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The Continuous Dependence and Numerical Approximation of the Solution of the Quasilinear Pseudo-Parabolic Problem with Periodic Boundary Condition

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ABSTRACT

In this paper we consider a pseudo- parabolic equation with a periodic boundary condition and we prove the stability of a solution on the data. We give a numerical example for the stability of the solution on the data.

Keywords:

Psedu-parabolic quasilinear equation; periodic-boundary condition; finite difference method.

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INTRODUCTION

Consider the following mixed problem

$$\boldsymbol{u_t = u_{xx} - \varepsilon u_{xxt} + f(x,t,u), (x,t)\epsilon D} := \{\boldsymbol{0} < x < \pi, 0 < t < T\} \quad (1)$$

$$u(0,t) = u(\pi,t), t\epsilon[0,T] \quad (2)$$

$$u_x(0,t) = u_x(\pi,t), t\epsilon[0,T] \quad (3)$$

$$u(x,0) = \varphi(x), x\epsilon[0,\pi] \quad (4)$$

for a quasilinear parabolic equation with the nonlinear source term $\boldsymbol{f(x,t,u)}$. The functions$\boldsymbol{\varphi(x)}$ and $\boldsymbol{f(x,t,u)}$ are given functions on$[\boldsymbol{0,\pi}]]$ and $\boldsymbol{D} \times (-\infty,\infty)$ respectively.Let $\boldsymbol{\varepsilon} > \boldsymbol{0}$ is small parameter.

Denote the solution of the problem (1)-(4) by $\boldsymbol{u = u(x,t,\varepsilon)}$.

The existence, uniqueness and convergence of the weak generalized solution the problem (1)-(4) are considered in [1]. The numerical solution of parabolic problem is consideredin [3].

In this study we prove the contiunous dependence of the solution $\boldsymbol{u = u(x,t,\varepsilon)}$ upon the data $\boldsymbol{\varphi(x)}$ and $\boldsymbol{f(x,t,u)}$.This kind of conditions arise from many important applications in heat transfer, life sciences, etc. For example the periodic conditions are used on lunar theory and in the system of rocket firing in [2]. Then we give a numerical example using the method for the stability.

CONTINUOUS DEPENDENCE UPON THE DATA

In this section, we shall prove the contiunous dependence of the solution $u = u(x,t,\varepsilon)$ using an a iteration method.The contiunous dependence upon the data for linear problems by different methods are shown.

Theorem: Under the following assumptions, the solution $u = u(x,t,\varepsilon)$ depends contiunously upon the data.

(A_1) Let the function $f(x,t,u)$ is continuous with respect to all arguments in $D \times (-\infty,\infty)$ and satisfies the following condition

$$|f(t,x,u) - f(t,x,u)| \le b(x,t)|u - u|,$$

Where $b(x,t) \in L^2(D), b(x,t) \ge 0,$

(A_2) $f(x,t,0) \in C^2[0,\pi], t\epsilon[0,\pi]$

(A_3) $\varphi(x) \in C^2[0,\pi]$.

Proof: Let $\phi = \{\varphi, f\}$ and $\bar{\phi} = \{\bar{\varphi}, \bar{f}\}$be two sets of data which satisfy the conditions (A_1)-(A_3)

Let $u = u(x,t,\varepsilon)$ and $v = v(x,t,\varepsilon)$ the solutions of the problem (1)-(4) corresponding to the data ϕ and $\bar{\phi}$ respectively and

$|f(t,x,0) - \bar{f}(t,x,0)| \le \varepsilon,$ for $\varepsilon \ge 0$.

The solutions of (1)-(4) $\boldsymbol{u = u(x,t,\varepsilon)}$ and $\boldsymbol{v = v(x,t,\varepsilon)}$ in the following form, respectively,

$$\boldsymbol{u_0(t,\varepsilon) = \varphi_0 + \frac{2}{\pi}\int_0^t \int_0^\pi f(\xi,\tau,u(\xi,\tau,\varepsilon))d\xi d\tau},$$

$$\boldsymbol{u_{ck}(t,\varepsilon) = \varphi_{ck} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi f(\xi,\tau,u(\xi,\tau,\varepsilon))e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} cos2k\xi d\xi d\tau},$$

$$u_{sk}(t,\varepsilon) = \varphi_{sk} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi f(\xi,\tau,u(\xi,\tau,\varepsilon))e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} sin2k\xi d\xi d\tau,$$

$$\boldsymbol{v_0(t,\varepsilon) = \bar{\varphi}_0 + \frac{2}{\pi}\int_0^t \int_0^\pi \bar{f}(\xi,\tau,v(\xi,\tau,\varepsilon))d\xi d\tau},$$

$$\boldsymbol{v_{ck}(t,\varepsilon) = \bar{\varphi}_{ck} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi \bar{f}(\xi,\tau,v(\xi,\tau,\varepsilon))e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} cos2k\xi d\xi d\tau},$$

$$v_{sk}(t,\varepsilon) = \bar{\varphi}_{sk} e^{\frac{-(2k)^2 t}{1+\varepsilon(2k)^2}} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi \bar{f}(\xi,\tau,v(\xi,\tau,\varepsilon))e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} sin2k\xi d\xi d\tau,$$

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From the condition of the theorem we have $\boldsymbol{u}^{(0)}(\boldsymbol{t},\boldsymbol{\varepsilon})$ and $\boldsymbol{v}^{(0)}(\boldsymbol{t},\boldsymbol{\varepsilon}) \in \boldsymbol{B}$. We will prove that the other sequentially approximations satisfy this condition.

$$\boldsymbol{u}_0^{(N+1)}(\boldsymbol{t},\boldsymbol{\varepsilon}) = \boldsymbol{u}_0^{(0)}(\boldsymbol{t},\boldsymbol{\varepsilon}) + \frac{\mathbf{2}}{\boldsymbol{\pi}}\int_0^t\int_0^\pi \boldsymbol{f}(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{u}^{(N)}(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{\varepsilon}))\boldsymbol{d\xi d\tau},$$

$$\boldsymbol{u}_{ck}^{(N+1)}(\boldsymbol{t},\boldsymbol{\varepsilon}) = \boldsymbol{u}_{ck}^{(0)}(\boldsymbol{t},\boldsymbol{\varepsilon}) + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi \boldsymbol{f}\left(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{u}^{(N)}(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{\varepsilon})\right)\boldsymbol{e}^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}}\boldsymbol{cos2k\xi d\xi d\tau}, \quad (5)$$

$$u_{sk}^{(N+1)}(t,\varepsilon) = u_{sk}^{(0)} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi f\left(\xi,\tau,u^{(N)}(\xi,\tau,\varepsilon)\right)e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} sin2k\xi d\xi d\tau,$$

$$\boldsymbol{v}_0^{(N+1)} = \boldsymbol{v}_0^{(0)}(\boldsymbol{t},\boldsymbol{\varepsilon}) + \frac{\mathbf{2}}{\boldsymbol{\pi}}\int_0^t\int_0^\pi \bar{\boldsymbol{f}}(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{v}^{(N)}(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{\varepsilon}))\boldsymbol{d\xi d\tau},$$

$$\boldsymbol{v}_{ck}^{(N+1)}(\boldsymbol{t},\boldsymbol{\varepsilon}) = \boldsymbol{v}_{ck}^{(0)}(\boldsymbol{t},\boldsymbol{\varepsilon}) + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi \bar{\boldsymbol{f}}\left(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{v}^{(N)}(\boldsymbol{\xi},\boldsymbol{\tau},\boldsymbol{\varepsilon})\right)\boldsymbol{e}^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}}\boldsymbol{cos2k\xi d\xi d\tau}, \quad (6)$$

$$v_{sk}^{(N+1)}(t,\varepsilon) = v_{sk}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2k)^2}\int_0^t \frac{2}{\pi}\int_0^\pi \bar{f}\left(\xi,\tau,v^{(N)}(\xi,\tau,\varepsilon)\right)e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} sin2k\xi d\xi d\tau,$$

where ${u_0}^{(0)}(t,\varepsilon) = \varphi_0, u_{ck}^{(0)}(t,\varepsilon) = \varphi_{ck}\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}}, u_{sk}^{(0)} = \varphi_{sk}\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}}$, and

${v_0}^{(0)}(t,\varepsilon) = \bar{\varphi}_0, v_{ck}^{(0)}(t,\varepsilon) = \bar{\varphi}_{ck}\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}} v_{sk}^{(0)}(t,\varepsilon) = \bar{\varphi}_{sk}\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}}$.

First of all , we write $N = 0$ in (5)-(6). we consider $u^{(1)}(t,\varepsilon) - v^{(1)}(t,\varepsilon)$

$u^{(1)}(t,\varepsilon) - v^{(1)}(t,\varepsilon) = \frac{{u_0}^{(1)}(t,\varepsilon) - {v_0}^{(1)}(t,\varepsilon)}{2} + \sum_{k=1}^{\infty}\left[\left(u_{ck}^{(1)}(t,\varepsilon) - v_{ck}^{(1)}(t,\varepsilon)\right)cos2kx + \left(u_{sk}^{(1)}(t,\varepsilon) - v_{sk}^{(1)}(t,\varepsilon)\right)sin2kx\right]$

$$=(\varphi_0 - \bar{\varphi}_0) + \frac{2}{\pi}\int_0^t\int_0^\pi\left[f\left(\xi,\tau,u^{(0)}(\xi,\tau,\varepsilon)\right) - \bar{f}\left(\xi,\tau,v^{(0)}(\xi,\tau,\varepsilon)\right)\right]d\xi d\tau + (\varphi_{ck} - \bar{\varphi}_{ck})\, e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}}$$

$$+\frac{1}{1+\varepsilon(2k)^2}\int_0^t\frac{2}{\pi}\int_0^\pi\left[f\left(\xi,\tau,u^{(0)}(\xi,\tau,\varepsilon)\right) - \bar{f}\left(\xi,\tau,v^{(0)}(\xi,\tau,\varepsilon)\right)\right]e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} cos2k\xi d\xi d\tau \quad (7)$$

$$+(\varphi_{sk} - \bar{\varphi}_{sk})e^{\frac{-(2k)^2t}{1+\varepsilon(2k)^2}} + \frac{1}{1+\varepsilon(2k)^2}\int_0^t\frac{2}{\pi}\int_0^\pi\left[f\left(\xi,\tau,u^{(0)}(\xi,\tau,\varepsilon)\right) - \bar{f}\left(\xi,\tau,v^{(0)}(\xi,\tau,\varepsilon)\right)\right]e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} sin2k\xi d\xi d\tau$$

Adding and subtracting

$$\frac{2}{\pi}\int_0^t\int_0^\pi f(\xi,\tau,0)d\xi d\tau, \frac{1}{1+\varepsilon(2k)^2}\int_0^t\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} cos2k\xi d\xi d\tau, \frac{1}{1+\varepsilon(2k)^2}\int_0^t\frac{2}{\pi}\int_0^\pi f(\xi,\tau,0)e^{\frac{-(2k)^2(t-\tau)}{1+\varepsilon(2k)^2}} sin2k\xi d\xi d\tau,$$

to both of side and applying Cauchy Inequality, Hölder Inequality, Lipshitzs Condition and Bessel Inequality to the right side of (7) respectively,we obtain:

$\left|u^{(1)}(t,\varepsilon) - v^{(1)}(t,\varepsilon)\right| \leq \frac{\left|u^{(1)}(t,\varepsilon) - v^{(1)}(t,\varepsilon)\right|}{2} + \sum_{k=1}^{\infty}\left[\left|u_{ck}^{(1)}(t,\varepsilon) - v_{ck}^{(1)}(t,\varepsilon)\right| + \left|u_{sk}^{(1)}(t,\varepsilon) - v_{sk}^{(1)}(t,\varepsilon)\right|\right] \leq \|\varphi -$
φ+3T+π6π0t0πb2ξ,τdξdτ12u0t,ε+3T+π6π0t0πb2ξ,τdξdτ12v0t,ε

$+\frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\left(\int_0^t\int_0^\pi\left[f^2(\xi,\tau,0) - \bar{f}^2(\xi,\tau,0)\right]d\xi d\tau\right)^{1/2}$,

where

$A_T = \|\varphi - \bar{\varphi}\| + \frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\|b(x,t)\|\left|u^{(0)}(t,\varepsilon)\right| + \frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\|\bar{b}(x,t)\|\left|v^{(0)}(t,\varepsilon)\right| + \frac{\sqrt{3T}+\pi}{\sqrt{6}\pi}\|f - \bar{f}\|$

$\|\varphi - \bar{\varphi}\| = max\frac{|\varphi_0 - \bar{\varphi}_0|}{2} + \sum_{k=1}^{\infty}|\varphi_{ck} - \bar{\varphi}_{ck}| + |\varphi_{sk} - \bar{\varphi}_{sk}|$.

For $N = 1$,

$|u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)| \le \frac{|u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)|}{2} + \sum_{k=1}^{\infty}\left[\left|u_{ck}^{(2)}(t,\varepsilon)-v_{ck}^{(2)}(t,\varepsilon)\right|+\left|u_{sk}^{(2)}(t,\varepsilon)-v_{sk}^{(2)}(t,\varepsilon)\right|\right] \le$
$\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}A_T + \frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\left(\int_0^t\int_0^\pi \bar{b}^2(\xi,\tau)d\xi d\tau\right)^{\frac{1}{2}}A_T.$

For $N=2$,

$|u^{(3)}(t,\varepsilon)-v^{(3)}(t,\varepsilon)| \le \frac{|u^{(3)}(t,\varepsilon)-v^{(3)}(t,\varepsilon)|}{2} + \sum_{k=1}^{\infty}\left[\left|u_{ck}^{(3)}(t,\varepsilon)-v_{ck}^{(3)}(t,\varepsilon)\right|+\left|u_{sk}^{(3)}(t,\varepsilon)-v_{sk}^{(3)}(t,\varepsilon)\right|\right] \le$
$\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left|u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} + \frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\left(\int_0^t\int_0^\pi \bar{b}^2(\xi,\tau)\left|u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)\right|^2 d\xi d\tau\right)^{\frac{1}{2}} \le$
$\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}A_T\left(\int_0^t\int_0^\pi b^2(\xi,\tau)\left(\int_0^\tau\int_0^\pi b^2(\xi_1,\tau_1)d\xi_1 d\tau_1\right)d\xi d\tau\right)^{\frac{1}{2}} + \frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}A_T\left(\int_0^t\int_0^\pi \bar{b}^2(\xi,\tau)\left(\int_0^\tau\int_0^\pi \bar{b}^2(\xi_1,\tau_1)d\xi_1 d\tau_1\right)d\xi d\tau\right)^{\frac{1}{2}} \le$
$\left(\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\right)^2 A_T\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{1}{2}} + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\right)^2 A_T\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^\pi \bar{b}^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{1}{2}}$

In the same way, for a general value of N we have

$|u^{(N+1)}(t,\varepsilon)-v^{(N+1)}(t,\varepsilon)| \le$
$\frac{|u^{(N+1)}(t,\varepsilon)-v^{(N+1)}(t,\varepsilon)|}{2} + \sum_{k=1}^{\infty}\left[\left|u_{ck}^{(N+1)}(t,\varepsilon)-v_{ck}^{(N+1)}(t,\varepsilon)\right|+\left|u_{sk}^{(N+1)}(t,\varepsilon)-v_{sk}^{(N+1)}(t,\varepsilon)\right|\right] \le A_T a_N \le$
$a_N\left(\|\varphi-\bar{\varphi}\|+C(t)+M_1\|f-\bar{f}\|\right)$

where

$$a_N = \left(\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\right)^N \frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}} + \left(\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\right)^N \frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^\pi \bar{b}^2(\xi,\tau)d\xi d\tau\right)^2\right]^{\frac{N}{2}},$$

$$M_1 = \left(\frac{\sqrt{3T}+\pi}{\sqrt{6\pi}}\right)^N.$$

(M =The sequence $\boldsymbol{a_N}$ is convergent then we can write $\boldsymbol{a_N \le N}, \forall \boldsymbol{N}$)

It follows from the estimation ([3]) that $\mathbf{lim}_{N\to\infty}\boldsymbol{u^{(N+1)}(t)=u(t)}$,

then let $\boldsymbol{N}\to\infty$ for last equation

$$|\boldsymbol{u}(\boldsymbol{t})-\boldsymbol{v}(\boldsymbol{t})| \le \boldsymbol{M}\|\boldsymbol{\varphi}-\boldsymbol{\varphi}^{-}\| + \boldsymbol{M}_2\|\boldsymbol{f}-\bar{\boldsymbol{f}}\|)$$

where $\boldsymbol{M_2 = M.M_1}$.

$\boldsymbol{If}\ \|\boldsymbol{f}-\bar{\boldsymbol{f}}\| \le \boldsymbol{\varepsilon},\ \|\boldsymbol{\varphi}-\boldsymbol{\varphi}^{-}\| \le \boldsymbol{\varepsilon}$ then $|\boldsymbol{u}(\boldsymbol{t})-\boldsymbol{v}(\boldsymbol{t})| \le \boldsymbol{\varepsilon}$.

NUMERICAL PROCEDURE FOR THE NONLINEAR PROBLEM (1)-(4)

We construct an iteration algorithm for the linearization of the problem (1)-(4):

$$\left(\frac{\partial u^{(n)}}{\partial t}\right) = \left(\frac{\partial^2 u^{(n)}}{\partial x^2}\right) - \varepsilon\left(\frac{\partial^3 u^{(n)}}{\partial x^2 \partial t}\right) + f(x,t,u^{(n-1)}), \quad (x,t)\in D \tag{8}$$

$$u^{(n)}(0,t) = u^{(n)}(\pi,t), \quad t\in[0,T] \tag{9}$$

$$u_x{}^{(n)}(0,t) = u_x{}^{(n)}(\pi,t), \quad t\in[0,T] \tag{10}$$

$$u^{(n)}(x,0) = \varphi(x), \quad x\in[0,\pi]. \tag{11}$$

Let $u^{(n)}(x,t)=v(x,t)$ and $f(x,t,u^{(n-1)})=\tilde{f}(x,t)$. Then the problem (8)-(11) can be written as a linear problem:

$$\left(\frac{\partial v}{\partial t}\right) = \left(\frac{\partial^2 v}{\partial x^2}\right) - \varepsilon\left(\frac{\partial^3 v}{\partial x^2 \partial t}\right) + \tilde{f}(x,t) \quad (x,t)\in D \tag{12}$$

$$v(0,t) = v(\pi,t), \quad t\in[0,T] \tag{13}$$

$$v_x(0,t) = v_x(\pi,t), \quad t\in[0,T] \tag{14}$$

$$v(x,0) = \varphi(x), \quad x\in[0,\pi]. \tag{15}$$

We use the finite difference method to solve (12)-(15).

We subdivide the intervals $[0,\pi]$ and $[0,T]$ into M and N subintervals of equal lengths $h=(\pi/M)$and $\tau=(T/N)$, respectively.

Then, we add two lines $x=0$ and $x=(M+1)\pi h$ to generate the fictitious points needed for dealing with the secondary boundary condition.

We choose the implicit scheme, which is absolutely stable and has a second order accuracy in h and a first order accuracy in τ.

The implicit monotone difference scheme for (12)-(15) is as follows:

$$\frac{\left(v_i^{j+1}-v_i^j\right)}{\tau}=\frac{1}{h^2}\left(v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1}\right)-\frac{\varepsilon}{h^2\tau}\left(v_{i-1}^{j+1}-2v_i^{j+1}+v_{i+1}^{j+1}\right)-\left(v_{i-1}^{j}-2v_i^{j}+v_{i+1}^{j}\right)+\tilde{f}_i^{j+1}$$

$$v_i^0=\phi_i, v_0^j=v_M^j, v_{x,0}^j=v_{x,M}^j$$

where$0\le i\le M$ and $1\le j\le N$ are the indices for the spatial and time steps, respectively, v_i^j is the approximation to $v(x_i,t_j)$, $\tilde{f}_{i,j}=\tilde{f}(x_i,t_j)$, $\phi_i=\varphi(x_i)$, $x_i=ih$, $t_j=j\tau$.

At the $t=0$ level, adjustment should be made according to the initial condition and the compatibility requirements.

Numerical Example

In this section, we will consider an example of numerical solution of the problem (1)-(3).

This problem was solved by applying the iteration scheme and the finite difference scheme which were explained in the Section 2. The condition

$$error(i,j)\coloneqq \|u_i^{j\,(n+1)}-u_i^{j\,(n)}\|_\infty$$

with

$$error(i,j):=10^{-3}$$

was used as a stopping criteria for the iteration process.

Example: Consider the problem

$$\left(\frac{\partial u}{\partial t}\right)-\left(\frac{\partial^2 u}{\partial x^2}\right)-\varepsilon\left(\frac{\partial^3 u}{\partial x^2\partial t}\right)=4(-cos2x+(sin2x)^2)(1-\varepsilon)u,(x,t)\epsilon D$$

$$u(x,0)=exp(cos2x), x\in[0,\pi],$$

$$u(0,t)=u(\pi,t),\ \ t\in[0,T], u_x(0,t)=u_x(\pi,t),\ \ t\in[0,T].$$

It is easy to see that the analytical solution of this problem is

$$u(x,t)=exp(t+cos2x)$$

The comparisons between the analytical solution and the numerical finite difference solution for differentε values when $T=1$ are shown in Figure 1 and 2.

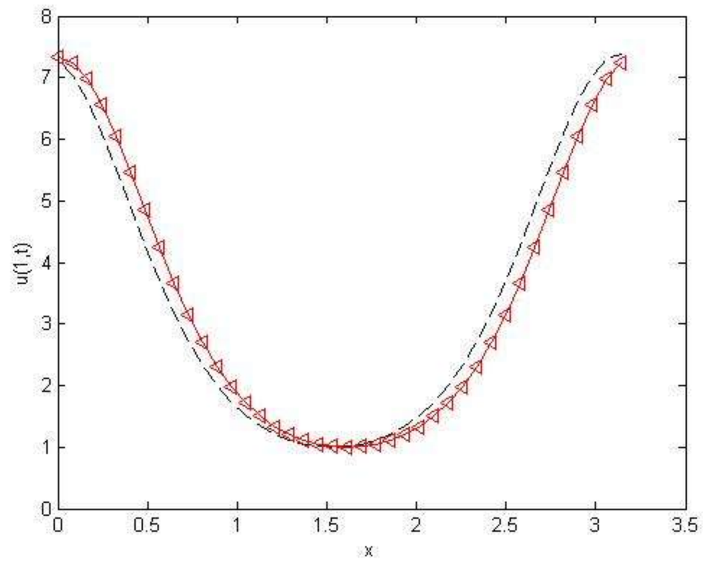

Figure 1 - The exact and numerical solutions of u(x,1)

The exact and numerical solutions of u(x,1) for ε=0, the exact solution is shown with dashes line.

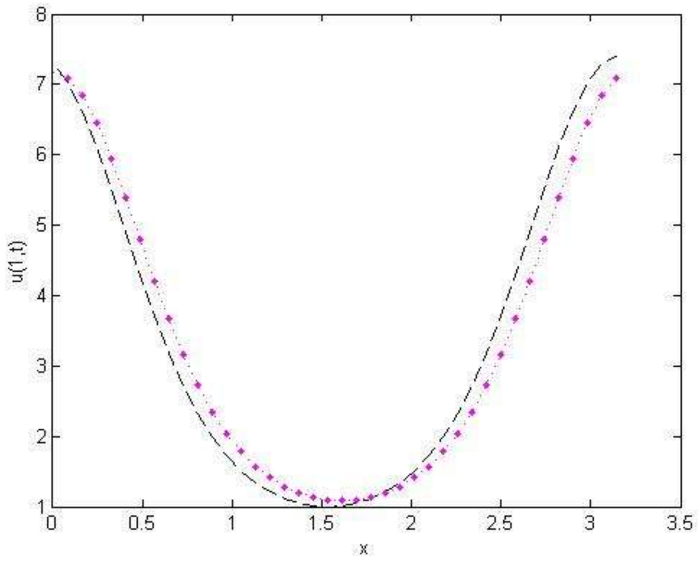

Figure 2 -The exact and numerical solutions of u(x,1)

The exact and numerical solutions of u(x,1) for ε=0.05, the exact solution is shown with dashes line.

In Figure3 we show that the analytical solution for $\varepsilon = 0$ and the numerical solution for $\varepsilon = 0\,, \varepsilon = 0{,}1, \varepsilon = 0{,}05$.

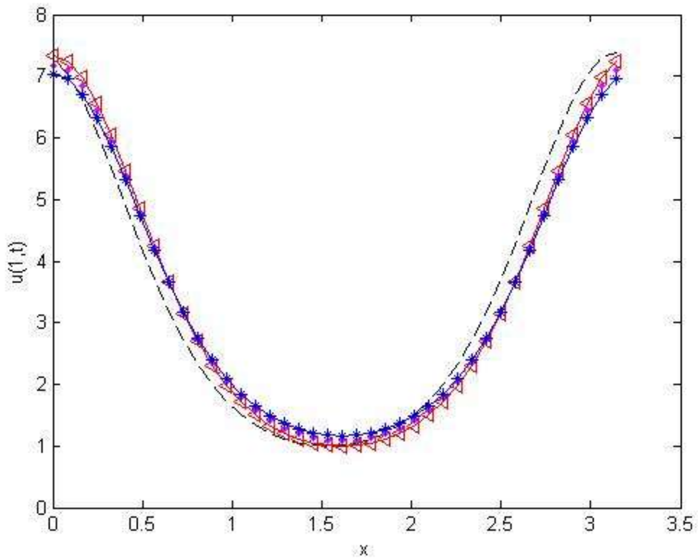

Figure 3 - The exact and numerical solutions of u(x,1)

The exact and numerical solutions of u(x,1), (-△) for ε=0, (-·) for ε=0.05, (-∗) for ε=0.1, the exact solution is shown with dashes line.

Relative errors obtained on different grids for different ε are shown on Table 1.

Table 1. The relative errors for different grids for $\boldsymbol{\varepsilon = 0, \varepsilon = 0.05}$ and $\boldsymbol{\varepsilon = 0.1}$.

h	τ	$\varepsilon = 0$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
0.1571	0.025	0.0142	0.0250	0.0553
0.1571	0.0125	0.0151	0.0269	0.0624
0.0785	0.025	0.0160	0.0334	0.0642
0.0785	0.0125	0.0179	0.0373	0.0720

From Figure 3 and Table 1,2 and 3 when ε approximates to zero the numerical solutions converge to the exact solution for $\varepsilon = 0$ like theoretical results .

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