



An Analytic Approximate Solution of the Matrix Riccati Differential Equation Arising from the LQ Optimal Control Problems

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ABSTRACT

Approximate analytical solution of the matrix Riccati differential equation related to the linear quadratic optimal control problems, is the main goal of this paper which has a specific importance in the optimal control theory. To this end, a modification of the parametric iteration method is used. This modification reduces the time consuming repeated calculations and improves the convergence rate of the iterative algorithm. Comparison with the existent solutions and also with the numerical Runge-Kutta (RK78) method confirms the high accuracy of the method, whilst accessibility to the analytical solutions is the preference of the new technique.

Keywords: Riccati differential equation; LQ optimal control problem; He's variational iteration method; parametric iteration method; Hamiltonian differential equation.

Mathematics Subject Classification: 49J15, 49K15, 93C15, 93C05.



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1. INTRODUCTION

Consider the following linear quadratic optimal control problem (LQ-OCP)

$$\text{Min } J = \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T Q x + u^T R u) dt, \quad (1.1)$$

$$\text{s. to: } \dot{x} = Ax + Bu, \quad x(t_0) = x_0.$$

Where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and control vectors, respectively. $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are constant matrix and x_0 is an initial state. The LQ-OCP is to find a control law $u^*(t)$ which minimizes the above quadratic cost functional where $S, Q \in \mathbb{R}^{n \times n}$ are symmetric positive semi-definite matrices and $R \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix.

The optimal control $u^*(t)$ for the above LQ-OCP is a linear feedback of the states as $u^*(t) = -K(t)x^*(t)$ with $K(t) = R^{-1}B^T P(t)$, where the matrix $P(t)$ is the solution of the following *matrix Riccati differential problem* (RDP) [6]

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \quad P(t_f) = S \quad (1.2)$$

In order to obtain the optimal control feedback we have to find the $P(t)$ from solving the RDP (1.2). A superseded alternative to find the solution $P(t)$ of the RDP (1.2) is solving the so-called *Hamiltonian differential problem* (HDP) [6]

$$\begin{aligned} \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} &= H \begin{bmatrix} X \\ Y \end{bmatrix} \text{ where } H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}, \\ \begin{bmatrix} X(t_f) \\ Y(t_f) \end{bmatrix} &= \begin{bmatrix} I_n \\ S \end{bmatrix}. \end{aligned} \quad (1.3)$$

Here I_n is an $n \times n$ identity matrix. The following theorem states the precise relation between the solutions of the RDP (1.2) and those of the HDP (1.3) (the proof in [7]).

Theorem 1.1 Let $X, Y: (-\infty, t_f] \rightarrow \mathbb{R}^{n \times n}$ be the solutions of the HDP (1.3). Then

1. $X(t)$ is non-singular for all $t \in (-\infty, t_f]$.
2. The solution $P(t)$ of the RDP (1.2) is $P(t) = Y(t)X^{-1}(t)$ when $t \in (-\infty, t_f]$.

This theorem provides us an analytical process to compute the solutions of the RDP (1.2) via the solutions of the HDP (1.3). This is theoretically good, but in practice, the solutions of the HDP (1.3) are

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = e^{H(t-t_f)} \begin{bmatrix} I_n \\ S \end{bmatrix} \quad (1.4)$$

and in general, there is no explicit way to express the exponential matrix e^{Ht} . Therefore, some additional techniques have to be applied to establish the solutions. For instance, the so-called *Davison–Maki numerical method* [8] proposes a procedure by partitioning the exponential matrix. Besides, from the early 1970s, many of researchers (see e.g. [5,7,14]) have been centered on the attempt to characterize more explicit representations for the solutions of the RDP (1.2).

One of the most general formulas which is introduced in [12], exploits the particular solutions of the associated algebraic Riccati equation. The authors established an explicit closed formulae for the solutions of the RDP (1.2) under some specific assumptions [12]. Theoretically, they proved some important existence theorems but in practice, there doesn't seem to be presented an easy to use computational procedure. Because, in order to solve the RDP (1.2), some lateral systems such as the Sylvester equations and the Lyapunov equations (same reference) have to be solved and then the general formula involving the exponential matrix can be employed. This states that such closed-form formulas may lead to time-consuming computations, except for certain special applications.

During the last four decades, beside such analytical solutions, approximate methods have been applied for finding the solutions of the RDP, for example, the above mentioned Davison–Maki numerical method [8]. Another typical sample is the numerical integration method [10]. Although the numerical integration methods are flexible, but they have their weaknesses, for instance, they react quite sensitively on the selection of time-step size (see e.g. [16]).

In recent times, due to the importance of the RDP in many fields of applied sciences, a vast amount of researches have been invested in the study of analytic approximate methods for solving the RDP [1-4,15]. Such methods expand the solution via a series or sequence of functions which provide better information of the solution in comparison with the numerical methods. Especially in the control theory, the numerical results may be not suitable for studying features like switching, limiting behavior of the solution and analysis of sensitivity of the solution in perturbations, which are fundamental arguments in both practical and theoretical viewpoints.

However, the analytic approximate methods have their demerits, for instance, their successive iterations may be very complex such that resulting integrations in their iterative relations may be impossible to perform analytically. Moreover, their application to the RDP may lead to the calculation of repeated terms or the terms that are not needed, consequently



more time is consumed in unnecessary calculations. Finally, the convergence region of the series or sequence of the solution may be rather small.

In this work, we aim to overcome the above mentioned difficulties by introducing a piecewise truncated parametric iteration method for solving the RDP (1.2). The numerical examinations confirm that this method provides an excellent approximations in a straightforward manner.

2. IMPLEMENTATION OF THE METHOD

The parametric iteration method (PIM) [9] is an analytic approximate method for solving linear and nonlinear problems. In this section, first, we introduce the PIM for solving the matrix Riccati differential problem (1.2), then, we explain a modification of the PIM to improve the accuracy of approximations. The idea of the PIM is simple. To explain the PIM, rewrite the matrix equation (1.2) as

$$\mathcal{L}[P(t)] + \mathcal{N}[P(t)] = g(t). \quad (2.1)$$

where \mathcal{L} with the property $\mathcal{L}[f] = 0$ when $f = 0$, denotes the so-called auxiliary linear differential operator with respect to P , \mathcal{N} is a nonlinear operator with respect to P and $g(t)$ is the source term.

Because of the terminal condition of the RDP (1.2), it is straightforward to expand the solution via a set of basis functions as $\{(t - t_f)^m \mid m = 0, 1, 2, \dots\}$. Each selection of \mathcal{L} will affect on the expression of the solution, therefore a natural choice for the auxiliary linear differential operator \mathcal{L} is $\mathcal{L}[P(t)] = \dot{P}(t)$.

Hence, a family of iterative formulas constructed by PIM [9,13] for the RDP (1.2) becomes

$$P_{k+1}(t) = P_k(t) + h \int_t^{t_f} H(s) [\dot{P}_k(s) + P_k(s)A + A^T P_k(s) - P_k(s)BR^{-1}B^T P_k(s) + Q] ds, \quad k = 0, 1, 2, \dots \quad (2.2)$$

Here $H(t) \neq 0$ is an auxiliary function which in view of the above solution expansion, is determined as $H(t) = 1$. Also $h \neq 0$ is an accelerating factor [9] and as will be shown in the next section, a suitable value of h , directly improves the convergence rate of the sequence of approximations. This is the main merit of the PIM over the methods in [1-4].

The first component $P_0(t)$ as an initial approximation may be selected by the solution of the corresponding linear homogeneous equation $\mathcal{L}[P_0(t)] = 0$. Here, according to the terminal condition in the RDP (1.2), we choose $P_0(t) = S$. Accordingly, the successive approximations $P_k(t); k \geq 1$ will be readily obtained satisfying the general property

$$P_{k+1}(t) = P_k(t) = \dots = P_0(t) = S. \quad (2.3)$$

Finally, the exact solution may be obtained by using

$$P(t) = \lim_{k \rightarrow \infty} P_k(t). \quad (2.4)$$

The successive iterations of the PIM to solve the RDP (1.2) may be very complex, so the integrals in (2.2) may not be performed analytically. Also, the implementation of the PIM may lead to the calculation of unnecessary or repeated terms, which causes to consume more time. For these reasons, we utilize the truncated PIM [13] for solving the RDP (1.2). Therefore, we have

$$P_{k+1}(t) = P_k(t) + h \int_t^{t_f} \Phi_k(s) ds, \quad k = 0, 1, 2, \dots \quad (2.5)$$

where $P_0(t)$ is the initial approximation and $\Phi_k(s)$ is obtained from the expansion of the integrand of (2.2) in Taylor series as

$$\dot{P}_k(s) + P_k(s)A + A^T P_k(s) - P_k(s)BR^{-1}B^T P_k(s) + Q = \Phi_k(s) + O[(s - t_f)^{k+1}] \quad (2.6)$$

Unfortunately, this solution gives a good approximation to the exact solution only on a small region of t . An easy and reliable way to improve the approximations for large t is to determine the solution on a sequence of equidistant subintervals of t , i.e. $I_{N-i} = [t_{N-(i+1)}, t_{N-i}]$, $i = 0, 1, \dots, N-1$ where $t_N = t_f$. Therefore, on $[t_{N-(i+1)}, t_{N-i}]$, we can construct the following piecewise approximations of truncated PIM (2.5), which is called the piecewise truncated parametric iteration method (PTP) [13]

$$P_{k+1, N-(i+1)}(t) = P_{k, N-(i+1)}(t) + h \int_t^{t_{N-i}} \Phi_{k, N-(i+1)}(s) ds = P_{kmax, N-(i+1)}(t), \quad (2.7)$$

$$P_{0, N-(i+1)}(t) = P_{kmax, N-i}(t_{N-i}) = S_{N-i}, \quad k = 0, 1, \dots, kmax - 1, \quad i = 0, \dots, N - 1.$$

where $\Phi_{k, N-(i+1)}(s)$ is as mentioned in (2.6) and $P_{kmax, N}(t_N) = S_N = S$. Now, the analytic approximate solution of the RDP (1.2) on the entire interval $[t_0, t_f]$ can easily be obtained and all solutions on $[t_{N-(i+1)}, t_{N-i}]$ are continuous at the end points of the each subinterval.



Remark 2.1 In the case of failure of convergence of the PTP algorithm, the presence of the parameter h could play an important role in the frame of the method. Although we can find a valid region of h for every physical problem by plotting the solution or its derivatives versus the parameter h in somepoints [11,13], but an optimal value of h can be determined at the order of approximation by the residual error

$$R_k^{N-i}(h_{N-i}) = \int_{t_{N-(i+1)}}^{t_{N-i}} \{ \mathcal{L}[P_k(t; h_{N-i})] + \mathcal{N}[P_k(t; h_{N-i})] - g(t) \}^2 dt, \quad i = 0, 1, 2, \dots, N - 1. \quad (2.8)$$

One can easily minimize (2.8) by imposing the requirement $\frac{\partial R_k^{N-i}(h_{N-i})}{\partial h_{N-i}} = 0$ which gives an approximate optimal h in the interval $[t_{N-(i+1)}, t_{N-i}]$ for $i = 0, 1, 2, \dots, N - 1$.

3. NUMERICAL EXPERIMENTS

3.1. An illustrative example

In this section, an example will be solved in details by the presented method. Consider the following optimal control problem from [17]

$$Min J = \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(x^T(t) \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} x(t) + u^2(t) \right) dt, \quad (3.1)$$

$$s. to: \dot{x}(t) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t).$$

According to (1.1), we have $t_0 = 0, t_f = \frac{\pi}{2}$ and

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad S = 0, \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \quad R = 1.$$

Then the matrix Riccati differential equation is

$$\dot{P}(t) = -P(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} P(t) + P(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P(t) - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}, \quad P\left(\frac{\pi}{2}\right) = 0. \quad (3.2)$$

or

$$\begin{cases} \dot{P}_{11}(t) = -P_{12} - P_{21} + P_{11}^2, & P_{11}\left(\frac{\pi}{2}\right) = 0 \\ \dot{P}_{12}(t) = -P_{22} + P_{11}P_{12}, & P_{12}\left(\frac{\pi}{2}\right) = 0 \\ \dot{P}_{21}(t) = -P_{22} + P_{11}P_{21}, & P_{21}\left(\frac{\pi}{2}\right) = 0 \\ \dot{P}_{22}(t) = P_{12}P_{21} - 4, & P_{22}\left(\frac{\pi}{2}\right) = 0 \end{cases} \quad (3.3)$$

To show the influence of h on the solutions obtained via the PTP algorithm, we have listed the maximum absolute error of the components $P(t)$ for various orders in Table 3.1. The results illustrate that the error decreases as the order of approximations increases for each considered value of h . This means that the corresponding sequences are convergent for all mentioned h , although some among them converge faster than others.

Table 3.1 The maximum absolute error of the $P(t)$ components using the PTP algorithm in solving (3.3)

h Order	-0.75	-0.80	-0.90	-1.00	-1.10	-1.20	-1.25
2	1E-1	9E-2	2E-2	4E-4	3E-2	1E-1	2E-1
3	4 E-2	2 E-2	3 E-3	4 E-6	3 E-3	2 E-2	4 E-2
4	9 E-3	4 E-3	3 E-4	5 E-8	3 E-4	4 E-3	1 E-2
5	2 E-3	8 E-4	3 E-5	8 E-10	3 E-5	9 E-4	3 E-3

The approximate optimal h obtained for the second order PTP solutions, according to (2.8), in all subintervals I_{N-i} when $N = 100$ and $i = 0, 1, \dots, N - 1$ can be observed in Figure 3.1.

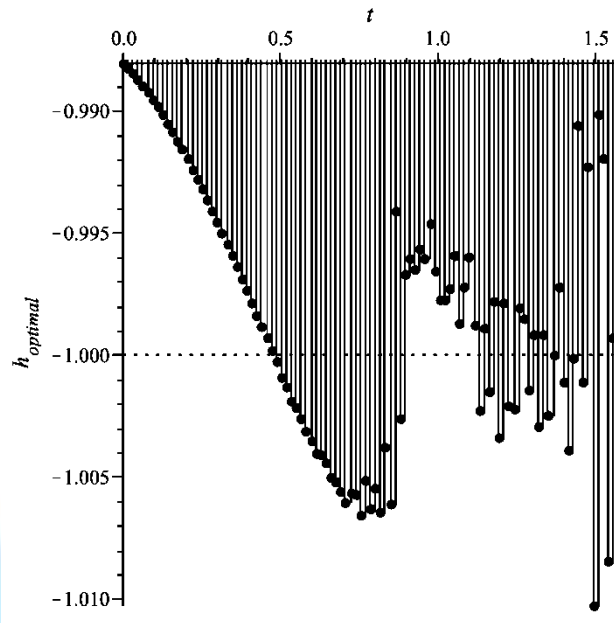


Figure 3.1 Approximate optimal h in $N = 100$ subintervals obtained by the residual error (2.8)

3.2. Comparison with variational iteration method and RK78

In order to show the accuracy of the PTP method, we compare the obtained solutions of the PTP algorithm and those of the **Maple RK78 solver**. The absolute error (i.e. $E_j(t) = |P_j^{RK78}(t) - P_j^{PTP}(t)|$, $j = 11, 12, 21, 22$) for the elements of the matrix $P(t)$ with $Kmax = 4$ and $N = 100$ is shown in Figure 3.2 and Figure 3.3. More completed report is presented in Table 3.2.

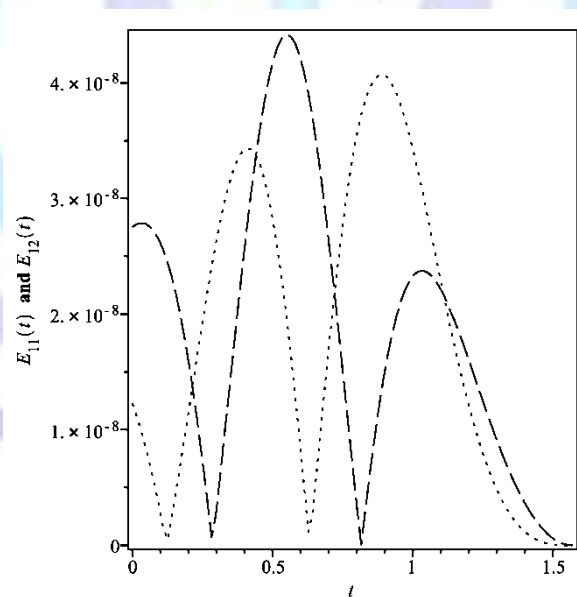


Figure 3.2 The absolute error of $P_{11}(t)$ (dot) and $P_{12}(t)$ (dash) of the PTP algorithm when $Kmax = 4$ and $N = 100$.

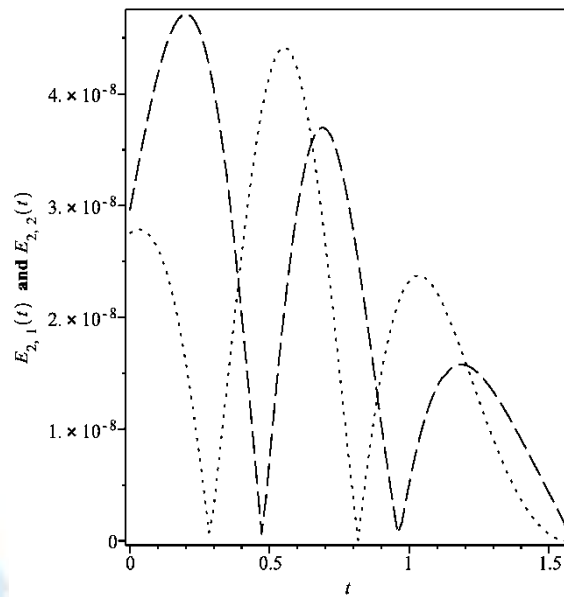


Figure 3.3 The absolute error of $P_{21}(t)$ (dot) and $P_{22}(t)$ (dash) of the PTP algorithm when $K_{max} = 4$ and $N = 100$.

Table 3.2 The maximum absolute error of the $P(t)$ components for various number of subintervals

iterations	10 intervals	50 intervals	100 intervals
2	4.0 E-2	1.6 E-3	3.8 E-4
3	4.3 E-3	3.6 E-5	4.5 E-6
4	5.3 E-4	7.7 E-7	4.8 E-8
5	6.4 E-5	2.1 E-8	8.3 E-10

In view of optimal control problems, using the computed matrix $P(t)$ as the solution of the matrix RDP(3.3), the main factor of the linear feedback $u^*(t) = -K(t)x^*(t)$ i.e. $K(t)$ is available via $K(t) = R^{-1}B^T P(t)$. A good criterion to check the validity and the accuracy of the obtained $K(t)$ is its exact value, which is fortunately, existent. From [17], we have

$$K(t) = \left(\frac{\sinh(\pi - 2t) - \sin(\pi - 2t)}{\cosh^2(\pi - 2t)/2 + \cos^2(\pi - 2t)/2}, \frac{\cosh(\pi - 2t) - \cos(\pi - 2t)}{\cosh^2(\pi - 2t)/2 + \cos^2(\pi - 2t)/2} \right) \tag{3.4}$$

The authors of [17] found $K(t)$ by solving the corresponding HDP (1.3) according to the theorem 1.1 by He's variational iteration method. The best reported error of $K(t)$ was about 10^{-4} on a very small region $[1.4, 1.7]$, while, such an accuracy is available just in 2 iterates on the entire interval $[0, \frac{\pi}{2}]$ as shown in Figure 3.4. This confirms the preference of the PTP algorithm over the He's variational iteration method.

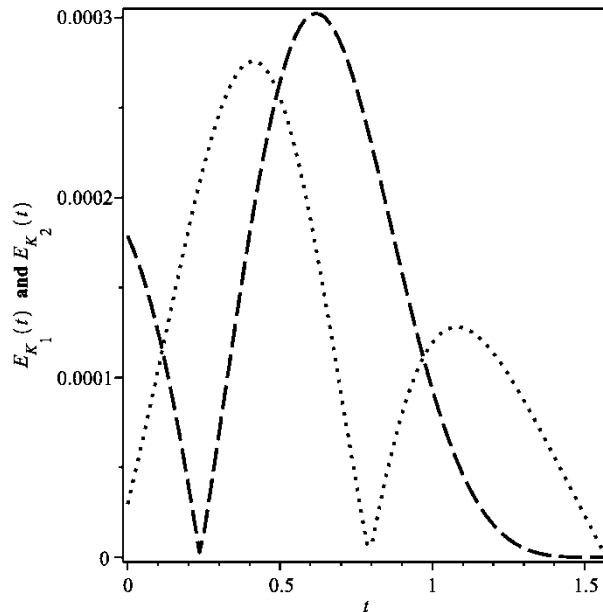


Figure 3.4 The absolute error of $K(t)$ of the PTP algorithm when $Kmax = 2$ and $N = 100$.

$E_{k1}(t)$: error in first coordinate (dot), $E_{k2}(t)$: error in second coordinate (dash).

These results for $Kmax = 2$, essentially emphasize that the convergence rate is rather high. But, in order to obtain more precise approximations, one can increase $Kmax$, For instance to earn the accuracy of the maple RK78 solver i.e. about 10^{-10} , it is sufficient to take $Kmax = 5$ as shown in Figure 3.5. Here we plot the absolute error of approximate $K(t)$ obtained from the PTP algorithm and the RK78 method in comparison with the exact $K(t)$ obtained from formula (3.4). As seen in the Figure 3.5 by $Kmax = 5$ the accuracy of the PTP algorithm is about that of the classic numerical RK78 method used in Maple, but It should be emphasized that the analytical representation of the solutions is the main preference of this method over the classic RK78 method.

Finally, we would like to mention that the analytical representation of the solutions is really worthwhile, especially in control theory, because, analysis of the nature of a dynamical system via the numerical solutions may lead to the incorrect or wrong consequents. Therefore, such analytic solutions are more desirable. Of course, in some problems we have to use the numerical methods, but we must remember that lack of a good theoretical method is its main reason. As the famous psychologist KourtLewin said:” There is nothing applicable as a good theory.”

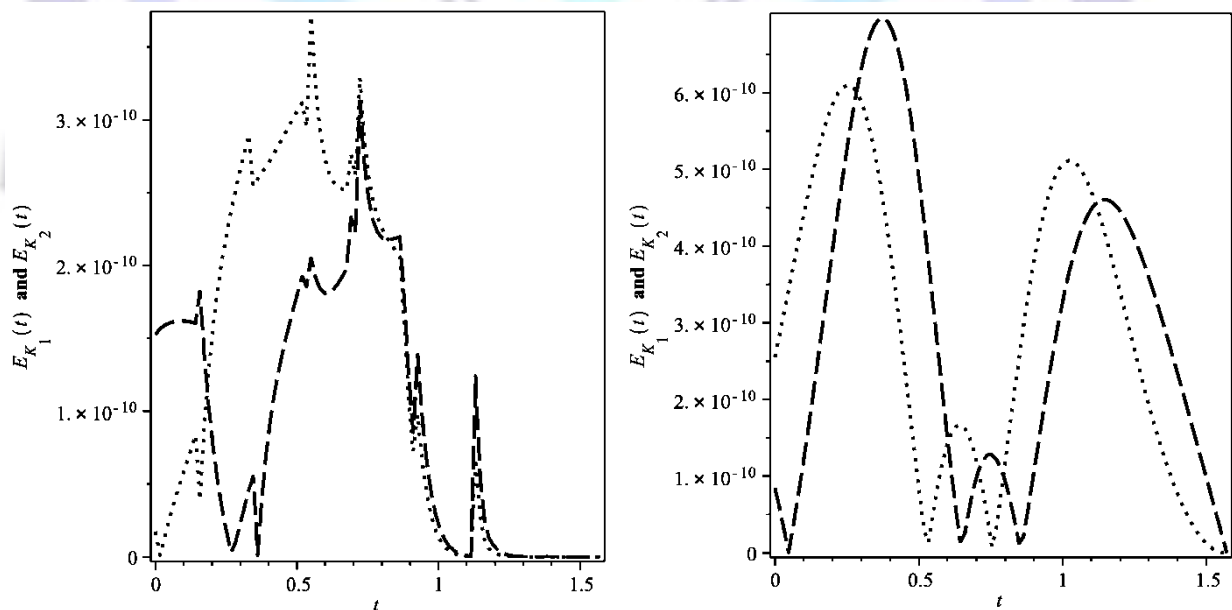


Figure 3.5 The absolute error of $K(t)$ from the RK78 method: (left); from the PTP algorithm when $Kmax = 5$ and $N = 100$: (right).

$E_{k1}(t)$: error in first coordinate (dot), $E_{k2}(t)$: error in second coordinate (dash).



4. CONCLUSION

In this article, in order to solve the matrix Riccati differential equation arising from the LQ optimal control problems, a modification of the parametric iteration method was utilized. The piecewise truncated PIM was proposed to reduce the repeated computations and to improve the accuracy of the PIM for a rather large time domain. Gaining High accuracy in less than five iterations is a worthwhile property of the method. Furthermore analytic representation of the solutions is a valuable consequent of the method which could be applied for solving LQ optimal control problems. In fact, using these solutions of the matrix RDE, we can obtain the analytic linear feedback of the optimal control. The accuracy of the method is about that of the RK78 and ability to represent analytical solutions is the preference of this method over the classic RK78 method. The presented method will be applicable for engineers of various branches such electronics, automatic control, signal processing, mechanics, etc.

REFERENCES

- [1] Abbasbandy, S. (2007), A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, *J. Comput. Appl. Math.* 207, 59–63.
- [2] Abbasbandy, S. (2006), Iterated He's homotopy perturbation method for quadratic Riccati differential equation, *Appl. Math. Comput.* 175, 581–589
- [3] Abbasbandy, S. (2006), Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method, *Appl. Math. Comput.* 172, 485–490.
- [4] Batiha, B., Noorani, M. S. M. & Hashim, I. (2007), Application of Variational Iteration Method to a General Riccati Equation, *International Mathematical Forum* 2, 2759–2770.
- [5] Callier, F. M., Winkin, J. J. & Willems, J. L. (1994), Convergence of the time-invariant Riccati differential equation and LQ problem: Mechanisms of attraction, *International Journal of Control* 59 (4), 983–1000.
- [6] Chen, G. & Chui, C. K. (1989), *Linear Systems and Optimal Control*, Springer-Verlag, Berlin, Heidelberg.
- [7] Coppel, W. A. (1974), Linear-quadratic optimal control, *Proceedings of the Royal Society of Edinburgh* 73A, 271–289.
- [8] Davison, E. J. & Maki, M. C. (1973), The numerical solution of the matrix Riccati differential equation, *IEEE Transactions on Automatic Control* AC-18 (1), 71–73.
- [9] Ghorbani, A. (2008), Toward a new analytical method for solving nonlinear fractional differential equations, *Comput. Meth. Appl. Mech. Engrg.* 197, 4173–4179.
- [10] Kenney, C. S. & Leipnik, R. B. (1985), Numerical integration of the differential matrix Riccati equation, *IEEE Transactions on Automatic Control* AC-30 (10), 962–970.
- [11] Liao, Sh. J. (2003), *Beyond perturbation: Introduction to the Homotopy Analysis Method*, Chapman & Hall=CRC Press, Boca Raton.
- [12] Ntogramatzidis, L. & Ferrante, A. (2010), On the solution of the Riccati differential equation arising from the LQ optimal control problem, *Systems & Control Letters* 59, 114–121.
- [13] Saberi-Nadjafi, J. & Ghorbani, A. (2010), Piecewise-truncated parametric iteration method: a promising analytical method for solving Abel differential equations, *Z Naturforsch* 65a, 529–539.
- [14] Sontag, E. (1990), *Mathematical Control Theory*, Springer-Verlag, New York.
- [15] Tan Y. & Abbasbandy, S. (2008), Homotopy analysis method for quadratic Riccati differential equation, *Communications in Nonlinear Science and Numerical Simulation* 13, 539–546.
- [16] Yamaguti, M. & Ushiki, S. (1981), Chaos in numerical analysis of ordinary differential equations, *Physica D* 3, 618–626.
- [17] Yousefi, S. A., Dehghan, M. & Lotfi, A. (2009), Finding the optimal control of linear systems via He's variational iteration method, *international journal of computer mathematics* AC-20.