ISSN 2347-1921

Poincarè type inequalities of Heisenberg group target for $p<2$

Fang-lan Li, Gao Jia
Shanghai Medical Instrumentation College, Shanghai 200093, China
lifl@smic.edu.cn
University of Shanghai for Science and Technology, Shanghai 200093, China
gaojia89@163.com

ABSTRACT

In this paper, we prove some Poincare type inequalities of the Heisenberg group target space in the case of $\frac{2m}{m+1} \leq p < 2.$ In order to overcome the obstacles which are due to the nonlinear structure of the group laws, there are some techniques in the arguments for proving the results.

Keywords:

Heisenberg Group; Sobolev Space; Poincare Type Inequality.

Mathematics Subject Classification:

22E20, 26D10, 46E35

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics
Vol 5, No. 3
editor@cirworld.com
www.cirworld.com, member.cirworld.com

1. INTRODUCTION

We firstly recall that the Heisenberg group $\mathbf{H}^n$ (see [1],[2]) is the Lie group whose underlying manifold is $\mathbf{C}^n \times \mathbf{R}, \mathbf{n} \in \mathbf{N}$, endowed with the group law algebra $\mathbf{g} = \mathbf{R}^{2n+1}$, and with a nonablian group law:

$$(z,t)\bullet(z',t') = (z+z', t+t' + 2\mathrm{Im} z\bullet z'), \quad (1.1)$$

where for $z, z' \in \mathbf{C}^n$, and $z\bullet z' = \sum_{j=1}^{n} z_j \bar{z}_j$. Setting $z_j = x_j + iy_j$, then $(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n)$ form a real coordinate system for $\mathbf{H}^n$. In this coordinate system the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, 2, \cdots, n, \quad (1.2)$$

and $T = \frac{\partial}{\partial t}$ generate the real Lie algebra of left-invariant vector fields on $\mathbf{H}^n$. It is easy to check that $[X_j, Y_k] = -4\delta_{jk} \frac{\partial}{\partial t}, j, k = 1, 2, \cdots, n,$ and that all other commutators are trivial. Furthermore, for the group, there are a natural dilation defined by

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad \lambda > 0, \quad (1.3)$$

and a metric $d(u, v)$ defined by (see [2],[3])

$$d(u,v) = |vu^{-1}| = \left[((x_v - x_u)^2 + (y_v - y_u)^2)^2 + (t_v - t_u + 2(x_u y_v - x_u y_v)^2\right]^{\frac{1}{4}}. \quad (1.4)$$

In particular, a homogeneous gauge $|u|_{\mathbf{H}^n}$ is defined as

$$[(x^2 + y^2)^2 + t^2]^{\frac{1}{4}} = (|z|^4 + t^2)^{\frac{1}{4}}.$$

We may see that $\mathbf{H}^n$ possesses the nonlinear structure about the group law which is one of the differences between $\mathbf{H}^n$ and general Riemann manifold. The fact that $\mathbf{H}^n$ is a singular space can be intuitively understood also in the light of a recent result of Christodoulou (see [2]) who proved that the Heisenberg group can be constructed as the continuum limit of a crystalline material.

Let $\Omega \subset R^m (m \geq 2)$ be a bounded and connected Lipschitz domain. For $2 \leq p < \infty$, Capogna and Lin ([4]) have provided the characterizations for the Sobolev space $W^{1,p}(\Omega, \mathbf{H}^n)$, proved the existence theorem for the minimizer, and established that all critical points for the energy are Lipschitz continuous in the 2-dimensional case. In [5], for the Sobolev maps of the Heisenberg group target, the following Poincare type inequality is obtained.

Theorem 1.1. (see [5]) Let Ω be a bounded and connected Lipschitz domain in R^m and $p \geq 2$. Then there exists a constant C depending only on Ω, m, n and p, such that for every function $u = (x, y, t) = (z, t) \in W^{1,p}(\Omega, \mathbf{H}^n)$, the inequality

$$\int_\Omega (d(u(h), \lambda_u))^p dh \leq C_\Omega E_{p,\Omega}(u) = C_\Omega \int_\Omega |\nabla z|^p (h) dh \quad (1.5)$$

holds, where $\lambda_u = (\lambda_x, \lambda_y, \lambda_t)$ and $\lambda_f = \frac{1}{|\Omega|} \int_\Omega f(h) dh.$

One of the main purposes of this paper is to derive a Poincare type inequality in the case of $\frac{2m}{m+1} \leq p < 2.$

The statements of these results are similar to the ones in the classical case. However, since the metric possesses the nonlinear structure of the group laws, we require a few of techniques in the proofs for our Poincare type inequalities.

This paper is organized as follows. In Section 2, we state some preliminaries and lemmas. In Section 3, we prove some Poincare type inequalities in the Heisenberg group target in $\frac{2m}{m+1} \le p < 2.$

2. PRELIMINARIES AND LEMMAS

In this section, to state our theorems, we need to introduce some notations, definitions and basic lemmas.

Definition 2.1. Let $1 \le p < \infty$. A function $u = (z,t) : \Omega \to \mathbf{H}^n$ is in $L^p(\Omega, \mathbf{H}^n)$ if for some $h_0 \in \Omega$, one has

$$\int_\Omega (d(u(h), u(h_0)))^p dh < \infty. \tag{2.1}$$

a function $u = (z,t) : \Omega \to \mathbf{H}^n$ is in the Sobolev space $W^{1,p}(\Omega, \mathbf{H}^n)$ if $u \in L^p(\Omega, \mathbf{H}^n)$ and

$$E_{p,\Omega}(u) = \sup_{f \in C_0(\Omega), 0 \le f \le 1} \limsup_{\varepsilon \to 0} \int_\Omega f(h) e_{u,\varepsilon}(h) dh < \infty, \tag{2.2}$$

where

$$e_{u,\varepsilon}(h) = \int_{|h-q|=\varepsilon} \left(\frac{d(u(h), u(q))}{\varepsilon} \right)^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}}.$$

$E_{p,\Omega}(u)$ is called an p -energy of u on Ω.

Lemma 2.1. (see [4]) Let $1 < p < \infty$, $u = (z,t) : \Omega \to \mathbf{H^n}$. Then $u = (z,t) = (x, y, t) \in L^p(\Omega, \mathbf{H}^n)$ if and only if $z \in L^p(\Omega, R^{2n}), t \in L^{p/2}(\Omega)$.

Lemma 2.2. (see [4]) Let $1 < p < \infty$, $u = (z,t) \in W^{1,p}(\Omega, \mathbf{H}^n)$. Then $z \in W^{1,p}(\Omega, R^{2n})$ and

$$E_p(u) \ge \sum_{i=1}^{n} || \nabla z_i ||_{L^p}^p .$$

Definition 2.2. Let $\Omega \subset R^m$ be a bounded and connected Lipschitz domain, $0 < p < 1$. A measurable function $f : \Omega \to R$ is called a p -integrable function if $\int_\Omega |f|^p \, dx < +\infty$. Denote $f \in L^p(\Omega)$.

Definition 2.3. Let $f \in L^1_{Loc}(\Omega) \cap L^p(\Omega)$, $0 < p < 1$. If there exists a function $g \in L^1_{Loc}(\Omega) \cap L^p(\Omega)$ such that

$$\int_\Omega f D^k \varphi dx = (-1)^k \int_\Omega g \varphi dx, \forall \varphi \in C_0^\infty(\Omega), \tag{2.3}$$

we say that g is the k -th weak derivative of f.

Definition 2.4. Let $0 < p < 1$. A measurable function $f \in W^{1,p}(\Omega)$ means $f \in L^1_{Loc}(\Omega) \cap L^p(\Omega)$ and $Du \in L^1_{Loc}(\Omega) \cap L^p(\Omega)$.

Now we discuss the properties of $W^{1,p}(\Omega, \mathbf{H}^n)$ in the case $\frac{2m}{m+1} \le p \le 2.$

Lemma 2.3. Let $\frac{2m}{m+1} \le p \le 2$. If $u = (z,t) = (x, y, t) : \Omega \to \mathbf{H}^n$ satisfies

⑴ $x, y \in W^{1,p}(\Omega, R^n)$,

ISSN 2347-1921

（2） $t \in W^{1,r}(\Omega)$ and $\nabla t = 2(y\nabla x - x\nabla y)$ $a.e.$ in $\Omega, r = \frac{mp}{2m-p}$.

Then we have $E_{p,\Omega}(u) < +\infty$.

Proof. From Definition 2.1, we get

$$E_{p,\Omega}(u) = \sup_{f\in C_0(\Omega), 0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left(\left(\frac{z(h)-z(q)}{\varepsilon} \right)^4 + \left(\frac{t(h)-t(q)}{\varepsilon^2} - 2y(h)\frac{x(h)-x(q)}{\varepsilon^2} + 2x(h)\frac{y(h)-y(q)}{\varepsilon^2} \right)^2 \right)^{\frac{p}{4}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh.$$

By using C_p -inequality ([6]), we have

$$\begin{aligned} E_{p,\Omega}(u) \le & \sup_{f\in C_0(\Omega), 0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{z(h)-z(q)}{\varepsilon} \right|^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh \\ & + \sup_{f\in C_0(\Omega), 0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{t(h)-t(q)}{\varepsilon^2} \right. \\ & \left. -2y(h)\frac{x(h)-x(q)}{\varepsilon^2} + 2x(h)\frac{y(h)-y(q)}{\varepsilon^2} \right|^{\frac{p}{2}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh \overset{\Delta}{=} I + J, \end{aligned} \tag{2.4}$$

where

$$I = \sup_{f\in C_0(\Omega), 0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{z(h)-z(q)}{\varepsilon} \right|^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh \tag{2.5}$$

and

$$\begin{aligned} J = & \sup_{f\in C_0(\Omega), 0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{t(h)-t(q)}{\varepsilon^2} \right. \\ & \left. -2y(h)\frac{x(h)-x(q)}{\varepsilon^2} + 2x(h)\frac{y(h)-y(q)}{\varepsilon^2} \right|^{\frac{p}{2}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh. \end{aligned} \tag{2.6}$$

To prove $E_{p,\Omega}(u) < +\infty$, we only need to show $I < +\infty$ and $J < +\infty$.

Firstly, we prove that $I < +\infty$. By C_p -inequality, we have

$$\begin{aligned} I \le & \sup_{f\in C_0(\Omega), 0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{x(h)-x(q)}{\varepsilon} \right|^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh \\ & + \sup_{f\in C_0(\Omega), 0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{y(h)-y(q)}{\varepsilon} \right|^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh. \end{aligned}$$

Case 1: If $z \in C^1(\Omega, R^{2n})$, then

$$I \le \sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h)\int_{S^{m-1}} \left|\frac{x(h)-x(h+\varepsilon\omega)}{\varepsilon}\right|^p d\omega dh$$

$$+ \sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h)\int_{S^{m-1}} \left|\frac{y(h)-y(h+\varepsilon\omega)}{\varepsilon}\right|^p d\omega dh.$$

Substituting the Taylor polynomial in the integral above we obtain

$$I \le \sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h)\int_{S^{m-1}} |\nabla x(h)|^p |cos\theta_1| d\omega dh$$

$$+ \sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h)\int_{S^{m-1}} |\nabla y(h)|^p |cos\theta_2| d\omega dh$$

$$+ \sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h)\int_{S^{m-1}} \left|\frac{o(\varepsilon)}{\varepsilon}\right| d\omega dh,$$

where θ_1 is the angle between $\nabla x(h)$ *and* ω, and θ_2 is the angle between $\nabla y(h)$ and ω.

Since $\limsup_{\varepsilon\to 0}\int_{S^{m-1}}\left|\frac{o(\varepsilon)}{\varepsilon}\right| d\omega = 0,$ and $f\in C_0(\Omega)$, $x(h), y(h)\in C^1(\Omega, R^n)$, we have $I < +\infty$.

Case 2: If $z = (x, y)\in L^p(\Omega, R^{2n})$, consider a mollifier $\chi_\delta(h) = \delta^{-m}\chi\left(\frac{h}{\delta}\right)$ with $\chi\in C_0^\infty(B_1(0))$ and $\int_{R^m}\chi dh = 1.$ Set $z_\delta(h) = z * \chi_\delta = (x_\delta, y_\delta)$. Since $\frac{2m}{m+1}\le p\le 2,$ by the C_p -inequality we get

$$|z(h)-z(q)|^p \le 2(|z_\delta(h)-z(h)|^p + |z_\delta(h)-z_\delta(q)|^p + |z_\delta(q)-z(q)|^p).$$

For a fixed $\varepsilon > 0$, using the properties of $z_\delta(h)$, one has

$$\lim_{\delta\to 0}\int_{|h-q|=\varepsilon}\left|\frac{z_\delta(q)-z(q)}{\varepsilon}\right|^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} = 0,$$

and

$$\lim_{\delta\to 0}\int_\Omega f(h)\left|\frac{z_\delta(h)-z(h)}{\varepsilon}\right|^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh$$

$$\le \lim_{\delta\to 0}\omega_{m-1}\int_\Omega\left|\frac{z_\delta(h)-z(h)}{\varepsilon}\right|^p dh = 0.$$

By the results in case 1, we have

$$\sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h)\int_{|h-q|=\varepsilon}\left|\frac{z_\delta(h)-z_\delta(q)}{\varepsilon}\right|^p \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh < +\infty.$$

Thus, we obtain $I < +\infty$.

Secondly, we prove that $J < +\infty$.

Case 1: If $z\in C^1(\Omega, R^{2n}), t\in C^1(\Omega)$, substituting the Taylor polynomial in the integral above we obtain

$$J=\sup_{f\in C_0(\Omega),0\le f\le 1}\limsup_{\varepsilon\to 0}\int_\Omega f(h)\int_{S^{m-1}}\left|\varepsilon^{-1}\left(\langle\nabla t(h)-2y(h)\nabla x(h)+2x(h)\nabla y(h),\omega\rangle\right)\right.$$

$$\left.+\varepsilon^{-2}\left(o(\varepsilon^{-2})-2y(h)o(\varepsilon^{-2})+2x(h)o(\varepsilon^{-2})\right)\right|^{\frac{p}{2}}d\omega dh.$$

On the other hand, since $t\in W^{1,\lambda}(\Omega),$ and $\nabla t\in L^{\lambda}(\Omega)\subseteq L^{1}(\Omega),$ $\lambda=\dfrac{mp}{2m-p}$, $\dfrac{2m}{m+1}\le p\le 2,|\Omega|<+\infty$, and $\nabla t=2(y\nabla x-x\nabla y),$ we have $J=0.$

Case 2: x,y,t satisfy the conditions (1) and (2) in Lemma 2.3. By the C_p-inequality we have

$$\left|(t(h)-t(q))-(t_\delta(h)-t_\delta(q))+2y(h)(x(h)-x(q))-2y_\delta(h)(x_\delta(h)-x_\delta(q))\right.$$

$$\left.+2x(h)(y(h)-y(q))-2x_\delta(q)(y_\delta(h)-y_\delta(q))\right|^{\frac{p}{2}}$$

$$\le\left|t_\delta(h)-t(h)\right|^{p/2}+\left|t_\delta(q)-t(q)\right|^{\frac{p}{2}}+K(\delta,h,q),$$

where

$$K(\delta,h,q)=2\left|y(h)(x(h)-x(q))-y_\delta(h)(x_\delta(h)\right.$$

$$\left.-x_\delta(q))+x(h)(y(h)-y(q))-x_\delta(q)(y_\delta(h)-y_\delta(q))\right|^{\frac{p}{2}}.$$

Since $t\in W^{1,\lambda}(\Omega),$ $|\Omega|<+\infty,$ noting that $0\le f\le 1,$ using the Sobolev embedded theorem, for a fixed $\varepsilon>0,$ we obtain that

$$\lim_{\delta\to 0}\int_\Omega f(h)\int_{|h-q|=\varepsilon}\left|t_\delta(h)-t(h)\right|^{\frac{p}{2}}d\sigma_\varepsilon(q)dh$$

$$\le C\lim_{\delta\to 0}\int_\Omega f(h)\int_{|h-q|=\varepsilon}\left|t_\delta(h)-t(h)\right|^{\frac{mp}{2m-p}}d\sigma_\varepsilon(q)dh=0$$

and

$$\lim_{\delta\to 0}\int_\Omega f(h)\int_{|h-q|=\varepsilon}\left|t_\delta(q)-t(q)\right|^{\frac{p}{2}}d\sigma_\varepsilon(q)dh$$

$$\le C\lim_{\delta\to 0}\int_\Omega f(h)\int_{|h-q|=\varepsilon}\left|t_\delta(q)-t(q)\right|^{\frac{mp}{2m-p}}d\sigma_\varepsilon(q)dh=0.$$

Similarly, it follows from the Sobolev embedded theorem and Poincare inequality that

$$\lim_{\delta\to 0}\int_\Omega f(h)\int_{|h-q|=\varepsilon}K(\delta,h,q)d\sigma_\varepsilon(q)dh=0.$$

To sum up the above arguments, $E_{p,\Omega}(u)<+\infty.$

Lemma 2.4. Let $x,y\in C^1(\Omega,R^n),t\in C^1(\Omega),$ $\dfrac{2m}{m+1}\le p\le 2.$ If $u=(z,t)=(x,y,t)\in L^p(\Omega,\mathbf{H}^n)$ and $E_{p,\Omega}(u)<+\infty.$ Then we have

(1) $x,y\in W^{1,p}(\Omega,R^n),$

(2) $t\in W^{1,r}(\Omega)$ and $\nabla t=2(y\nabla x-x\nabla y)$ a.e. in $\Omega,$ where $r=\dfrac{mp}{2m-p}.$

Proof. Since $u=(x,y,t)\in L^p(\Omega,\mathbf{H}^n),$ $E_{p,\Omega}(u)<+\infty,$ and

$$\int_\Omega d^p(u(h),u(q))dh=\int_\Omega \left|(z(h)-z(q))^4+(t(h)-t(q)+2(y(q)x(h)-x(q)y(h)))^2\right|^{\frac{p}{4}}dh,$$

we get that $x,y\in L^p(\Omega,R^n),\ t\in L^{p/2}(\Omega)$ and

$$I=\sup_{f\in C_0(\Omega),0\le f\le 1}\limsup_{\varepsilon\to 0}\int_\Omega f(h)\int_{|h-q|=\varepsilon}\left|\frac{z(h)-z(q)}{\varepsilon}\right|^p\frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}}dh<\infty,$$

$$J=\sup_{f\in C_0(\Omega),0\le f\le 1}\limsup_{\varepsilon\to 0}\int_\Omega f(h)\int_{|h-q|=\varepsilon}\left|\frac{t(h)-t(q)}{\varepsilon^2}+2y(h)\frac{x(h)-x(q)}{\varepsilon^2}\right.$$

$$\left.+2x(h)\frac{y(h)-y(q)}{\varepsilon^2}\right|^{\frac{p}{2}}\frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}}dh<+\infty.$$

Part 1. Prove $\nabla x,\nabla y\in L^p(\Omega,R^n)$. With a change of variables we can write

$$\int_{|h-q|=\varepsilon}\left|\frac{z(h)-z(q)}{\varepsilon}\right|^p\frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}}=\int_{S^{m-1}}\left|\frac{z(h)-z(h+\varepsilon\omega)}{\varepsilon}\right|^p d\omega.$$

By using Taylor formula $z(h+\varepsilon\omega)=z(h)+\varepsilon\langle\nabla z(h),\omega\rangle+o(\varepsilon^2)$, we get that

$$\int_{|h-q|=\varepsilon}\left|\frac{z(h)-z(q)}{\varepsilon}\right|^p\frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}}\ge\int_{S^{m-1}}\left|\langle\nabla x(h),\omega\rangle+o(\varepsilon)\right|^p d\omega.$$

According to $E_{p,\Omega}(u)<+\infty$, we obtain that $\nabla x\in L^p(\Omega,R^n)$. Similarly, we get $\nabla y\in L^p(\Omega,R^n)$.

Part 2. Prove $\nabla t\in L^{p/2}(\Omega,R^n)$, and $\nabla t=2(y\nabla x-x\nabla y)$ a.e. in $L^{p/2}(\Omega)$. Since $x,y\in C^1(\Omega,R^n),\ t\in C^1(\Omega)$ and $J<+\infty$, by the Taylor formula, we have

$$\sup_{f\in C_0(\Omega),0\le f\le 1}\int_\Omega f(h)\int_{S^{n-1}}\left|\frac{<\nabla t(h)-2y(h)\nabla x(h)+2x(h)\nabla y(h),\omega>}{\varepsilon}\right.$$

$$\left.+\frac{(o(\varepsilon^2)-2y(h)o(\varepsilon^2)+2x(h)o(\varepsilon^2))}{\varepsilon^2}\right|^{\frac{p}{2}}d\omega dh<+\infty.$$

For arbitrary ε, this yields $\nabla t=2(y\nabla x-x\nabla y)$ and $t\in W^{1,p/2}(\Omega)$.

By the embedded theorem, we obtain $x,y\in L^{\frac{mp}{m-p}}(\Omega,R^n)$. Using C_p -inequality and Holder inequality, we deduce

$$\begin{aligned}&\int_\Omega|y(h)\nabla x(h)-x(h)\nabla y(h)|^{\frac{mp}{2m-p}}dh\\ &\le C_p\left(\int_\Omega|y(h)|^{\frac{mp}{m-p}}dh\right)^{\frac{m-p}{2m-p}}\left(\int_\Omega|\nabla x(h)|^p dh\right)^{\frac{m}{2m-p}}\\ &+C_p\left(\int_\Omega|x(h)|^{\frac{mp}{m-p}}dh\right)^{\frac{m-p}{2m-p}}\left(\int_\Omega|\nabla y(h)|^p dh\right)^{\frac{m}{2m-p}}.\end{aligned}\tag{2.7}$$

This implies that $\nabla t \in L^{\frac{mp}{2m-p}}(\Omega)$.

Part 3. Prove $t \in L^r(\Omega)$, and $r = \dfrac{mp}{2m-p}$. Let $t(h) = t_1(h) + t_2(h)$, and $t_1(h) = t(h) \bullet I_{\Omega_k}$, $t_2(h) = t(h) \bullet I_{\Omega \backslash \Omega_k}$, where $\Omega_k = \{h \,||\, t(h) |\le k\}$, and $I_D(h) = 1$ if $h \in D$, $I_D(h) = 0$ if $h \notin D$, k is a fixed positive number.

If $t = t_1$, i.e. $|t| \le k$, then $t \in L^1(\Omega)$. By using the results in Part 2, we have $t \in W^{1,1}(\Omega)$, By the embedded theorem, we get $t \in L^{\frac{m}{m-1}}(\Omega) \searrow L^{\frac{mp}{2m-p}}(\Omega)$. This implies that $t \in W^{1,r}(\Omega)$.

If $t = t_2$, according to $\nabla |t|^{p/2} = \dfrac{p}{2} |t|^{\frac{p}{2}-2} t \bullet \nabla t$, and $0 < p < 2$, then

$$\int_\Omega |\nabla |t|^{p/2}| \, dh \le \frac{p}{2} k^{\frac{p}{2}-1} \int_\Omega |\nabla t| \, dh < +\infty. \tag{2.8}$$

Thus, $t \in L^{\frac{mp}{2(m-1)}}(\Omega) \searrow L^{\frac{mp}{2m-p}}(\Omega)$.

Using $L^p -$ theory and the mollifier method, we get easily that

Corollary 2.1. If $E_{p,\Omega}(u) < +\infty$, we then have $x, y \in W^{1,p}(\Omega, R^n)$.

In order to study the properties about the function $t(h)$ in the general cases, we need the following lemma.

Lemma 2.5. Let $u = (x, y, t) \in W^{1,p}(\Omega, \mathbf{H^n})$, $\dfrac{2m}{m+1} \le p < 2$, $u_\delta = (x_\delta, y_\delta, t_\delta)$. One has

$$\lim_{\delta \to 0} \int_\Omega \left| \partial_i t_\delta - 2(y_\delta \partial_i x_\delta - x_\delta \partial_i y_\delta) \right|^{p/2} dh = 0, \tag{2.9}$$

Where ∂_i means $\dfrac{\partial}{\partial h_i}$.

Proof. Since $u = (x, y, t) \in W^{1,p}(\Omega, \mathbf{H}^n)$, by the fact $\left| |f|^p - |g|^p \right| \le |f - g|^p$ with $0 < p < 1$, we have $u = (x, y, t) \in L^p(\Omega, \mathbf{H}^n)$, $E_{p,\Omega}(u) < +\infty$, and

$$E_{p,\Omega}(u) \ge \sup_{f \in C_0(\Omega), 0 \le f \le 1} \limsup_{\varepsilon \to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{t(h) - t(q)}{\varepsilon} \right|^{\frac{p}{2}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh$$

$$\ge \sup_{f \in C_0(\Omega), 0 \le f \le 1} \limsup_{\varepsilon \to 0} \int_\Omega f(h) \varepsilon^{\frac{p(2+2\beta-p)}{2(p-2\beta)}} \int_{|h-q|=\varepsilon} \left| \frac{|t(h)|^{\frac{p}{2}-\beta} - |t(q)|^{\frac{p}{2}-\beta}}{\varepsilon} \right|^{\frac{p}{p-2\beta}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh,$$

Where $0 < \beta \ll 1$, $E_{p,\Omega}(u) < +\infty$ implies that

$$\sup_{f \in C_0(\Omega), 0 \le f \le 1} \limsup_{\varepsilon \to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left| \frac{|t(h)|^{\frac{p}{2}-\beta} - |t(q)|^{\frac{p}{2}-\beta}}{\varepsilon} \right|^{\frac{p}{p-2\beta}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh < +\infty. \tag{2.10}$$

Furthermore, we get $|t(h)|^{\frac{p}{2}-\beta} \in L^{\frac{p}{2}-\beta}(\Omega)$. Let $T_\delta(h) = \left(|t(h)|^{\frac{p}{2}-\beta}\right)_\delta$. If δ sufficiently small, it follows from (2.10) that

$$\sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left|\frac{T_\delta(h)-T_\delta(q)}{\varepsilon}\right|^{\frac{p}{p-2\beta}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh < +\infty$$

and

$$\sup_{f\in C_0(\Omega),0\le f\le 1} \limsup_{\varepsilon\to 0} \int_\Omega f(h) \int_{|h-q|=\varepsilon} \left|\frac{T_\delta(h)-T_\delta(q)}{\varepsilon}\right|^{\frac{p}{p-2\beta}} \frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}} dh$$
$$= C(m,p) \sup_{f\in C_0(\Omega),0\le f\le 1} \int_\Omega f(h) |\nabla T_\delta(h)|^{\frac{p}{p-2\beta}} < +\infty.$$

By using the Sobolev theory, we get $|t(h)|^{\frac{p}{2}-\beta} \in W^{1,\frac{p}{p-2\beta}}(\Omega)$. By the Sobolev embedded theorem, we have $|t(h)|^{\frac{p}{2}-\beta} \in L^{\frac{mp}{(m-1)p-2m\beta}}(\Omega)$, or $t(h) \in L^{\frac{mp(p-2\beta)}{2((m-1)p-2m\beta)}}(\Omega)$.

On the other hand,

$$\begin{aligned} K(\delta) = \int_\Omega \Bigg| & \frac{t_\delta(h)-t_\delta(q)}{\varepsilon} - \frac{t(h)-t(q)}{\varepsilon} - 2y_\delta(h)\frac{x_\delta(h)-x_\delta(q)}{\varepsilon} \\ & +2y(h)\frac{x(h)-x(q)}{\varepsilon} + 2x_\delta(h)\frac{y_\delta(h)-y_\delta(q)}{\varepsilon} - 2x(h)\frac{y(h)-y(q)}{\varepsilon}\Bigg|^{\frac{p}{2}} dh \\ \le & \int_\Omega \left|\frac{t_\delta(h)-t(h)}{\varepsilon}\right|^{\frac{p}{2}} dh + \int_\Omega \left|\frac{t_\delta(q)-t(q)}{\varepsilon}\right|^{\frac{p}{2}} dh + \int_\Omega \Bigg| 2y_\delta(h)\frac{x_\delta(h)-x_\delta(q)}{\varepsilon} \\ & -2y(h)\frac{x(h)-x(q)}{\varepsilon} + 2x_\delta(h)\frac{y_\delta(h)-y_\delta(q)}{\varepsilon} - 2x(h)\frac{y(h)-y(q)}{\varepsilon}\Bigg|^{\frac{p}{2}} dh. \end{aligned} \quad (2.11)$$

For fixed $\varepsilon > 0$, by the properties of the mollifier function and Holder inequality, we then deduce from (2.11) that

$$\int_\Omega \left|\frac{t_\delta(h)-t(h)}{\varepsilon}\right|^{\frac{p}{2}} dh \le C(\Omega) \int_\Omega \left|\frac{t_\delta(h)-t(h)}{\varepsilon}\right| dh \to 0 (\delta\to 0)$$

and

$$\int_\Omega \left|\frac{t_\delta(q)-t(q)}{\varepsilon}\right|^{\frac{p}{2}} dh \le C(\Omega) \int_\Omega \left|\frac{t_\delta(q)-t(q)}{\varepsilon}\right| dh \to 0 (\delta\to 0).$$

By Lemma 2.4, we then deduce from (2.11) that

$$\begin{aligned} \int_\Omega \Bigg| & 2y_\delta(h)\frac{x_\delta(h)-x_\delta(q)}{\varepsilon} - 2y(h)\frac{x(h)-x(q)}{\varepsilon} \\ & +2x_\delta(h)\frac{y_\delta(h)-y_\delta(q)}{\varepsilon} - 2x(h)\frac{y(h)-y(q)}{\varepsilon}\Bigg|^{\frac{p}{2}} dh \to 0 (\delta\to 0). \end{aligned}$$

Thus, we get $\lim_{\delta\to 0} K(\delta)=0.$ Using the definition of $W^{1,p}(\Omega,\mathbf{H^n})$, one has

$$\int_\Omega\int_{|h-q|=\varepsilon}\left|\frac{t(h)-t(q)}{\varepsilon^2}-2y(h)\frac{x(h)-x(q)}{\varepsilon^2}+2x(h)\frac{y(h)-y(q)}{\varepsilon^2}\right|^{\frac{p}{2}}\frac{d\sigma_\varepsilon(q)}{\varepsilon^{m-1}}dh=o(\varepsilon).$$

Thus, we have

$$\lim_{\delta\to 0}\int_\Omega\left|\partial_i t_\delta-2(y_\delta\partial_i x_\delta-x_\delta\partial_i y_\delta)\right|^{\frac{p}{2}}dh=0.$$

Lemma 2.6. Let $u=(x,y,t)\in W^{1,p}(\Omega,\mathbf{H^n})$, $\dfrac{2m}{m+1}\le p<2.$ One has $t\in W^{1,r}(\Omega)$ and $\nabla t=2(y\nabla x-x\nabla y)$, a.e. in Ω, where $r=\dfrac{mp}{2m-p}.$

Proof. Let $u_\delta=(x_\delta,y_\delta,t_\delta)$, $\varphi\in C_0^\infty(\Omega)$, $t(h)\in L^{\frac{mp(p-2\beta)}{2((m-1)p-2m\beta)}}(\Omega).$ Using Lemma 2.5, one gets

$$\int_\Omega t(h)\partial_i\varphi(h)dh=-2\int_\Omega(x(h)\partial_i y(h)-y(h)\partial_i x(h))\varphi(h)dh.$$

i.e., $\nabla t=2(y\nabla x-x\nabla y).$ By Lemma 2.4, we conclude $x,y\in W^{1,p}(\Omega,R^n).$ This implies from Sobolev embedded theorem that $\nabla t\in L^{\frac{mp}{2m-p}}(\Omega).$ On the other hand, by Lemma 2.5 and its proof, we get $t\in L^{\frac{mp}{2m-p}}(\Omega)$, which implies that $t\in W^{1,\frac{mp}{2m-p}}(\Omega).$

3. POINCARE TYPE INEQUALITIES OF HEISENBERG GROUP TARGET FOR $\dfrac{2m}{m+1}<p<2$

Theorem 3.1. (Poincare type inequality) Let Ω be a bounded and connected Lipschitz domain in R^m, $\dfrac{2m}{m+1}<p<2.$ Then there exists a constant C depending only on Ω,n,m and p, such that for every function $u=(x,y,t)=(z,t)\in W^{1,p}(\Omega,\mathbf{H}^n)$, the inequality

$$\int_\Omega (d(u(h),\lambda_u))^p\,dh\le C\int_\Omega|\nabla z|^p\,(h)dh, \tag{3.1}$$

holds, where $\lambda_u=(\lambda_x,\lambda_y,\lambda_t)$ and $\lambda_f=\dfrac{1}{|\Omega|}\int_\Omega f(h)dh.$

Proof. Obviously, $\lambda_u\in W^{1,p}(\Omega,\mathbf{H}^m).$ From (1.4), using C_p-inequality, we have

$$\begin{aligned}(d(u(h),\lambda_u))^p&=\left[|z(h)-\lambda_z|^4+(t(h)-\lambda_t+2(\lambda_x y(h)-\lambda_y x(h)))^2\right]^{\frac{p}{4}}\\&\le\left[|x(h)-\lambda_x|^p+|y(h)-\lambda_y|^p+|t(h)-\lambda_t+2(\lambda_x y(h)-\lambda_y x(h))|^{\frac{p}{2}}\right].\end{aligned} \tag{3.2}$$

By the Poincare inequality in the classical case, noting that

$$2(\lambda_x y(h)-\lambda_y x(h))=2\lambda_x(y(h)-\lambda_y)-2\lambda_y(x(h)-\lambda_x),$$

we obtain

$$\int_\Omega (d(u(h),\lambda_u))^p dh$$
$$\leq \left[\int_\Omega (|x(h)-\lambda_x|^p + |y(h)-\lambda_y|^p + |t(h)-\lambda_t + 2(\lambda_x y(h) - \lambda_y x(h))|^{\frac{p}{2}}) dh \right]$$
$$\leq C\int_\Omega |\nabla x|^p (h)dh + C\int_\Omega |\nabla y|^p (h)dh + C\int_\Omega |\nabla t + 2(\lambda_x \nabla y - \lambda_y \nabla x)|^{\frac{p}{2}} dh.$$

By virtue of Lemma 2.6, using the Holder inequality, noting $|\nabla x| \leq |\nabla z|$ and $|\nabla y| \leq |\nabla z|$, we have

$$\int_\Omega (d(u(h),\lambda_u))^p dh$$
$$\leq C\int_\Omega |\nabla x|^p dh + C\int_\Omega |\nabla y|^p (h)dh + C\int_\Omega |\nabla y(x-\lambda_x) - \nabla x(y-\lambda_y)|^{\frac{p}{2}} dh$$
$$\leq C\int_\Omega |\nabla x|^p dh + C\int_\Omega |\nabla y|^p dh + C\left(\int_\Omega |\nabla x|^{\frac{p}{2}} |y-\lambda_y|^{\frac{p}{2}} dh + \int_\Omega |\nabla y|^{\frac{p}{2}} |x-\lambda_x|^{\frac{p}{2}} dh \right)$$
$$\leq C\int_\Omega |\nabla x|^p dh + C_2\int_\Omega |\nabla y|^p dh + C_5\left(\int_\Omega |\nabla x|^p dq \int_\Omega |\nabla y|^p dh \right)^{\frac{1}{2}}$$
$$\leq C\int_\Omega |\nabla z|^p (h)dh,$$

where C is dependent on Ω, m, n and p.

Corollary 3.2. Let $B(h_0,r) \subset R^m$, $\frac{2m}{m+1} < p < 2$. Then for any $u \in W^{1,p}(B(h_0,r),\mathbf{H}^n)$, we have

$$\int_{B(h_0,r)} (d(u(h),\lambda_u))^p dh \leq Cr^p \int_{B(h_0,r)} |\nabla z|^p (h)dh. \quad (3.3)$$

ACKNOWLEDGMENTS

We wish to thank Professor X.P. Yang for useful discussions and suggestions. This work was supported by the National Natural Science Foundation of China (11171220) and Shanghai Leading Academic Discipline Project (XTKX2012).

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