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# Solution of nonlinear equations using Mann iteration

Tawfik Younis Elmasry
Cairo University, ISSR, Department of Operations Research, Cairo, Egypt
e-mail: tawfik – younis@hotmail.com

**ABSTRACT**

In this paper, we recall some basic concepts, properties of the spaces and some types of iteration approaches. Also, we give algorithm - fixed point iteration scheme and examples. Finally, we obtain the solution of nonlinear equations of the form $Ax = f$ using Mann iteration.

**Keywords:** Fixed points; Mann iteration.

**Mathematics Subject Classification :** 47 H 10, 54 H 25.

## INTRODUCTION

Let $X$ be a nonempty set and $T : X \to X$ a self-mapping. We say that $x \in X$ is a fixed point of $T$ if $T(x) = x$ and denote the set of all fixed points of $T$ by $F_T = \{x \in X : T(x) = x\}$ or by $FixT$.

**Example 1.1**

1- If $X = R$ and $T(x) = x^2 + 5x + 4,$ then $F_T = \{-2\};$

2- If $X = R$ and $T(x) = x^2 - x,$ then $F_T = \{0,2\};$

3- If $X = R$ and $T(x) = x + 2,$ then $F_T = \varnothing;$

4- If $X = R$ and $T(x) = x,$ then $F_T = R.$

Now we introduce some basic definitions and results on the spaces considered throughout this paper such as, metric spaces, normed spaces, Hilbert spaces and Banach spaces. We begin with the following definitions:

**Definition 1.1 [11]** *A metric space is a pair* $(M, d),$ *where* $M$ *is a set and* $d$ *is a metric on* $M$ *if for all* $x, y, z \in M,$

$(M_1)$ *d is real-valued function on* $M \times M,$ *finite and nonnegative, i.e.,* $0 \le d(x, y) < \infty$ *for all* $x, y \in M;$

$(M_2)$ $d(x, y) = 0 \Leftrightarrow x = y;$

$(M_3)$ $d(x, y) = d(y, x)$ *(Symmetry);*

$(M_4)$ $d(x, y) < d(x, z) + d(z, y)$ *(Triangle inequality).*

**Definition 1.2 [2]** *A sequence* $\{x_n\}$ *in a metric space* $(M, d)$ *converges to* $x$ *if for all* $\varepsilon > 0,$ *there exists* $n_0(\varepsilon) \in N(N :=$ *the set of all positive integers) such that* $d(x_n, x) \le \varepsilon$ *for all* $n \ge n_0(\varepsilon).$

**Definition 1.3 [11]** *The sequence* $\{x_n\}$ *in a metric space* $(M, d)$ *is called a Cauchy sequence iff* $d(x_m, x_n) \to 0$ *as* $m, n \to \infty,$ *i.e., for every* $\varepsilon > 0,$ *there exists* $N(\varepsilon)$ *such that* $d(x_n, x_m) < \varepsilon$ *for all* $n, m > N(\varepsilon).$

**Definition 1.4 [11]** *A metric space* $(M, d)$ *is said to be complete if every Cauchy sequence in* $M$ *converge to a point in* $M.$

**Definition 1.5 [11]** *A normed space* $E$ *is a vector space with a norm defined on it. Here, a norm on a (real or complex) vector space* $E$ *is a real-valued function on* $E$ *whose valued at an element* $x \in E$ *is denoted by* $\| x\|$ *and has the following properties:*

$(N_1)$ $\| x \| \ge 0;$

$(N_2)$ $\| x \| = 0 \Leftrightarrow x = 0;$

$(N_3)$ $\| \alpha x \| = | \alpha | \| x \|;$

$(N_4)$ $\| x + y \| \le \| x \| + \| y \|$ *(triangle inequality); where* $y$ *is an arbitrary vector in* $E$ *and* $\alpha$ *is any scalar. A norm on* $E$ *defines a metric* $d$ *on* $E$ *where*

$$d(x, y) = \| x - y \|, \quad x, y \in E,$$

and is called the metric induced by the norm. The normed space always denoted by $(E, \|.\|).$

**Definition 1.6 [11]** *A Banach Space is a complete normed space.*

**Definition 1.7 [11]** *Let* $V$ *be a vector space. The inner product* $\langle x, y\rangle$ *of* $x, y \in V$ *is defined as a function from* $V \times V$ *into a field* $K$ *(where K is real or complex* $R$ *or* $C$*) satisfying the following axioms:* $(I_1)$ $\langle x, y\rangle = \overline{\langle y, x\rangle};$ $(I_2)$ $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle;$ $(I_3)$ $\langle x, x\rangle \ge 0$ *and* $\langle x, x\rangle = 0$ *iff* $x = 0;$ *for all* $x, y, z \in V$ *and*

$\alpha, \beta \in K$. *An inner product defines a norm on* $V$ *and a metric* $d$ *as the following given by*

$$\| x\| = \sqrt{\langle x, x\rangle}, \quad x \in V$$

**Definition 1.8 [11]** *A Hilbert space is a complete inner product space.*

**Definition 1.9 [11]** *(Strong convergence) A sequence* $(x_n)$ *in a normed space* $E$ *is said to be strongly convergent (or convergent in the norm) if there is an* $x \in E$ *such that*

$$\lim_{n\to\infty} \| x_n - x\| = 0.$$

This can be written as follows

$$\lim_{n\to\infty} x_n = x$$

or simply

$$x_n \to x.$$

$x$ is called the strong limit of $(x_n)$, and we say that $(x_n)$ converges strongly to $x$.

Weak convergence is defined in terms of bounded linear functionals on $E$ as follows.

**Definition 1.10 [11]** *(Weak convergence) A sequence* $(x_n)$ *in a normed space* $E$ *is said to be weakly convergent if there is an* $x \in E$ *such that for every* $f \in E^*$,

$$\lim_{n\to\infty} f(x_n) = x$$

This can be written as follows

$$x_n \rightharpoonup x.$$

The element $x$ is called the weak limit of $(x_n)$, and we say that $(x_n)$ converges weakly to $x$.

**Theorem 1.1 [25]** *(Brouwer's Theorem) Every continuous map of a closed bounded convex in* $R^n$ *into itself has a fixed point.*

**Definition 1.11** *Suppose* $(M, d)$ *is a complete metric space and* $T : M \to M$ *is any mapping. The mapping* $T$ *is said to satisfy a Lipschitz condition with a real number* $L$ *if*

$$d(Tx, Ty) \le Ld(x, y)$$

holds for all $x, y \in M$ such mapping $T$ is called a contraction if $L < 1$ and a nonexpansive if $L = 1$. We call $T$ contractive if for all $x, y \in M$ and $x \neq y$, we have

$$d(Tx, Ty) < d(x, y)$$

**Remark 1.1** *Observe that contraction* $\Rightarrow$ *contractive* $\Rightarrow$ *nonexpansive* $\Rightarrow$ *Lipschitz, and a mapping satisfying any of these conditions is continuous (see [4, 25]).*

We now give the important theorem which known as Banach fixed point theorem ( see [3]).

**Theorem 1.2** *(Banach fixed point theorem). Let* $T$ *be a contraction mapping of a complete metric space* $M$ *into itself. Then* $T$ *has a unique fixed point* $\bar{x}$. *Moreover, If* $x_0$ *is any point in* $M$ *and the sequence* $\{x_n\}$ *is defined iteratively by the formula*

$$x_n = Tx_{n-1}, \quad n = 1,2,..., \text{then} \quad \lim_{n} x_n = \bar{x}.$$

In 1930, J. Schauder gave the generalization of Brouwer's fixed point theorem which as follows:

Theorem 1.3 *([25])(Schauder's fixed point theorem) A continuous mapping* $T$ *that transforms a compact convex set* $K$ *in* *a Banach space $X$ into itself has a fixed point.*

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## 2 Some iterative methods

In this section, we recall some iterative methods and we give some references using these methods to obtain fixed point theorems.

Consider the equation $x = Tx.$ This equation is called fixed point equation, where $T$ is a continuous linear operator on a set $M$ and $x$ is an unknown element in $M.$ One of the prevalent methods for finding the solutions of the equation $x = Tx$ is so-called a method of successive approximations or Picard iteration. To describe this method, we introduce the definition:

**Definition 2.1 ([4])** *(Picard iteration):*

Let $(M, d))$ be a metric space, $D \subset M$ a closed subset of $M$ (we often have $D = M$ ) and $T : D \to D$ a self map possessing at least one fixed point $p \in F_T.$ For a given $x_0 \in M$ we consider the sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by the successive iteration method

$$x_n = Tx_{n-1}, \quad n = 1,2,...$$

This sequence is known as Picard iteration.

All the next fixed point iteration schemes are defined in a real normed space $(E, ||.||).$

**Definition 2.2 [4]** *(Krasnoselskii iteration )*

Let $T : E \to E$ be a normed space, $x_0 \in E$ and $\lambda \in [0,1].$ The sequence $\{x_n\}$ given by

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \quad n = 0,1,2,...$$

will be called the Krasnoselskii iteration.

The Mann iterative scheme was invented in 1953 (see [13]) and was used to obtain convergence to a fixed point for many classes of mappings see ([1, 4, 5, 6, 10, 16, 17, 19, 20] and others).

Form the next example the idea of considering fixed point iteration procedures with errors comes from practical numerical computations. This topic of research plays important role in the stability problem of fixed point iterations. In 1995, Liu [12] initiated a study of fixed point iterations with errors. Several authors have proved some fixed point theorems for Mann type iteration with errors using several classes of mappings (see [6, 7, 8, 9, 14, 15, 24, 26] and others).

Definition 2.3 *[4, 13] (Mann iteration)*

The Mann iteration starting from $x_0 \in E$ , is the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-a_n)x_n + a_n Tx_n, \quad n = 0,1,2,...$$

where $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ satisfies certain appropriate conditions.

## 3 Algorithm - Fixed Point Iteration Scheme

Let the initial guess be $x_0$ such that

$$x_{i+1} = T(x_i)$$

### Numerical Example

**Example 3.1** *Find a root of* $x^4 - x - 10 = 0$

**Proof:** Consider $T(x) = \dfrac{10}{(x^3 - 1)}$ and the fixed point iterative scheme

$$x_{i+1} = \frac{10}{(x_i^3 - 1)}, \qquad i = 0,1,2,...,$$

let the initial guess $x_0$ be 2.0

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 2 | 1.429 | 5.214 | 0.071 | -10.004 | -0.009 | -10 | -0.00999 | -10 |

**Table 1: Example 3.1**

So the iterative process with $T$ gone into an infinite loop without converging.

**Example 3.2** *Suppose* $T(x) = \frac{(x+10)^{\frac{1}{2}}}{x}$ *and the fixed point iterative scheme*

$$x_{i+1} = \frac{(x_i + 10)^{\frac{1}{2}}}{x_i}, \qquad i = 0,1,2,...$$

let the initial guess $x_0$ be 1.8,

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 98 |
|---|---|---|---|---|---|---|---|---|
| $x_i$ | 1.8 | 1.908 | 1.808 | 1.900 | 1.8152 | 1.8935 | 1.8212 | 1.8555 |

**Table 2: Example 3.2**

**Example 3.3** *Find the root of* $\cos x - xe^x = 0$

**Proof:** Consider $T(x) = \frac{\cos x}{e^x}$ The graph of $T(x)$ and $x$ are given in the figure. let the initial guess $x_0$ be $2$

| i | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 31 | 32 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_i$ | 1 | 0.199 | 0.803 | 0.311 | 0.698 | 0.381 | 0.634 | 0.427 | 0.594 | 0.458 | 0.567 | 0.518 | 0.518 |

**Table 3: Example 3.3**

That is for $T(x) = \frac{\cos x}{e^x}$ the iterative process is converged to 0.518.

## 4 Iterative solution of the equation Ax=f

Let $X$ be a a real Banach space and $X^*$ its dual. For $1 < p < \infty$, the duality mapping. $J_p : X \to 2^{X^*}$, is defined by

$$J_p(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^p, \|f^*\|^p = \|x\|^{(p-1)}\}, \quad x \in X.$$

Where $\langle .,. \rangle$ denotes the generalized duality pairing between $X$ and $X^*$. Recall that a map $A: X \to X$ is said to be accretive if $\forall\ x,y\in D(A)\ \exists\ j_p(x-y)\in J_p(x-y)$ such that

$$\langle Ax-Ay\ ,\ j_p(x-y)\rangle\geq 0 \qquad (1)$$

and is said to be strongly accretive if $A-kI$ is accretive where $k\in(0,1)$ is a constant and $I$ denotes the identity operator on $X$. Let $S(T)=\{x^*\in D(A): Ax^*=f\}\neq\emptyset$ denote the solution set of the equation $Ax=f$.

**Theorem 4.1** *Let $X$ be a real $p-$uniformly smooth Banach space and let $A:D(A)\subset X\to X$ be locally Lipschitz and strongly quasi-accretive operator with open domain $D(A)$ in $X$ such that the equation $Ax=f$ has a solution $x^*\in D(A)$ for $f\in R(A)$ arbitrary but fixed. Define $A_\lambda: D(A)\to X$ by*

$$A_\lambda x = x-\lambda(Ax-f)\quad \forall\ x\in D(A)$$

Then there exist a neighbourhood $B$ of $x^*$ and a real number $\lambda\in(0,1)$ such that starting with an arbitrary $x_0\in B$ the Mann sequence $\{x_n\}$ generated by

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n Ax_n \qquad (2)$$

remains in $B$ and converges strongly to $x^*$ with convergence being at least as fast as geometric progression.

Proof: Since $A$ is locally Lipschitz, there is an $r>0$ such that $A$ is Lipschitz on $B=\overline{B}_r(x_0)=\{x\in X: \|x-x^*\|\leq r\}\subset D(A)$ let $k\in(0,1)$ and $L>1$ denote the strong accretivity and Lipschitz constant of $A$ respectively. Observe that $f=Ax^*$. Pick an arbitrary $x_0\in B,$ choose

$$\alpha_n\lambda=\left(\frac{k}{L^pC_p}\right)^{\frac{1}{p-1}}$$

and generate the sequence $\{x_n\}_{n\geq 0}$ as in (1). We now prove that $x_n\in B,\ \forall\ n\geq 0.$ Suppose that $x_n\in B.$ Then

$$\|x_{n+1}-x^*\|^p=\|(1-\alpha_n)x_n+\alpha_n)Ax_n-x^*\|^p$$
$$=\|(1-\alpha_n)x_n+\alpha_n[x_n-\lambda(Ax_n-f)]-x^*\|^p$$
$$=\|(1-\alpha_n)x_n+\alpha_n[x_n-\lambda(Ax_n-Ax^*)]-x^*\|^p$$
$$=\|x_n-\alpha_n\lambda(Ax_n-Ax^*)-x^*\|^p$$
$$=\|x_n-x^*\|^p+p\alpha_n\lambda\langle Ax_n-Ax^*,J_p(x_n-x^*)\rangle+\alpha_n^p\lambda^pC_p\|Ax_n-Ax^*\|$$
$$=(1-pk\alpha_n\lambda+L^p\alpha_n^p\lambda^pC_p)\|x_n-x^*\|^p$$
$$\leq(1-(pk-L^pC_p(\alpha_n\lambda)^{p-1})\alpha_n\lambda)\|x_n-x^*\|^p$$
$$=(1-(p-1)k\left(\frac{k}{L^pC_p}\right)^{\frac{1}{p-1}})\|x_n-x^*\|^p\leq r^p.$$

Hence, since $x_0\in B$ by choice of the initial guess, it follows by the inductive hypothesis that the sequence $\{x_n\}$ remains in $B.$ set

$$w=\left[1-(p-1)k\left(\frac{k}{L^pC_p}\right)^{\frac{1}{p-1}}\right]^{\frac{1}{p}}$$

and observe that $w \in (0,1)$ since

$$k < \frac{LC_p^{\frac{1}{p}}}{(p-1)^{\frac{p-1}{p}}}, \quad \forall\ 1 < p < \infty$$

Hence, iterating further from (9), we obtain

$$\|x_n - x^*\|^p \leq (w^p)^n \|x_n - x^*\|^p$$

or, equivalntly,

$$\|x_n - x^*\| \leq w^n \|x_n - x^*\|$$

since $w^n \to 0$ as $n \to \infty$ the assertions of the theorem follows and the proof is complete.

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