ISSN 2347-1921

Characterization Of Exponential and Power Function Distributions Using s^{th} Truncated Moments Of Order Statistics

Ahmed Afify[1] Zohdy M. Nofal[2] Abdul–Hadi N. Ahmed[3]

(1) Department of Statistics, Faculty of Commerce, Benha University, EGYPT
Ahmed.Afify@fcom.bu.edu.eg[1]

(2) Department of Statistics, Faculty of Commerce, Benha University, EGYPT
dr_znofal@hotmail.com[2]

(3) The Institute of Statistical Studies and Research, Cairo University, EGYPT
drhadi@cu.edu.eg[3]

Abstract

Characterization results have great importance in statistics and probability applications. New characterizations of Exponential and Power Function distributions are presented using the $s\,th$ conditional expectation of order statistics in terms of their failure (hazard) rate. Our results generalize some of the known results of Ahsanullah (2009). A simulation study has been conducted to help an engineer or a practitioner to check whether the underlying distribution belongs to the hypothesized family.

Keywords: Characterization; Failure Rate; Conditional Expectation; Order Statistics; Exponential; Power Function Distributions.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 4, No. 3
editor@cirworld.com
www.cirworld.com, member.cirworld.com

ISSN 2347-1921

1. Introduction

In recent years order statistics and their moments have assumed considerable interest, the moments of order statistics have been tabulated quite extensively for several distributions, for example see Arnold et al (1992) and David (1981).

Many papers dealing with characterization through properties of order statistics are appeared, see for example Khan and Abouammoh (1999), Malik et al., (1988), Lin (1988), Kamps (1995) and Mohie El-Din et al., (1991), Ahsanullah (2009).

Khan and Abu-Salih (1989) have characterized many well-known continuous probability distributions such as Pareto and power function distributions through conditional expectation of functions of order statistics. Ahsanullah and Raqab (2004) have characterized continuous distributions by conditional expectation of some functions of generalized order statistics. Ahsanullah and Hamedani (2007) characterized beta of the first kind and the power function distribution using 1 th order statistics and n th order statistics respectively. Hamedani et al., (2008) characterized certain univariate distributions using truncated moments of $X_{(1)}$. We like to mention here the works of Galambos and Kotz (1978), Kotz and Shanbag (1980), Ahsanullah (1989), Oncel et al., (2005) and Wesolowski and Ahsanullah (2004). Ahsanullah (2009) characterized several univariate distributions using truncated moments of the i th order statistic. Ahsanullah (2009) characterized several univariate distributions by the moments of the $i+1(\ 1 \le i \ge n\ )$ th order statistic given i th order statistic$=t$. In this paper characterizations of some univariate distributions using the s th moments of the $r+1$ th order statistic given r th order statistic $= x$ are given. Besides, our results generalize some of the known results in the literature (see, e. g., Ahsanullah (2009)).

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from an absolutely continuous distribution with cumulative distribution function $(cdf)\, F(x)$ and the corresponding probability density function $(pdf)\, f(x)$. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the corresponding order statistics. Then the pdf of $X_{(r)}$, the joint pdf of $X_{(r)}$ and $X_{(r+1)}$ and the conditional pdf of $X_{(r+1)}$ given $X_{(r)} = x$ are, respectively, see Arnold et al. (1992).

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f(x)[F(x)]^{r-1}[1-F(x)]^{n-r}, a < x < b. \tag{1.1}$$

$$f_{X_{(r)},X_{(r+1)}}(x,y) = \frac{n!}{(r-1)!(n-r-1)!} f(x)f(y)[F(x)]^{r-1}[1-F(y)]^{n-r-1}, a < x < y < b. \tag{1.2}$$

$$f_{X_{(r+1)}|X_{(r)}}(y|x) = \frac{f_{X_{(r)},X_{(r+1)}}(x,y)}{f_{X_{(r)}}(x)} = (n-r)\frac{[1-F(y)]^{n-r-1}}{[1-F(x)]^{n-r}} f(y). \tag{1.3}$$

In section 2, the exponential and Power Function distributions are to be characterized through truncated moments of order statistics given by:

$$\mathrm{E}\left(\mathrm{X^s}_{(r+1)}|\mathrm{X}_{(r)} = \mathrm{x}\right) = \int_{\mathrm{x}}^{\infty} \mathrm{y}^{\mathrm{s}} f_{X_{(r+1)}|X_{(r)}}(\mathrm{y}|\mathrm{x})\,\mathrm{dy}, \mathrm{s} = 1,2,3,\dots, \qquad \mathrm{r} = 1,2,\dots,\mathrm{n}-1.$$

2. Characterization Theorems

2.1 Characterization of the Exponential distribution

In this section characterization of the exponential distribution through truncated moments of order statistics is presented.

Theorem 2.1:

Let X be a nonnegative continuous random variable with distribution function $F(.)$, survival (reliability) function $\overline{F}(.)$, density function $f(.)$ and Failure (hazard) rate function $h(.)$. Let $X_{(1)} \le X_{(2)} \cdots \le X_{(n)}$ denote the order statistics of a random sample of size n from $F(.)$. Then X has the exponential distribution with positive parameter λ if and only if

$$E\left(X^s{}_{(r+1)}|X_{(r)} = x\right) = \frac{s!}{[h(x)]^s}\sum_{j=0}^{s}\frac{[x.h(x)]^{s-j}}{(s-j)!(n-r)^j}, s = 1,2,\dots, r = 1,2,3,\dots, n-1, and\ h(x) = \lambda. \tag{2.1}$$

The following two lemmas are used to prove the sufficiency of theorem 2.1.

The two lemmas are proved in the appendix.

Lemma 1:

$$\frac{d}{dx}\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,p^j} = \frac{s!}{\lambda^{s-1}}\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^j}.$$

Lemma 2:

$$\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,p^j}=x^s+\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^{j+1}}.$$

Proof. (Necessity): Observe that

$$E\left(X^s{}_{(r+1)}|X_{(r)}=x\right)=\int_x^\infty y^s f_{X_{(r+1)|X_{(r)}}}(y|x)\,dy.$$

Using Equation (1.3), we obtain

$$E\left(X^s{}_{(r+1)}|X_{(r)}=x\right)=\frac{n-r}{\left(e^{-\lambda x}\right)^{n-r}}\int_x^\infty \lambda y^s\left(e^{-\lambda y}\right)^{n-r}dy=\frac{n-r}{\left(e^{-\lambda x}\right)^{n-r}}A. \quad (2.2)$$

Where

$$A=\frac{1}{\lambda^{s-1}}\int_x^\infty(\lambda y)^s\left(e^{-\lambda y}\right)^{n-r}dy.$$

Let $u=\lambda y$ and let, for simplification, $q=\lambda x,\ \ p=n-r.$

Hence

$$A=\frac{1}{\lambda^s}\int_q^\infty u^s\,e^{-up}\,du. \quad (2.3)$$

Integrating Equation (2.3) by parts, we obtain

$$A=\frac{1}{\lambda^s}\left(\frac{q^s}{p}e^{-qp}+\frac{s}{p}H_1\right). \quad (2.4)$$

But

$$H_1=\int_q^\infty u^{s-1}\,e^{-up}\,du.=\frac{q^{s-1}}{p}e^{-qp}+\frac{s-1}{p}H_2. \quad (2.5)$$

And

$$H_2=\int_q^\infty u^{s-2}\,e^{-up}\,du=\frac{q^{s-2}}{p}e^{-qp}+\frac{s-2}{p}H_3. \quad (2.6)$$

Substituting from Equation (2.5) and Equation (2.6) into Equation (2.4), we obtain

$$A=\frac{1}{\lambda^s}\left(\frac{q^s}{p}e^{-qp}+\frac{s!q^{s-1}}{(s-1)!p^2}e^{-qp}+\frac{s!q^{s-2}}{(s-2)!p^3}e^{-qp}+\frac{s!}{(s-3)!p^3}H_3\right). \quad (2.7)$$

After integrating Equation (2.7) for a number of times, the following recurrence relation is obtained

$$H_i=\int_q^\infty u^{s-i}\,e^{-up}\,du$$

$$=\frac{q^{s-i}}{p}e^{-qp}+\frac{s-i}{p}H_{i+1},\ i=1,2,\ldots,s-1. \quad (2.8)$$

And

$$H_s=\frac{1}{p}e^{-qp}. \quad (2.9)$$

Substituting from Equation (2.8) and Equation (2.9) into Equation (2.7), we obtain

$$A=\frac{1}{\lambda^s}\sum_{j=0}^{s}\frac{s!q^{s-j}}{(s-j)!p^{j+1}}e^{-qp}. \quad (2.10)$$

ISSN 2347-1921

Substituting from Equation (2.10) into Equation (2.2), we obtain

$$E\left(X^s{}_{(r+1)}|X_{(r)}=x\right)=\frac{1}{\lambda^s}\sum_{j=0}^{s}\frac{s!q^{s-j}}{(s-j)!p^j}, s=1,2,\dots,r=1,2,\dots,n-1. \quad (2.11)$$

Where $q=\lambda x, p=n-r,$ then

$$E\left(X^s{}_{(r+1)}|X_{(r)}=x\right)=\frac{s!}{[h(x)]^s}\sum_{j=0}^{s}\frac{[x.h(x)]^{s-j}}{(s-j)!(n-r)^j}, s=1,2,\dots,r=1,2,3,\dots,n-1, and\ h(x)=\lambda. \quad (2.12)$$

(Sufficiency): Notice that Equation (2.12) can be rewritten as follows

$$\int_x^\infty py^s f(y)\left(\bar{F}(y)\right)^{p-1}dy=\left(\bar{F}(x)\right)^p\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!p^j}. \quad (2.13)$$

Differentiating both sides of Equation (2.13) with respect to x, we obtain

$$-px^s f(x)\left(\bar{F}(x)\right)^{p-1}=\left\{\left(\bar{F}(x)\right)^p\frac{d}{dx}\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,p^j}-pf(x)\left(\bar{F}(x)\right)^{p-1}\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,p^j}\right\}$$

Using Lemma (1), we obtain

$$-px^s f(x)\left(\bar{F}(x)\right)^{p-1}=\left\{\left(\bar{F}(x)\right)^p\frac{s!}{\lambda^{s-1}}\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^j}-pf(x)\left(\bar{F}(x)\right)^{p-1}\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,p^j}\right\}$$

Using Lemma (2), we obtain

$$\left\{pf(x)\left(\bar{F}(x)\right)^{p-1}\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^{j+1}}\right\}=\left\{\left(\bar{F}(x)\right)^p\frac{s!}{\lambda^{s-1}}\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^j}\right\}$$

Dividing both sides of the above equation by $\left(\bar{F}(x)\right)^{p-1}$, we obtain

$$f(x)\frac{ps!}{\lambda^s}\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^{j+1}}=\bar{F}(x)\frac{ps!}{\lambda^{s-1}}\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^{j+1}},$$

Or equivalently

$$\frac{f(x)}{\bar{F}(x)}=\lambda. \quad (2.14)$$

Integrating both sides of Equation (2.14) with respect to x, we obtain

$$ln\,\bar{F}(x)=-\lambda x+ln\,c,\ \text{where } c \text{ is constant}$$

Hence

$$ln\left(\frac{\bar{F}(x)}{c}\right)=ln\,e^{-\lambda x}.$$

Or equivalently

$$\overline{F}(x) = ce^{-\lambda x}.$$

Using the fact that $\overline{F}(0) = 1$, then $c = 1$, hence

$$\overline{F}(x) = e^{-\lambda x}, x > 0, \lambda > 0 \blacksquare$$

Which is the sf of the exponential distribution with positive parameter λ .

This completes the proof.

Remark 1 *Specifying $s = 1$ and $s = 2$ in (2.1) yields the following results*

(i) $E\left(X_{(r+1)}|X_{(r)} = x\right) = x + \frac{1}{\lambda(n-r)}$.

(ii) $E\left(X^2{}_{(r+1)}|X_{(r)} = x\right) = x^2 + \frac{2x}{\lambda(n-r)} + \frac{2}{\lambda^2(n-r)^2}$.

Then

$$Var\left(X_{(r+1)}|X_{(r)} = x\right) = \frac{1}{\lambda^2(n-r)^2}.$$

Remark 2 *Specifying $s = 1$ in (2.1) gives the result of Ahsanullah (2009).*

2.2 Characterization of the Power Function distribution

In this section characterization of the Power Function distribution through truncated moments of order statistics is presented.

In the sequel, we shall use the following symbol, which is known by the Pochhammer symbol $(L)_r$; see Mathai and Haubold (2008).

$$(L)_r = L(L+1)(L+2)\ldots(L+r-1), (L)_0 = 1, L \neq 0, r = 1,2,3,\ldots$$

Theorem 2.2

Let X be a positive continuous rv with df $F(.)$, sf $\overline{F}(.)$, pdf $f(.)$ and HR function $h(.)$. Let $X_{(1)} \leq X_{(2)} \cdots \leq X_{(n)}$ denote the order statistics of a random sample of size n from $F(.)$.Then X has the power function distribution if and only if

$$E\left(X^s{}_{(r+1)}|X_{(r)} = x\right) = \sum_{j=0}^{s} \frac{(L)_1 s!\, \alpha^j x^{s-j}}{(L)_{j+1}(s-j)!\left(h(x)\right)^j}, \quad s = 1,2,\ldots, r = 1,2,\ldots, n-1, L = \alpha(n-r), h(x) = \frac{\alpha}{1-x}. \tag{2.15}$$

The following two lemmas are used to prove the sufficiency of theorem 2.2.

The two lemmas are proved in the appendix.

Lemma 3:

$$\frac{d}{dx}\sum_{j=0}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!} = \frac{(L)_1}{1-x}\sum_{j=1}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}.$$

Lemma 4:

$$\sum_{j=0}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!} = x^s + \sum_{j=1}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}.$$

Proof. (Necessity): Observe that

$$E\left[X^s{}_{(r+1)}|X_{(r)} = x\right] = \int_x^\infty y^s f_{X_{(r+1)}|X_{(r)}}\, dy.$$

Using Equation (1.3), we obtain

$$E\left(X^s{}_{(r+1)}|X_{(r)}=x\right)=\frac{\alpha(n-r)}{((1-x)^{\alpha})^{n-r}}\int_x^1 y^s(1-y)^{\alpha(n-r)-1}dy$$

$$E\left(X^s{}_{(r+1)}|X_{(r)}=x\right)=\frac{1}{(1-x)^L}\int_x^1 \mathrm{L}y^s(1-y)^{L-1}dy=\frac{A}{(1-x)^L}\,. \quad (2.16)$$

Where $L=\alpha(n-r)$,

$$A=\int_x^1 Ly^s(1-y)^{L-1}dy. \quad (2.17)$$

Integrating Equation (2.17) by parts, we obtain

$$A=x^s(1-x)^L+H_1. \quad (2.18)$$

But

$$H_1=s\int_x^1 y^{s-1}(1-y)^L dy=\frac{sx^{s-1}(1-x)^{L+1}}{(L+1)}+H_2. \quad (2.19)$$

Where

$$H_2=\frac{s(s-1)}{(L+1)}\int_x^1 y^{s-2}(1-y)^{L+1}dy=\frac{s(s-1)x^{s-2}(1-x)^{L+2}}{(L+1)(L+2)}+H_3. \quad (2.20)$$

Substituting from Equation (2.19) and Equation (2.20) into Equation (2.18), we obtain

$$A=\left(x^s(1-x)^L+\frac{sx^{s-1}(1-x)^{L+1}}{(L+1)}+\frac{s(s-1)x^{s-2}(1-x)^{L+2}}{(L+1)(L+2)}+H_3\right)$$

$$=x^s(1-x)^L+\frac{(L)_1 s!x^{s-1}(1-x)^{L+1}}{(L)_2(s-1)!}+\frac{(L)_1 s!x^{s-2}(1-x)^{L+2}}{(L)_3(s-2)!}+H_3. \quad (2.21)$$

But

$$H_3=\frac{(L)_1 s!}{(L)_3(s-3)!}\int_x^1 y^{s-3}(1-y)^{L+2}dy=\frac{(L)_1 s!x^{s-3}(1-x)^{L+3}}{(L)_4(s-3)!}+H_4. \quad (2.22)$$

After integrating Equation (2.22) for many times, we note the following recurrence relation

$$H_i=\frac{(L)_1 s!}{(L)_i(s-i)!}\int_x^1 y^{s-i}(1-y)^{L+i-1}dy$$

$$H_i=\frac{(L)_1 s!x^{s-i}(1-x)^{L+i}}{(L)_{i+1}(s-i)!}+H_{i+1}, i=1,2,\ldots,s-1. \quad (2.23)$$

Finally, we find that

$$H_s=\frac{(L)_1 s!(1-x)^{L+s}}{(L)_{s+1}}. \quad (2.24)$$

Substituting from Equation (2.23) and Equation (2.24) into Equation (2.21), we obtain

$$A=(1-x)^L\sum_{j=0}^{s}\frac{(L)_1 s!x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}. \quad (2.25)$$

Substituting from Equation (2.25) into Equation (2.15), we obtain

$$E\left(X^s{}_{(r+1)}|X_{(r)}=x\right)=\sum_{j=0}^{s}\frac{(L)_1 s!\,x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}$$

$$= \sum_{j=0}^{s} \frac{(L)_1 s! \alpha^j x^{s-j}}{(L)_{j+1}(s-j)!(h(x))^j}. \quad (2.26)$$

(Sufficiency): Notice that Equation (2.26) can be rewritten as follows

$$\int_x^\infty (n-r) y^s f(y) \left(\bar{F}(y)\right)^{n-r-1} dy = \left(\bar{F}(x)\right)^{n-r} \sum_{j=0}^{s} \frac{(L)_1 s! x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}. \quad (2.27)$$

Differentiating both sides of Equation (2.27) with respect to x, we obtain

$$-(n-r)x^s f(x)\left(\bar{F}(x)\right)^{n-r-1} = \left\{\left(\bar{F}(x)\right)^{n-r} \frac{d}{dx}\sum_{j=0}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!} - (n-r) f(x)\left(\bar{F}(x)\right)^{n-r-1} \sum_{j=0}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}\right\}$$

Using Lemma (3) and Lemma (4) and simplifying, we obtain

$$\left\{-(n-r)x^s f(x)\left(\bar{F}(x)\right)^{n-r-1}\right\}$$

$$= \left\{\frac{(L)_1}{1-x}\left(\bar{F}(x)\right)^{n-r} \sum_{j=1}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}\right\} - \left\{(n-r) f(x)\left(\bar{F}(x)\right)^{n-r-1}\left(x^s + \sum_{j=1}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}\right)\right\}$$

$$(n-r) f(x) \sum_{j=1}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!} = \frac{(L)_1}{(1-x)} \bar{F}(x) \sum_{j=1}^{s} \frac{(L)_1 s!\, x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}$$

$$(n-r) f(x) = \frac{(L)_1}{(1-x)} \bar{F}(x)$$

or equivalently

$$\frac{f(x)}{\bar{F}(x)} = \frac{\alpha}{(1-x)}. \quad (2.28)$$

Integrating both sides of Equation (2.28) with respect to x, we obtain

$$\int \frac{f(x)}{\bar{F}(x)} dx = \int \frac{\alpha}{(1-x)} dx$$

$ln\,\bar{F}(x) = \alpha\, ln(1-x) + ln\, k$, where k is constant

$$ln\left(\frac{\bar{F}(x)}{k}\right) = ln(1-x)^\alpha$$

$$\bar{F}(x) = k(1-x)^\alpha.$$

Using the fact that $\bar{F}(0) = 1$. Then $k = 1$.

Hence

$$\bar{F}(x) = (1-x)^\alpha, 0 < x < 1, \alpha > 0 \blacksquare$$

Which is the sf of the power Function distribution. This completes the proof.

Remark 3 *Specifying* $s = 1$ *and* $s = 2$ *in (2.15) yields the following results*

(i) $E\left(X_{(r+1)} | X_{(r)} = x\right) = x + \frac{1-x}{L+1}$.

(ii) $E\left({X^2}_{(r+1)} | X_{(r)} = x\right) = x^2 + \frac{2x(1-x)}{L+1} + \frac{2(1-x)^2}{(L+1)(L+2)}$.

Then

$$Var\left(X_{(r+1)} | X_{(r)} = x\right) = \frac{L\alpha^2 (h(x))^{-2}}{(L+1)^2(L+2)}.$$

Remark 4 *Specifying* $s = 1$ *in (2.15) gives the result of Ahsanullah (2009).*

3. Simulation Study

This section illustrates the practical importance of the results above through an experimental validation, using simulated data. The objective of the simulation study is to show that these results pave the way for simple and easily checks, from any given data set, enabling an engineer or practitioner to identify the present distribution. Although the work in this section can be done for the two characterization results presented in this paper, the focus is on Theorem (2.1). This is so, since the objective is just to show that all the characterization results give an easy way for the practitioner to test whether the available data follows a particular distribution, rather than going to sophisticated hypothesis testing analysis. To validate the correctness of the theoretical results obtained in this paper, a simulation study has been conducted with m=20 and n=30, and once more with m=100 and n=100, and the results of these two choices are presented in Tables (3.1) and (3.2), respectively

Table 3.1: Verification of the results

λ, r, s, x	$\mathrm{E}(X^s{}_{(r+1)} \mid X_{(r)} = x)$	$\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,(n-r)^j}$	$\left\lvert\frac{R.H.S - L.H.S}{R.H.S}\right\rvert$
0.2, 22, 1, 5.6687	6.4644	6.2937	0.0271
0.2, 22, 2, 6.3561	49.8862	49.1264	0.0155
0.5, 22, 1, 2.5417	2.8212	2.7917	0.0106
0.5, 22, 2, 2.6600	8.8063	8.7594	0.0054
1, 22, 1, 1.2086	1.3053	1.3336	0.0212
1, 22, 2, 1.2270	2.0310	1.8435	0.1017
3, 22, 1, 0.4057	0.4255	0.4474	0.0489
3, 22, 2, 0.3982	0.1997	0.1952	0.0231
6, 22, 1, 0.2200	0.2361	0.2408	0.0195
6, 22, 2, 0.2059	0.0545	0.0518	0.0521

Table 3.2: Verification of the results

λ, r, s, x	$E(X^s{}_{(r+1)} \mid X_{(r)} = x)$	$\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,(n-r)^j}$	$\left\lvert\frac{R.H.S - L.H.S}{R.H.S}\right\rvert$
0.2, 80, 1, 7.9686	8.1956	8.2186	0.0028
0.2, 80, 2, 7.8521	66.4844	65.7065	0.0118
0.5, 80, 1, 3.1938	3.2873	3.2938	0.0019
0.5, 80, 2, 3.1582	10.7353	10.6259	0.0103
1, 80, 1, 1.6204	1.6705	1.6704	0.00006
1, 80, 2, 1.5821	2.6700	2.6663	0.0014
3, 80, 1, 0.5344	0.5551	0.5511	0.0073
3, 80, 2, 0.5374	0.3103	0.3073	0.0098
6, 80, 1, 0.2649	0.2733	0.2732	0.0004
6, 80, 2, 0.2601	0.0734	0.0721	0.0180

The second column of Tables (3.1) and (3.2), contains the values of the left-hand side (LHS), while the third column contains the corresponding values of the right- hand side (RHS) and the right most column contains the absolute relative difference between the two sides of the characterizing equation. This is done for several choices of the parameter λ, r, s.

The right most column shows that the absolute relative difference between the two sides of the characterizing equation has a maximum value less than 10%, but the bound of the absolute difference goes down to only 1:8% when the sample size is increased from 30 to 100 and the number of samples is increased from 20 to 100, as seen from table (3.2).

This remark shows that as the sample size (and the number of samples) increases, the relative absolute difference decreases, and is going to eventually diminish, supporting the correctness of the results.

References

[1] Ahsanullah, M. (2009). On some characterizations of univariate distributions based on truncated moments of order statistics. Pak. J. statist. Vol. 25(2), 83-91.

[2] Ahsanullah, M. (1989). On characterizations of the uniform distribution based on functions of order statistics. Aligarh J. Statistics, 9, 106.

[3] Ahsanullah, M. and Hamedani, G.G. (2007). Certain characterizations of power function and Beta distributions based on order statistics. J. Statist. Theor. Appl., 6, 220-226.

[4] Ahsanullah, M. and Raqab M. Z. (2004) Characterizations of distributions by conditional expectations of generalized order statistics. J. Appl. Statist. Sc., 13, No. l, 41-48.

[5] Arnold, B.C., Balakrishnan, N. and Nagaraja, H. N. (1992). A First Course in Order Statistics, John Wiley & Sons, New York.

[6] David, H .A. (1981) Order statistics, 2nd Ed. John Wiley & Sons, New York.

[7] Galambos, J. and Kotz, S. (1978). Characterizations of probability distributions.A unified approach with an emphasis on exponential and related models. Lecture Notes in Mathematics, 675, Springer Verlag.

[8] Hamedani, G.G., Ahsanullah, M. and Sheng, R. (2008). Characterizations of certain continuous univariate distributions based on truncated moment of the first order statistics. Aligarh J Stat. 28. 75-81.

[9] Kamps, U. (1991). A general recurrence relation for moments of order statistics in a class of probability distributions and characterizations. Metrika 38, 215-225.

[10] Kamps, U. (1995). A concept of generalized order statistics. Teubner, Stuttgart.

[11] Khan, A. H. and Abouammoh, A. M. (1999). Characterizations of distributions by conditional expectation of order statistics, 9, 159-168.

[12] Khan, A.H. and Abu-Salih MS (1989). Characterizations of probability distributions by conditional expectation of order statistics. Metron, 47, 171-181.

[13] Kotz, S. and Shanbhag, D.N. (1980). Some new approaches to probability distributions. Adv. in Appl. Probab., 12, 903-912.

[14] Lin, G. D. (1988). Characterizations of distributions via relationships between two moments of order statistics. J. Statist.Plan. Inf. 19, 73-80.

[15] Malik, H. J., Balakrishnan, N. and Ahmed, S. E. (1988). Recurrence relations and identities for moments of order statistics, I: Arbitrary continuous distributions. Commun. Statist. Theor. Meth. 17(8), 2623-2655.

[16] Mathai, A. M. and Haubold, H. J. (2008). Special functions for applied scientists. Springer.

[17] Mohie El-Din, M.M., Mahmoud, M.A.W. and Abu-Youssef, S.E. (1991). Moments of order statistics from parabolic and skewed distributions and characterization of Weilbull distribution. Commun. Statist. Simul. Comput., 20(2,3), 639-645.

[18] Mohie El-Din, M. M., Mahmoud, M. A. W., Abu-Youssef, S. E. and Sultan, K. S. (1997). Order Statistics from the doubly truncated linear exponential distribution and its characterizations. Commun. Statist.-Simula. 26,281-290.

[19] Oncel, S.Y., Ahsanullah, M., Aliev, F.A. and Aygun, F. (2005). Switching record and order statistics via random contraction. Statist. Probab. Lett., 73, 207-217.

[20] Wesolowski, J. and Ahsanullah, M. (2004). Switching order statistics through random power contraction. Aust. N.Z.J. Statist. 6, 297-303.

APPENDIX

Proof of Lemma 1:

$$\frac{d}{dx}\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,p^j}=\left(\frac{s!}{\lambda^{s-1}}\right)\left\{\frac{(\lambda x)^{s-1}}{(s-1)!}+\frac{(\lambda x)^{s-2}}{(s-2)!\,p^1}+\frac{(\lambda x)^{s-3}}{(s-3)!\,p^2}+\cdots+\frac{1}{p^{s-1}}\right\}$$

$$=\left(\frac{s!}{\lambda^{s-1}}\right)\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^j}\;\blacksquare$$

Proof of Lemma 2:

$$\left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s}\frac{(\lambda x)^{s-j}}{(s-j)!\,p^j}=x^s+\left(\frac{s!}{\lambda^s}\right)\left\{\frac{(\lambda x)^{s-1}}{(s-1)!\,p^1}+\frac{(\lambda x)^{s-2}}{(s-2)!\,p^2}+\cdots+\frac{\lambda x}{p^{s-1}}+\frac{1}{p^s}\right\}$$

$$= x^s + \left(\frac{s!}{\lambda^s}\right)\sum_{j=0}^{s-1}\frac{(\lambda x)^{s-j-1}}{(s-j-1)!\,p^{j+1}}.$$

Proof of Lemma 3:

$$\frac{d}{dx}\sum_{j=0}^{s}\frac{(L)_1 s!\,x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}$$

$$= \left\{s x^{s-1} + \frac{(L)_1 s!\,x^{s-2}(1-x)}{(L)_2(s-2)!} - \frac{(L)_1 s!\,x^{s-1}(1-x)^0}{(L)_2(s-1)!} + \frac{(L)_1 s!\,x^{s-3}(1-x)^2}{(L)_3(s-3)!} - \frac{2(L)_1 s!\,x^{s-2}(1-x)}{(L)_3(s-2)!} + \cdots + \frac{(L)_1 s!\,x^0(1-x)^{s-1}}{(L)_s 0!} - \frac{(s-1)(L)_1 s!\,x(1-x)^{s-2}}{(L)_s 1!} - \frac{s(L)_1 s!\,x^0(1-x)^{s-1}}{(L)_{s+1}0!}\right\}$$

$$= \left\{\left(\frac{(L)_1 s!\,x^{s-1}}{(L)_1(s-1)!} - \frac{(L)_1 s!\,x^{s-1}}{(L)_2(s-1)!}\right) + \left(\frac{(L)_1 s!\,x^{s-2}(1-x)}{(L)_2(s-2)!} - \frac{2(L)_1 s!\,x^{s-2}(1-x)}{(L)_3(s-2)!}\right) + \left(\frac{(L)_1 s!\,x^{s-3}(1-x)^2}{(L)_3(s-3)!} - \frac{3(L)_1 s!\,x^{s-3}(1-x)^2}{(L)_5(s-3)!}\right) + \cdots + \left(\frac{(L)_1 s!\,x(1-x)^{s-2}}{(L)_{s-1}1!} - \frac{(s-1)(L)_1 s!\,x(1-x)^{s-2}}{(L)_s 1!}\right) + \left(\frac{(L)_1 s!\,x^0(1-x)^{s-1}}{(L)_s 0!} - \frac{s(L)_1 s!\,x^0(1-x)^{s-1}}{(L)_{s+1}0!}\right)\right\}.$$

$$= \left\{(L)_1\frac{(L)_1 s!\,x^{s-1}(1-x)^0}{(L)_2(s-1)!} + (L)_1\frac{(L)_1 s!\,x^{s-2}(1-x)}{(L)_3(s-2)!} + (L)_1\frac{(L)_1 s!\,x^{s-3}(1-x)^2}{(L)_5(s-3)!} + \cdots + (L)_1\frac{(L)_1 s!\,x(1-x)^{s-1}}{(L)_s 1!} + (L)_1\frac{(L)_1 s!\,x^0(1-x)^s}{(L)_{s+1}0!}\right\}$$

$$\frac{d}{dx}\sum_{j=0}^{s}\frac{(L)_1 s!\,x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!} = \frac{(L)_1}{1-x}\sum_{j=1}^{s}\frac{(L)_1 s!\,x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}\ \blacksquare$$

Proof of Lemma 4:

$$\sum_{j=0}^{s}\frac{(L)_1 s!\,x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}$$

$$= x^s + \left\{\frac{(L)_1 s!\,x^{s-1}(1-x)}{(L)_2(s-1)!} + \frac{(L)_1 s!\,x^{s-2}(1-x)^2}{(L)_3(s-2)!} + \cdots + \frac{(L)_1 s!\,x(1-x)^{s-1}}{(L)_s 1!} + \frac{(L)_1 s!\,(1-x)^s}{(L)_{s+1}0!}\right\}$$

$$\sum_{j=0}^{s}\frac{(L)_1 s!\,x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!} = x^s + \sum_{j=1}^{s}\frac{(L)_1 s!\,x^{s-j}(1-x)^j}{(L)_{j+1}(s-j)!}\ \blacksquare$$