, and $B: \boldsymbol{Z}^+ \to \boldsymbol{C}^{k\times k}$ is a matrix function satisfying

$$||B(n)|| \to 0 \ when \ n \to \infty, \quad (1.6)$$

where $||.||$ denotes any matrix norm which is associated with the vector norm; $||.||$ also denotes the Euclidean norm in R^2 given by

$$||x|| = ||(x, y)|| = \sqrt{x^2 + y^2} \quad (1.7)$$

Theorem 1.2([15]) Assume that condition (1.6) holds. If $\boldsymbol{x}_n$ is a solution of system (1.5), then either $\boldsymbol{x}_n$ for all large n or

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|x_n\|} \quad (1.8)$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 1.2([15]) Assume that condition (1.6) holds. If x_n is a solution of system (1.5), then either $x_n = 0$ for all large n or

$$\rho = \lim_{n\to\infty} \frac{\|x_{n+1}\|}{\|x_n\|} \quad (1.9)$$

ISSN 2347-1921

exists and is equal to the modulus of one of the eigenvalues of matrix A.

2. DYNAMICS OF THE SYSTEM (1.4).

In this section we consider system of difference equations (1.4).

Theorem 2.1 System (1.4) has the unique positive equilibrium $E = (\bar{x}, \bar{y})$ which is globally asymptotically stable.

The equilibrium point of the system (1.4) satisfies the following system of equations

$$\begin{cases} \bar{y}^b \bar{x} = 1 + a \\ \bar{x}^b \bar{y} = 1 + a \end{cases} \tag{2.1}$$

where a is the real number that for $a > -1$.

From system (2.1) we have

$$\begin{cases} \bar{x} = (1 + a)^{1/(1+b)} \\ \bar{y} = (1 + a)^{1/(1+b)} \end{cases} \tag{2.2}$$

The map T associated to the system (1.4) is

$$T(x, y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \frac{x}{y^b x - a} \\ \frac{x}{x^b y - a} \end{pmatrix} \tag{2.3}$$

The Jacobian matrix of T is

$$J_T = \begin{pmatrix} \frac{-a}{(y^b x - a)^2} & \frac{-b x^2 y^{b-1}}{(y^b x - a)^2} \\ \frac{-b x^2 y^{b-1}}{(y^b x - a)^2} & \frac{-a}{(x^b y - a)^2} \end{pmatrix} \tag{2.4}$$

By using the system (2.2), value of the Jacobian matrix of T at the equilibrium point $E = (\bar{x}, \bar{y})$ is

$$J_T(\bar{x}, \bar{y}) = \begin{pmatrix} -a & -b(1+a) \\ -b(1+a) & -a \end{pmatrix} \tag{2.5}$$

The determinant of (2.5) is given by

$$det J_T(\bar{x}, \bar{y}) = a^2 - b^2(1 + a)^2$$

The trace of (2.5) is

$$Tr J_T(\bar{x}, \bar{y}) = -2a$$

The characteristic equation has the form

$$\lambda^2 + 2a\lambda + a^2 - b^2(1 + a)^2 = 0$$

Instead of proving local stability by standard test, which is a fairly complicated task, we will prove global asymptotic stability which will implies the local stability as well. We will use Theorem 1.1.

First, let

R $= [a, b] \times [c, d], (x_0, y_0) \in$ R

and

f: R→ $[a, b]$, g: R→ $[c, d]$

$f(x, y)$ and $g(x, y)$ are continuous functions in R.

It is easy to see that $f(x, y) = \frac{x}{y^b x - a}$ and $g(x, y) = \frac{y}{x^b y - a}$ are decreasing in both variables for each (x, y)$\in R$.

If $(m_1, M_1, m_2, M_2) \in R$ is a solution of

$$\begin{cases} M_1 = f(m_1, m_2), \; m_1 = f(M_1, M_2) \\ M_2 = g(m_1, m_2), \; m_2 = g(M_1, M_2) \end{cases} \tag{2.6}$$

we have:

$$\begin{cases} M_1 = \frac{m_1}{m_2^b m_1 - a}, M_2 = \frac{m_2}{m_1^b m_2 - a} \\ m_1 = \frac{M_1}{M_2^b M_1 - a}, m_2 = \frac{M_1}{M_1^b M_2 - a} \end{cases} \tag{2.7}$$

also, we get:

$$\begin{cases} (m_1 - M_1)(1 - a) = m_1 M_1 (m_2^b - M_2^b) \\ (m_2 - M_2)(1 - a) = m_2 M_2 (m_1^b - M_1^b) \end{cases}$$

Assuming that $M_2 > m_2$ this implies $m_1 > M_1$, which is a contradiction. Since $m_2 = M_2$ and $m_1 = M_1$. The conclusion of this theorem follows from Theorem 1.1 and the fact that Theorem 1.1 does not give only global attractivity but global stability as well.

3. RATE OF CONVERGENCE

Our goal in this Section is to determine the rate of convergence of every solution of the system (1.1) in the regions where the parameters a$\in (0, \infty)$, b$\in (0, \infty)$ and initial conditions x_0 and y_0 are arbitrary, nonnegative numbers.

Theorem 3.1 The error vector

$$e_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$$

of every solution $x_n \neq 0$ of (1.1) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \sqrt[n]{\|e_n\|} = \left|\lambda_i\left(J_T(E)\right)\right| \text{ for some i=1, 2,...} \tag{3.1}$$

And

$$\lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = \left|\lambda_i\left(J_T(E)\right)\right| \text{ for some i=1, 2,...} \tag{3.2}$$

where $\left|\lambda_i\left(J_T(E)\right)\right|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $J_T(E)$.

Proof

First, we will find a system satisfied by the error terms. The error terms are given as

$$x_{n+1}-\bar{x}=\frac{x_n}{y_n^b x_n-a}-\frac{\bar{x}}{\bar{y}^b\bar{x}-a}=\frac{-x_n\bar{x}(y_n^b-\bar{y}^b)-a(x_n-\bar{x})}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)}$$

$$=\frac{-x_n\bar{x}}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)}(y_n^b-\bar{y}^b)-\frac{a}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)}(x_n-\bar{x}) \qquad (3.3)$$

and

$$y_{n+1}-\bar{y}=\frac{y_n}{x_n^b y_n-a}-\frac{\bar{y}}{\bar{x}^b\bar{y}-a}=\frac{-y_n\bar{y}(x_n^b-\bar{x}^b)-a(y_n-\bar{y})}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)}$$

$$=\frac{-y_n\bar{y}}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)}(x_n^b-\bar{x}^b)-\frac{a}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)}(y_n-\bar{y}) \qquad (3.4)$$

We calculate $x_n^b-\bar{x}^b$ as following:

$$x_n^b-\bar{x}^b=\bar{x}^b\left[\left(\frac{x_n}{\bar{x}}\right)^b-1\right]=\bar{x}^b\left\{\left[1+\left(\frac{x_n}{\bar{x}}-1\right)\right]^b-1\right\}=\bar{x}^b\left\{\left[1+b\left(\frac{x_n}{\bar{x}}-1\right)+\right.\right.$$
$$\left.\left.\frac{b(b-1)}{2}\left(\frac{x_n}{\bar{x}}-1\right)^2+\cdots\right]-1\right\}=\bar{x}^b\left[b\left(\frac{x_n}{\bar{x}}-1\right)+\frac{b(b-1)}{2}\left(\frac{x_n}{\bar{x}}-1\right)^2+\cdots\right]=$$
$$b\bar{x}^{b-1}(x_n-\bar{x})+\frac{b(b-1)}{2}\bar{x}^{b-2}(x_n-\bar{x})^2+\cdots=b\bar{x}^{b-1}(x_n-\bar{x})+O_1[(x_n-\bar{x})^2] \qquad (3.5)$$

Similarly, we have:

$$y_n^b-\bar{y}^b=b\bar{y}^{b-1}(y_n-\bar{y})+O_2[(y_n-\bar{y})^2] \qquad (3.6)$$

Then from relation (3.3), (3.4), (3.5) and (3.6) we get:

$$x_{n+1}-\bar{x}=\frac{-x_n\bar{x}b\bar{y}^{b-1}(y_n-\bar{y})}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)}-\frac{a(x_n-\bar{x})}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)}-\frac{-x_n\bar{x}O_2[(y_n-\bar{y})^2]}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)} \qquad (3.7)$$

and

$$y_{n+1}-\bar{y}=\frac{-y_n\bar{y}b\bar{x}^{b-1}(x_n-\bar{x})}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)}-\frac{a(y_n-\bar{y})}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)}-\frac{y_n\bar{y}O_1[(x_n-\bar{x})^2]}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)} \qquad (3.8)$$

That is

$$\begin{cases} x_{n+1}-\bar{x}\approx-\frac{a}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)}(x_n-\bar{x})-\frac{x_n\bar{x}b\bar{y}^{b-1}}{(y_n^b x_n-a)(\bar{y}^b\bar{x}-a)}(y_n-\bar{y}) \\ y_{n+1}-\bar{y}\approx-\frac{y_n\bar{y}b\bar{x}^{b-1}}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)}(x_n-\bar{x})-\frac{a}{(x_n^b y_n-a)(\bar{x}^b\bar{y}-a)}(y_n-\bar{y}) \end{cases} \qquad (3.9)$$

Set

$e_n^1=x_n-\bar{x}$ and $e_n^2=y_n-\bar{y}$

Then system (3.9) can be represented as:

$$e_{n+1}^1\approx a_n e_n^1+b_n e_n^2$$
$$e_{n+1}^2\approx c_n e_n^1+d_n e_n^2$$

where

$$a_n \approx -\frac{a}{(y_n^b x_n - a)(\bar{y}^b \bar{x} - a)}, \quad b_n \approx -\frac{x_n \bar{x} b \bar{y}^{b-1}}{(y_n^b x_n - a)(\bar{y}^b \bar{x} - a)}$$
$$c_n \approx -\frac{y_n \bar{y} b \bar{x}^{b-1}}{(x_n^b y_n - a)(\bar{x}^b \bar{y} - a)}, \quad d_n \approx -\frac{a}{(x_n^b y_n - a)(\bar{x}^b \bar{y} - a)}$$

Taking the limmits of a_n, b_n, c_n and d_n as n→ ∞, we obtain

$$\lim_{n\to\infty} a_n = -a, \lim_{n\to\infty} b_n = -b, \lim_{n\to\infty} c_n = -b(1+a), \lim_{n\to\infty} d_n = -a,$$

that is

$$a_n = -a + \propto_n, b_n = -b + \beta_n, c_n = -b(1+a) + \gamma_n, d_n = -a + \delta_n$$

where $\propto_n \to 0$, $\beta_n \to 0$, $\gamma_n \to 0$ and $\delta_n \to 0$ as n→ ∞.

Now, we have system of the form (1.1):

$$\boldsymbol{e}_{n+1} = (A + B(n))\boldsymbol{e}_n$$

where

$$A\begin{pmatrix} -a & -b(1+a) \\ -b(1+a) & -a \end{pmatrix}, B(n) = \begin{pmatrix} \alpha_n & \beta_n \\ \delta_n & \gamma_n \end{pmatrix}$$

and

$$||B(n)|| \to 0 \text{ as } n \to \infty.$$

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{pmatrix} = \begin{pmatrix} -a & -b(1+a) \\ -b(1+a) & -a \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix}$$

The system is exactly linearized system of (1.1) evaluated at the equilibrium $E = \left((1+a)^{1/(1+b)}, (1+a)^{1/(1+b)}\right)$. Then Theorem 1.2 and Theorem 1.3 imply the result.

When $E = \left((1+a)^{1/(1+b)}, (1+a)^{1/(1+b)}\right)$, we also obtain the following result.

Corollary 3.1

Assume that a∈ $(0, \infty)$, b∈ $(0, \infty)$. Then the positive equilibrium point $(\bar{x}, \bar{y}) = \left((1+a)^{1/(1+b)}, (1+a)^{1/(1+b)}\right)$

is globally asymptotically stable. The error vector

$$\boldsymbol{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

of every solution x_n of (1.1) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \sqrt[n]{\|\boldsymbol{e}_n\|} = \lim_{n\to\infty} \sqrt[2n]{x_n^2 + y_n^2} = \left|\lambda_i\big(J_T(E)\big)\right|$$

and

$$\lim_{n\to\infty} \frac{\|\boldsymbol{e}_{n+1}\|}{\|\boldsymbol{e}_n\|} = \lim_{n\to\infty} \sqrt{\frac{x_{n+1}^2 + y_{n+1}^2}{x_n^2 + y_n^2}} = \left|\lambda_i\big(J_T(E)\big)\right|$$

where $\lambda_i\big(J_T(E)\big)$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium E.

REFERENCES

[1] E. Camouzis, M.R.S. Kulenovi'c, G. Ladas and O. Merino, Rational systems in the plane - open problems and conjectures, J. Difference Equ. Appl., 15(2009), 303-323.

[2] D. Clark, M.R.S. Kulenovi'c and J.F. Selgrade, Global asymptotic behavior of a two dimensional difference equation modelling competition, Nonlinear Anal., Theory Methods Appl, 52 (2003), 1765-1776.

[3] TavelJ.M. Cushing, S. Levarge, N. Chitnis and S.M. Henson, Some discrete competition models and the competitive exclusion principle, J. Difference Equ. Appl., 10 (2004), 1139-1152.

[4] M. Hirsch and H. Smith, Monotone Dynamical Systems. Handbook of Differential Equations: Ordinary differential Equations, Vol. II, 239-357, Elsevier B. V., Amsterdam, 2005.

[5] S. Kalabusi'c and M.R.S. Kulenovi'c, Dynamics of certain anti-competitive systems of rational difference equations in the plane, 2009. (to appear)

[6] M.R.S. Kulenovi'c and G. Ladas, Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures, Chapman and Hall/CRC, Boca Raton, London, 2001.

[7] M. R. S. Kulenovi'c and O. Merino, Discrete Dynamical Systems and Difference Equations with Mathematica, Chapman\& Hall/CRC, Boca Raton, Fla, USA, 2002.

[8] M. R. S. Kulenovi\'{c} and O. Merino, A global attractivity result for maps with invariant boxes, Discrete Conti. Dyn. Syst.,Ser.B,6(2006),97-110.

[9] M. R. S. Kulenovi'c and O. Merino, Competitive-exclusion versus competitive- coexistance for systems in the plane, Discrete Conti. Dyn. Syst., Ser. B, 6 (2006), 1141-1156.

[10] M. R. S. Kulenovi'c and O. Merino, Global bifurcation for competitive systems in the plane, Discrete Conti. Dyn. Syst., Ser. B, 12 (2009), 133-149.

[11] M. R. S. Kulenovi'c and M. Nurkanovi'c, Asymptotic behavior of two dimensional linear fractional system of difference equations, Rad. Mat., 11 (2002), 59-78.

[12] M. R. S. Kulenovi'c and M. Nurkanovi'c, Asymptotic behavior of a linear fractional system of difference equations, J. Inequal. Appl., (2005), 127-143.

[13] M. R. S. Kulenovi'c and M. Nurkanovi'c, Asymptotic behavior of a competitive system of linear fractional difference equations, Adv. Difference Equ. , (2006), Art. ID 19756, 13pp.

[14] M. R. S. Kulenovi'c and Z. Nurkanovi'c, The rate of convergence of solution of a three dimensional linear fractional systems of difference equations, Zbornik radova PMF Tuzla - Svezak Matematika, 2 (2005), 1-6.

[15] M. Pituk, More on Poincare's and Peron's theorems for difference equations, J. Difference Equ. Appl., 8 (2002), 201-216.

[16] C. Robinson, Stability, Symbolic Dynamics and Chaos, CRC Press, Boca Raton, 1995.

[17] H.L. Smith, Planar competitive and cooperative difference equations, J. Difference Equ. Appl., 3 (1998), 335-357.

[18] DZ. Burgi'c and Z. Nurkanovi'c, An example of globally asymptotically stable anti-monotonic system of rational difference equations in the plane, Sarajevo Journal of Mathematics, 5 (18) (2009), 235-245.