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Existence results for quasilinear degenerated equations in unbounded domains

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ABSTRACT

In this paper, we study the existence of solutions for quasilinear degenerated elliptic operators $Au+g(x,u,\nabla u)=f$, in unbounded domains $\mathcal{O}$, where A is a Lerray-Lions operator from the Weighted Sobolev space $W_0^{1,p}(\mathcal{O},\omega)$ to its dual, while $g(x,s,\xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s, but it satisfies a sign condition on s, and $f \in W^{-1,p'}(\mathcal{O},\omega^*)$.

Keywords:

Unbounded domains; Quasilinear degenerated elliptic operators; Weighted sobolev space.

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Introduction

Let $\mathcal{O} \subset \mathbb{R}^N$, be a domain (not necessarily bounded) with boundary $\partial\mathcal{O}$. Let A be the non linear operator from the Weighted Sobolev space $W_0^{1,p}(\mathcal{O},\omega)$ into its dual $W^{-1,p'}(\mathcal{O},\omega^*)$ defined as

$$Au = -\operatorname{div}(a(x,u,\nabla u))$$

where a: $\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-function satisfying:

$$|a_i(x,s,\xi)| \leq \beta\omega_i^{1/p}(x)\,[\,c_1(x) + \sigma^{1/p'}|s|^{q/p'} + \sum_{j=1}^{N}\omega_j^{1/p'}(x)|\xi_j|^{p-1}]; \text{ for } i = 1,\dots,N, \quad (1.1)$$

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)>0 \forall\xi \neq \eta \in \mathbb{R}^N \quad (1.2)$$

$$a(x,s,\xi).\xi \geq \alpha\sum_{i=1}^{N}\omega_{i|\xi_i|^p}, \quad (1.3)$$

where $c_1(x)$ is a positive function in $L^{p'}(\mathcal{O})$, and α, β are positive constants.

Let $g(x,s,\xi)$ is a Carathéodory function satisfying

$$g(x,s,\xi)s \geq 0 \quad (1.4)$$

$$|g(x,s,\xi)| \leq b(|s|)(\sum_{i=1}^{N}\omega_i|\xi_i|^p + c(x)), \quad (1.5)$$

where b : $\mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and c(x) is positive function which in $L^1(\mathcal{O})$.

The purpose in this paper is to prove the existence of solutions for quasilinear elliptic equations of the type

$$\begin{cases} Au + g(x,u,\nabla u) = f \text{ in } \mathcal{O} \\ \quad u \in W_0^{1,p}(\mathcal{O},\omega) \end{cases} \quad (1.6)$$

Where $Au= -\operatorname{div}(a(x,u,\nabla u))$ is a Lerray-Lions operator from the weighted Sobolev space $W_0^{1,p}(\mathcal{O},\omega)$ into its dual $W^{-1,p'}(\mathcal{O},\omega^*)$ and $f \in W^{-1,p'}(\mathcal{O},\omega^*)$ in unbounded domain.

The study of elliptic equations proved very important given their application in various fields of physics, biology, astronomy. The case of nonlinear elliptic equations whith second order of Lerray-lions type has been the subject of numerous studied since the early fifties years.

In the bounded domainsΩ, it is well known that equation Au=f is solvable by Drabek, Kufner and Mustonen in [6] in the case where $f \in W^{-1,p'}(\Omega,\omega^*)$ and in [1] where $f \in L^1(\Omega)$, see also [3], where A is of the form $-\operatorname{div}(a(x,u,\nabla u)) + a_0(x,u,\nabla u)$. In the case where a(x,u,∇u) is replaced by $A(x,u)\nabla u$ (problems with diffusion matrix) and $f \in L^1(\Omega)$, existence and a partial uniqueness result have been established on the Sobolev spaces $H_0^1(\Omega)$ in D.Blanchard and O.Guibé [5]. The problem (1.6) is solvable by Akdim, Azroul and Benkirane [2] in bounded domain.

The paper is organized as follows: In section 2, we precise some basic properties of Weighted Sobolev space. In section 3, we prove some technical lemmas concerning some convergences in Weighted Sobolev space. In section 4,we study the existence of a solution (1.6) in unbounded domain.

Preliminaries

Let $\mathcal{O} \subset \mathbb{R}^N$ be a domain (not necessarily bounded) whith boundary $\partial\mathcal{O}$ and $\omega = \{\omega_i(x) : 0 \leq i \leq N\}$ be a vector of weight functions, i.e. every component $\omega_i(x)$ is a measurable function which is strictly positive a.e. in $\mathcal{O}$. Furthermore, we suppose in all our conditions that

$$\omega_i \in L^1_{loc}(\mathcal{O}), \quad (2.1)$$

$$\omega_i^{\frac{-1}{p-1}} \in L^1_{loc}(\mathcal{O}), \text{for any } 0 \leq i \leq N. \quad (2.2)$$

We define the weighted space $L^p(\mathcal{O},\gamma)$, where γ is a weighted function on $\mathcal{O}$, by $L^p(\mathcal{O},\gamma) = \{u = u(x), u\gamma^{\frac{1}{p}} \in Lp(\mathcal{O})$.

We denote by $W^{1,p}(\mathcal{O},\omega)$ the space of all real − valued functions $u \in L^p(\mathcal{O},\omega_0)$ such that the derivatives in the sense of distributions satisfies

$$\frac{\partial u}{\partial x_i} \in L^p(\mathcal{O}, \omega_i) \text{ for } i = 1, \dots, N,$$

which is a Banach space under the norm

$$\|u\| = \left[\int_{\mathcal{O}} |u(x)|^p \omega_0 dx + \sum_{i=1}^{N} \int_{\mathcal{O}} \left|\frac{\partial u(x)}{\partial x_i}\right|^p \omega_i(x) dx\right]^{\frac{1}{p}}. \quad (2.3)$$

Since we shall deal with the Dirichlet problem, we shall use the space $X = W_0^{1,p}(\mathcal{O}, \omega)$

defined as the closure of $C_0^{\infty}(\mathcal{O})$ with respect to the norm (2.3). Note that, $C_0^{\infty}(\mathcal{O})$ is dence in $W_0^{1,p}(\mathcal{O}, \omega)$ and $(X, \|.\|_{1,p,\omega})$ is a reflexive Banach spaces.

We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\mathcal{O}, \omega)$ is equivalent to $W^{-1,p'}(\mathcal{O}, \omega^*)$, where $\omega^* = \{\omega_i^* = \omega_i^{1-p'},\ i = 0, \dots, N\}$ and where p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$.

The expression

$$|||u||| = \left(\sum_{i=1}^{N} \int_{\mathcal{O}} \left|\frac{\partial u}{\partial x_i}\right|^p \omega_i(x) dx\right)^{\frac{1}{p}} \quad (2.4)$$

is a norm defined on X and it's equivalent to (2.3).

There exists a weight function σ on $\mathcal{O}$ and a parameter q, $1 < q < \infty$, such that the Hardy inequality:

$$\left(\int_{\mathcal{O}} |u(x)|^q \sigma dx\right)^{\frac{1}{q}} \le c \left(\sum_{i=1}^{N} \int_{\mathcal{O}} \left|\frac{\partial u}{\partial x_i}\right|^p \omega_i(x) dx\right)^{\frac{1}{p}}, \quad (2.5)$$

holds for every $u \in X$ with a constant $c > 0$ independent of u.

The imbedding $X \hookrightarrow L^q(\mathcal{O}, \sigma)$ (2.6)

determined by the inequality (2.5) is compact (see [6], [4], [7]).

Main Results

Let $\mathcal{O} = \cup_{l=1}^{\infty} \mathcal{O}_l$, $\overline{\mathcal{O}_l} \subseteq \mathcal{O}_{l+1} \subseteq \overline{\mathcal{O}_{l+1}} \subset \mathcal{O}$, each $\mathcal{O}_l \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\mathcal{O}_l$. Let A be the non linear operator from $W_0^{1,p}(\mathcal{O}_l, \omega)$ into its dual $W^{-1,p'}(\mathcal{O}_l, \omega^*)$ defined as

$$Au = -\mathrm{div}(a(x, u, \nabla u))$$

Where a: $\mathcal{O}_l \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory vector-function satisfying:

$$|a_i(x, s, \xi)| \le \beta \omega_i^{\frac{1}{p}}(x) \left[c_1(x) + \sigma^{\left(\frac{1}{p'}\right)} |s|^{\left(\frac{q}{p'}\right)} + \sum_{j=1}^{N} \omega_j^{\frac{1}{p'}}(x) \left|\xi_j\right|^{p-1}\right], \text{ For } i = 1,\dots,N, \quad (3.1)$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \; \forall \xi \neq \eta \in \mathbb{R}^N \quad (3.2)$$

$$a(x, s, \xi).\xi \ge \alpha \sum_{i=1}^{N} \omega_i |\xi_i|^p, \quad (3.3)$$

where $c_1(x)$ is a positive function in $L^{p'}(\mathcal{O}_l)$, and α, β are some strictly positive constants.

Let $g(x, s, \xi)$ is a Carathéodory function satisfying

$$g(x, s, \xi)s \ge 0, \quad (3.4)$$

$$|g(x, s, \xi)| \le b(|s|) \left(\sum_{i=1}^{N} w_i |\xi_i|^p + c(x)\right), \quad (3.5)$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function and $c(x)$ is positive function which in $L^1(\mathcal{O}_l)$.

Lemma 3.1. [2] Let $F: \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitz, with $F(0)=0$. Let $u \in W_0^{1,p}(\mathcal{O}_l, \omega)$, then $F(u) \in W_0^{1,p}(\mathcal{O}_l, \omega)$.

Lemma 3.2. [2] Let (u_n) be a sequence of $W_0^{1,p}(\mathcal{O}_l, \omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\mathcal{O}_l, \omega)$. Then, $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\mathcal{O}_l, \omega)$.

Lemma 3.3. [2] Let $g \in L^r(\mathcal{O}_l,\gamma)$ and let $g_n \in L^r(\mathcal{O}_l,\gamma)$, with . $\|g_n\|_{r,\gamma} \leq c, 1 < r < \infty$

If $g_n(x) \to g(x)$ a.e. in $\mathcal{O}_l$, then $g_n \rightharpoonup g$ in $L^r(\mathcal{O}_l,\gamma)$, where $\rightharpoonup$ denotes weak sequence and γ is a weight function on $\mathcal{O}_l$.

Lemma 3.4. [2] Let (u_n) be a sequence in $W_0^{1,p}(\mathcal{O}_l,\omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\mathcal{O}_l,\omega)$ and

$\int_{\mathcal{O}_l}[a(x,u_n,\nabla u_n) - a(x,u_n,\nabla u)]\nabla(u_n - u)dx \to 0.$ Then, $u_n \to u$ in $W_0^{1,p}(\mathcal{O}_l,\omega)$. (3.6)

Theorem 3.1. [2] if $f \in W^{-1,p'}(\mathcal{O}_l,\omega^*)$ there exists a solution of the problem

$$\begin{cases} Au + g(x,u,\nabla u) = f \text{ in } \mathcal{O}_l, \\ \quad u \in W_0^{1,p}(\mathcal{O}_l,\omega). \end{cases} \tag{3.7}$$

Existence Result in Unbounded Domains

Theorem 4.1. Let $\mathcal{O} = \bigcup_{l=1}^{\infty}\mathcal{O}_l$, $\mathcal{O}_l \subseteq \bar{\mathcal{O}}_l \subseteq \mathcal{O}_{l+1} \subseteq \bar{\mathcal{O}}_{l+1}$ be bounded domains in $\mathcal{O}$, if $f \in W^{-1,p'}(\mathcal{O},\omega^*)$, there exists a solution of the problem

$$\begin{cases} Au + g(x,u,\nabla u) = f \text{ in } \mathcal{O}, \\ \quad u \in W_0^{1,p}(\mathcal{O},\omega). \end{cases} \tag{4.1}$$

Proof: Let $\{u_k\}$ be the sequence of solutions of (3.7) in $W_0^{1,p}(\mathcal{O}_k,\omega), (k \geq 1)$. Since g verifies the sign condition, using (3.3) we obtain $\alpha\sum_{i=1}^{N}\int_{\mathcal{O}_l}\omega_i\left|\frac{\partial u_k}{\partial x_i}\right|^p \leq \langle f,u_k\rangle,$ (4.2)

$\text{i.e. } \alpha\|u_k\|^p \leq \|f\|_{X^*}\|u_k\|, \text{then } \|u_k\| \leq \beta_1,$ (4.3)

for all $k \geq l$ and β_1 is independent of k.

Let $\tilde{u}_k$, for $k \geq 1$, denote the extension of u_k by zero outside $\mathcal{O}_k$, which we continue to denote it by u_k.

From (4.3), we have $\|u_k\| \leq \beta_1$, for $k \geq l$.

Then, $\{u_k\}$ has a subsequence $\{u_{k_m^1}\}$ which converges weakly to u^1, as $m \to \infty$, in $W_0^{1,p}(\mathcal{O}_1,\omega)$.

Since $\{u_{k_m^1}\}$ is bounded $W_0^{1,p}(\mathcal{O}_2,\omega)$, it has a convergent subsequence $\{u_{k_m^2}\}$ converging weakly to u^2 in $W_0^{1,p}(\mathcal{O}_2,\omega)$. By induction, we have $\{u_{k_m^{l-1}}\}$ has a subsequence $\{u_{k_m^l}\}$ which weakly converges to u^l in $W_0^{1,p}(\mathcal{O}_l,\omega)$; i.e., in short, we have $u_{k_m^l} \rightharpoonup u^l$ in $W_0^{1,p}(\mathcal{O}_l,\omega), l \geq 1$.

Define $u: \mathcal{O} \to \mathbb{R}$ by $u(x) := u^l(x)$, for $x \in \mathcal{O}_l$ (Here, there is no confusion since $u^l(x) = u^m(x)$ for any $m \geq l$.

Let M be any fixed (but arbitrary) bounded domain such that $M \subseteq \mathcal{O}$. Then, there exists an integer l such that $M \subseteq \mathcal{O}_l$. We note that, the diagonal sequence $\{u_{k_m^m}; m \geq l\}$ converges weakly to $u = u^l$ in $W_0^{1,p}(M,\omega)$ as $m \to \infty$. We still need to show that u is the solution. It is sufficient to show that u is a solution of (3.7) for an arbitrary bounded domain M in $\mathcal{O}$.

Since $u_{k_m^m} \to u^l$ in $W_0^{1,p}(M,\omega)$,we can pass to the limit in $\langle A u_{k_m^m}, v\rangle + \int_M g(x,u_{k_m^m},\nabla u_{k_m^m})v = \langle f,v\rangle$

and we obtain $\langle Au,v\rangle + \int_M g(x,u,\nabla u)v = \langle f,v\rangle$ as $m \to \infty$ for any $v \in W_0^{1,p}(\mathcal{O},\omega) \cap L^\infty(M)$ [2].

This concludes the proof of Theorem 4.1.

REFERENCES

[1] K.Ammar, H.Redwane. 2009. Renormalized solutions for nonlinear degenerate elliptic problems whith L^1 data, Rev. Mat. Complut (22) 37-52.

[2] Y. Akdim, E. Azroul and A. Benkirane. 2001. Existence of solution for quasilinear degenerated elliptic equations, ElectronicJ.Diff.Equ., 1-19.

[3] Y. Akdim, E. Azroul and A. Benkirane. 2001. Pseudo-monotonicity and degenerate elliptic operators of second order, Electronic J.Diff.Equ., 9-24.

[4] O.T. Bengt. 2000. Nonlinear Potential Theory and Weighted Sobolev Spaces, Spinger-Verlag Berlin Heidelberg .

[5] D.Blanchard and O.Guibé. 2004. Infinite valued solutions of non-uniformly elliptic problems, Anal. Appl. (2) 227-246.

[6] P. Drabek, A. Kufner and V. Mustonen. 1998. Pseudo-monotonicity and degenerate or singular elliptic operators, Bull. Austral. Math. Soc. Vol. (58) 213-221.

[7] P. Drabek, A. Kufner and F. Nicolosi. 1996. Non linear elliptic equations, singular and degenerate cases, University of West Bohemia.

[8] J. L. Lions. 1969. Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris.

[9] J. Leray and J. L. Lions. 1965. Quelques résultats de Visik sur des problèmes elliptiques non linéaires par les méthodes de Minty-Browder, Bul. Soc. Math. France (93), 97-10