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Two Characterizations of Gamma Distribution in Terms of $s\ th$ Conditional Moments

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Abstract

The gamma distribution is highly important in applications and data modeling. It is usually used to model waiting times, the size of insurance claims, and rainfalls. In this paper we state and prove two new characterizations of the two parameter gamma distribution by establishing a connection between s-right truncated moments (s-left truncated moments) and the reversed hazard rate (hazard rate). These characterization results are easier to check in data analysis. Besides, our results generalize some of the well-known theoretical results of Koicheva (1993) and Ahsanullah et al., 2012.

Keywords: characterization; Failure rate; Reversed failure rate; Right censored mean function; Left censored mean function; Gamma distribution.

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1. Introduction and Motivation

In Reliability analysis the hazard rate, also known as failure rate, of a life distribution, by "life distributions" we mean those for which negative values do not occur, i.e., $F(t) = 0$ for $x < 0$, plays an important role for stochastic modeling and classification. Being a ratio of probability density function and the corresponding survival function, it uniquely determines the underlying distributions and exhibits different monotonic behaviors.

Let X bea random variable usually representing the lifetime for a certain unit or component in a system, i.e., time to first failure, with reliability (survival) function and probability density function as follows

$$\overline{F}(t) = P(X \geq t), f(t) \geq 0, \forall\, t \in R.$$

In life testing situations, the additional lifetime given that the component has survived up to time t is called the residual life function(RLF) of the component. More specifically if X is the lifetime of component, then the random variable $X_t = (X - t|X \geq t)$ is called the residual life random variable. In the insurance business, this random variable represents the amount of claim if the deductible for a particular policy is t and X is the random variable representing the loss. The quantity

$$\mu(t) = E(X_t) = E(X - t|X \geq t),$$

is called the mean residual life function($MRLF$) or the life expectancy at age t and has been employed in life length studies by various authors e.g. Hollander and Proschan (1975), Hall and Wellner (1981) have characterized the class of mean residual life functions. Limiting properties (behavior) of the $MRLF$ have been studied by Bradley and Gupta (2003). Several functions are defined related to the residual life and it plays a crucial role in reliability and survival analysis. The failure (hazard) rate function, defined by

$$h(t) = \frac{f(t)}{\overline{F}(t)} = -\frac{d}{dt}\ln \overline{F}(t), \text{for all } x \text{ such that } \overline{F}(t) > 0.$$

And it can be easily verified that

$$h(t) = \frac{1 + d/dt\big(\mu(t)\big)}{\mu(t)}.$$

The reversed hazard rate(RHR), being the ratio of probability density function and the corresponding distribution function, $r(t)$ is defined by the following equation

$$r(t) = \frac{f(t)}{F(t)}.$$

Reversed hazard rate is useful among other ways in the estimation of the survival function for left censored life times. Making simple transformations we arrive at an important relation between $r(t)$ and $h(t)$ by the following

$$r(t) = \frac{h(t)\overline{F}(t)}{1-\overline{F}(t)}.$$

Applications of hazard functions are quite well known in the statistical literature. Recently the reversed hazard function also become quite popular among the statisticians, see for example Gupta and Han (2001). Anderson et al. (1993) show that the reversed hazard function plays the same role in the analysis of left-censored data as the hazard function plays in the analysis of right-censored data.

The concept of failure rate, mean residual life, reversed failure rate and mean inactivity time, also known as reversed mean residual life, are extensively used in modeling and analysis in reliability studies and applicable in other disciplines such as economics, biostatistics, actuarial sciences, engineering, biology, biometry and applied probability areas. They also are useful in survival analysis studies when we take are faced with left or right censored data. The relationship between the failure rate (reversed failure rate) and the mean residual life (reversed mean residual life) or left (right) truncated expectations of functions of X were found to be quite useful in studying the comparative behavior of these functions and in characterizing the probability distributions, especially when any of these functions does not have a simple closed form for analytic treatment.

Several authors have considered different techniques, such as truncated conditional expectation from the left, truncated conditional expectation from the right, conditional variance, record values and truncated moments of order statistics, to characterize probability distributions of interest, See, e.g., Arnold (1980), Galambos and Kotz (1978), Kagan , Linnik and Rao (1973).Yehia and Ahmed (1993) introduced two characterizations of Geometric distributions. EL Arishi (2005) presented the A conditional variance characterization of some discrete probability distributions. Ahmed (1991) presented the characterization of beta, binomial and Poisson distributions. Shanbhag (1970) has characterized the exponential distribution using the linearity of the conditional expectation. Talwalker (1977) has characterized various distributions like Pareto, Power and Burr distributions in terms of conditional expectation of a function of the absolutely continuous random variable. Also, characterizations based on the properties of the failure rate function have been considered by many authors. Xekalaki (1983) has identified the Pareto distribution through decreasing failure rate. Royand Mukherjee (1986) have characterized the Weibull distribution via increasing failure rate. Osaki and Li (1988) characterized the gamma distribution using the relationship between the failure rate and the vitality function.Koicheva(1993) has characterized the gamma distribution via $k\,th$ conditional moment. El Batal et al., (2012) have characterized the gamma distribution by properties relating to the reversed hazard rate and the mean inactivity time.

2. The First Characterization of the Gamma distribution

The following result characterizes the gamma distribution by properties relating tos-right censored mean function and reversed failure rate.

Theorem 2.1

Let X be a non-negative random variable(rv) with probability density function(pdf)

$$f(x) = \frac{\alpha(\alpha x)^{\beta-1}e^{-\alpha x}}{\Gamma(\beta)}, x > 0, \alpha, \beta > 0, (2.1)$$

and let $F(x), r(x)$be the distribution function(df)and reversed failure (hazard) rate(RFR) respectively. Then the random variable X has the pdf in (2.1) if and only if

$$E(X^s|X \le y) = \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} - r(y)\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}},\ s = 1,2,3,\ldots \tag{2.2}$$

The following two lemmas are used to prove the sufficiency of theorem 2.1. The two lemmas are proved in the appendix.

Lemma (1)

$$\frac{d}{dy}\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}} = \sum_{j=0}^{s-1}\frac{(s-j)\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}.$$

Lemma (2)

$$\sum_{j=0}^{s-1}\frac{(s-j)\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}} = \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} - y^s - (\beta-1-\alpha y)\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}.$$

Proof

1. **Necessity**

Observe that

$$E(X^s|X \le y) = \frac{1}{F(y)}\int_0^y x^s f(x)dx.$$

Letting

$$A_s = \int_0^y x^s f(x)dx, \qquad s = 0,1,2,\ldots, A_0 = F(y).$$

Then

$$E(X^s|X \le y) = \frac{A_s}{F(y)},\ s = 0,1,2,\ldots \tag{2.3}$$

Hence

$$A_s = \int_0^y x^s f(x)dx = \int_0^y x^s \frac{\alpha(\alpha x)^{\beta-1}e^{-\alpha x}}{\Gamma(\beta)}dx \tag{2.4}$$

Integrating Equation (2.4) by parts

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$$A_s = \left[\frac{-(\alpha x)^{\beta+s-1}e^{-\alpha x}}{\alpha^s\Gamma(\beta)}\right]_0^y + \frac{(\beta+s-1)}{\alpha^{s-1}\Gamma(\beta)}\int_0^y (\alpha x)^{\beta+s-2}e^{-\alpha x}dx$$

$$= \frac{-y^s}{\alpha}f(y) + \frac{(\beta+s-1)}{\alpha}\int_0^y x^{s-1}f(x)dx$$

$$= \frac{-y^s}{\alpha}f(y) + \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-1)\alpha}\int_0^y x^{s-1}f(x)dx. \quad (2.5)$$

Rewriting A_s as a recurrence relation between the terms of the sequence $\{A_s\}$, gives

$$A_s = \frac{-y^s}{\alpha}f(y) + \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-1)\alpha}A_{s-1},\ s = 1,2,\ldots \quad (2.6)$$

Hence

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+s-1)\alpha}A_{s-1} = \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-1)\alpha}\int_0^y x^{s-1}f(x)dx$$

$$= \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-1)\alpha}\int_0^y \frac{\alpha(\alpha x)^{\beta+s-2}e^{-\alpha x}}{\alpha^{s-1}\Gamma(\beta)}dx, \quad s = 1,2,\ldots \quad (2.7)$$

Integrating Equation (2.7) by parts, we obtain

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+s-1)\alpha}A_{s-1} = \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-1)}\left\{\frac{-(\alpha y)^{\beta+s-2}e^{-\alpha y}}{\alpha^s\Gamma(\beta)} + \frac{\alpha(\beta+s-2)}{\alpha^s}\int_0^y \frac{(\alpha x)^{\beta+s-3}e^{-\alpha x}}{\Gamma(\beta)}dx\right\}$$

$$= \frac{-\Gamma(\beta+s)y^{s-1}}{\Gamma(\beta+s-1)\alpha^2}f(y) + \frac{(\beta+s-2)\Gamma(\beta+s)}{(\beta+s-2)\Gamma(\beta+s-2)\alpha^2}\int_0^y x^{s-2}f(x)dx$$

$$= \frac{-\Gamma(\beta+s)y^{s-1}}{\Gamma(\beta+s-1)\alpha^2}f(y) + \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-2)\alpha^2}A_{s-2}. \quad (2.8)$$

Substituting from Equation (2.8) into Equation (2.6), we obtain

$$A_s = \frac{-y^s}{\alpha}f(y) - \frac{\Gamma(\beta+s)y^{s-1}}{\Gamma(\beta+s-1)\alpha^2}f(y) + \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-2)\alpha^2}A_{s-2}. \quad (2.9)$$

Hence

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+s-2)\alpha^2}A_{s-2} = \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-2)\alpha^2}\int_0^y x^{s-2}f(x)dx$$

$$= \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-2)\alpha^2}\int_0^y \frac{\alpha(\alpha x)^{\beta+s-3}e^{-\alpha x}}{\alpha^{s-2}\Gamma(\beta)}dx. \quad (2.10)$$

Integrating Equation (2.10) by parts, we find

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+s-2)\alpha^2}A_{s-2} = \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-2)}\left\{\frac{-(\alpha y)^{\beta+s-3}e^{-\alpha y}}{\alpha^s\Gamma(\beta)} + \frac{\alpha(\beta+s-3)}{\alpha^s}\int_0^y \frac{(\alpha x)^{\beta+s-4}e^{-\alpha x}}{\Gamma(\beta)}dx\right\}$$

$$= \frac{-\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-2)\alpha^3}f(y) + \frac{(\beta+s-3)\Gamma(\beta+s)}{(\beta+s-3)\Gamma(\beta+s-3)\alpha^3}\int_0^y x^{s-3}f(x)dx$$

$$= \frac{-\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-2)\alpha^3}f(y) + \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-3)\alpha^3}A_{s-3}. \quad (2.11)$$

Substituting from Equation (2.11) into Equation (2.9), we obtain

$$A_s = \frac{-y^s}{\alpha} f(y) - \frac{\Gamma(\beta+s)y^{s-1}}{\Gamma(\beta+s-1)\alpha^2} f(y) - \frac{\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-2)\alpha^3} f(y) + \frac{\Gamma(\beta+s)}{\Gamma(\beta+s-3)\alpha^3} A_{s-3}. \tag{2.12}$$

Therefore, noting that

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s-1}} A_1 = \frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s-1}} \int_0^y x f(x)dx$$

$$= \frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s-1}} \int_0^y \frac{(\alpha x)^\beta e^{-\alpha x}}{\Gamma(\beta)} dx. \tag{2.13}$$

Integrating Equation (2.13) by parts, we find

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s-1}} A_1 = \frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s-1}} \left\{ \frac{-(\alpha y)^\beta e^{-\alpha y}}{\alpha\Gamma(\beta)} + \frac{\beta}{\alpha} \int_0^y \frac{\alpha(\alpha x)^{\beta-1} e^{-\alpha x}}{\Gamma(\beta)} dx \right\}$$

$$= \frac{-\Gamma(\beta+s)y}{\Gamma(\beta+1)\alpha^s} f(y) + \frac{\beta\Gamma(\beta+s)}{\beta\Gamma(\beta)\alpha^s} \int_0^y f(x)dx$$

$$= \frac{-\Gamma(\beta+s)y}{\Gamma(\beta+1)\alpha^3} f(y) + \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} A_0. \tag{2.14}$$

Where, by definition

$$A_0 = \int_0^y f(x)dx = F(y). \tag{2.15}$$

Therefore by using Equations (2.12), (2.14) and (2.15), the following representation is obtained:

$$A_s = \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} F(y) - f(y) \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}}. \tag{2.16}$$

Substituting from Equation (2.16) into Equation (2.3), we obtain

$$E(X^s | X \le y) = \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} - r(y) \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}}. \tag{2.17}$$

2. Sufficiency

Notice that Equation (2.17) can be rewritten as follows

$$\int_0^y x^s f(x)dx = \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} F(y) - f(y) \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}}. \tag{2.18}$$

Differentiating both sides of Equation (2.18), with respect to y, we obtain

$$y^s f(y) = \left\{ \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} f(y) - f(y) \frac{d}{dy} \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}} - f'(y) \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}} \right\}$$

Using Lemma (1), we get

$$y \left\{ \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}} \right\} f'(y) = \left\{ \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} - y^s - \sum_{j=0}^{s-1} \frac{(s-j)\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}} \right\} f(y)$$

Using Lemma (2), we get

$$y \left\{ \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}} \right\} f'(y) = (\beta - 1 - \alpha y) \left\{ \sum_{j=0}^{s-1} \frac{\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}} \right\} f(y)$$

$$\frac{f'(y)}{f(y)} = \left(\frac{\beta-1}{y} - \alpha \right). \tag{2.19}$$

Solving the differential Equation (2.19), gives

$$\int \frac{f'(y)}{f(y)}dy = \int (\frac{\beta-1}{y} - \alpha)dy$$

$$\ln f(y) - \ln k = \ln y^{\beta-1} + \ln e^{-\alpha y}$$

$$\ln\left(\frac{f(y)}{c}\right) = \ln\,(y^{\beta-1}\,e^{-\alpha y})$$

$$\frac{f(y)}{k} = \; y^{\beta-1}e^{-\alpha y}.$$

$$f(y) = k\,y^{\beta-1}e^{-\alpha y}.$$

Using the fact that $\int_0^\infty f(y)dy = 1$, then $k = \frac{\alpha^\beta}{\Gamma(\beta)}$.

Hence

$$f(y) = \frac{\alpha(\alpha y)^{\beta-1}e^{-\alpha y}}{\Gamma(\beta)}.$$

This completes the proof.

Remark 1 *Specifying $s = 1$ and $s = 2$ in Equation (2.2) yields the following results of El Batal et al., (2012).*

(i) $E(X|X \le y) = \frac{\beta}{\alpha} - \frac{y}{\alpha}r(y)$.

(ii) $E(X^2|X \le y) = \frac{\beta(\beta+1)}{\alpha^2} - r(y)\left(\frac{y^2}{\alpha} + \frac{(\beta+1)y}{\alpha^2}\right)$.

Remark 2 *Specifying $\beta = 1$ in Equation (2.2), we get*

$$E(X^s|X \le y) = \frac{s!}{\alpha^s} - r(y)\sum_{j=0}^{s-1}\frac{s!\,y^{s-j}}{(s-j)!\,\alpha^{j+1}}, s = 1,2,3, \ldots$$

which characterizes the exponential distribution with parameter α.

Remark 3 *Using Remark 1, we get*

$$\mathrm{V}(X|X \le y) = \frac{\beta}{\alpha^2} - \frac{y}{\alpha^2}r(y)[\alpha y - (\beta-1)] - \frac{y^2}{\alpha^2}r^2(y).$$

This is the result of Nofal (2003).

Remark 4 *Specifying $\beta = 1$ in Remark 3, we get*

$$V(X|X \le y) = \frac{1}{\alpha^2} - \frac{y^2}{\alpha}r(y) - \frac{y^2}{\alpha^2}r^2(y).$$

This is the result of Nofal (2003), which characterizes the exponential distribution.

3. The Second Characterization of the Gamma distribution

The following result characterizes the gamma distribution by establishing a connection between s-left truncated moments and hazard rate.

Theorem 3.1

Let X be a non-negative random variable (rv) with probability density function(pdf)

$$f(x) = \frac{\alpha(\alpha x)^{\beta-1}e^{-\alpha x}}{\Gamma(\beta)}, x > 0, \alpha, \beta > 0, \quad (3.1)$$

and let $\overline{F}(x), h(x)$be the survival (reliability) function(sf) and failure (hazard) rate(FR)respectively. Then the random variable X has the pdf in (3.1) if and only if

$$E(X^s|X > y) = \frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^s} + h(y)\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}},\ s = 1,2,3,\ldots,(3.2)$$

Proof It can be obtained similarly as Theorem 2.1.

Remark 5 *Specifying $s = 1$ in Equation (3.2) gives the result of Osaki and Li (1988).*

Remark 6 *Specifying $\alpha = 1$ in Equation (3.2) yields Koicheva's (1993) result.*

Remark 7 *Specifying $s = 1$and $s = 2$ in (3.2) yields the following results*

(i) $E(X|X > y) = \frac{\beta}{\alpha} + \frac{y}{\alpha}h(y)$.

(ii) $E(X^2|X > y) = \frac{\beta(\beta+1)}{\alpha^2} + h(y)\left(\frac{y^2}{\alpha} + \frac{(\beta+1)y}{\alpha^2}\right)$.

Remark 8 *Using Remark 7, we obtain*

$$V(X|X > y) = \frac{\beta}{\alpha^2} + \frac{y}{\alpha^2}h(y)[\alpha y - (\beta - 1)] - \frac{y^2}{\alpha^2}h^2(y).$$

Remark 9 *Specifying $\beta = 1$ in Remark 8, we get*

$$V(X|X > y) = \frac{1}{\alpha^2} + \frac{y^2}{\alpha}h(y) - \frac{y^2}{\alpha^2}h^2(y).$$

This characterizes the exponential distribution.

Remark 10 *Specifying $\beta = 1$ in (3.2), we get*

$$E(X^s|X > y) = h(y)\sum_{j=0}^{s-1}\frac{\Gamma(s+1)y^{s-j}}{\Gamma(s-j+1)\alpha^{j+1}} + \frac{\Gamma(s+1)}{\alpha^s}, s = 1,2,3,\ldots,$$

This characterizes the exponential distribution.

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APPENDIX

Proof of Lemma 1

$$\frac{d}{dy}\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j}}{\Gamma(\beta+s-j)\alpha^{j+1}}=\left\{\frac{sy^{s-1}}{\alpha}+\frac{(s-1)\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-1)\alpha^{2}}+\cdots+\frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s}}\right\}$$

$$=\sum_{j=0}^{s-1}\frac{(s-j)\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}.$$

Proof of Lemma 2

Let $B=(\beta-1-\alpha y)\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}+\sum_{j=0}^{s-1}\frac{(s-j)\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}$

$$\begin{aligned}=&\left\{(\beta-1-\alpha y)\frac{y^{s-1}}{\alpha}+(\beta-1-\alpha y-1)\frac{\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-1)\alpha^{2}}+(\beta-1-\alpha y-2)\frac{\Gamma(\beta+s)y^{s-3}}{\Gamma(\beta+s-2)\alpha^{3}}+\cdots\right.\\&\left.+(\beta-1-\alpha y-(s-1))\frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s}}\right\}+s\left\{\frac{y^{s-1}}{\alpha}+\frac{\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-1)\alpha^{2}}+\frac{\Gamma(\beta+s)y^{s-3}}{\Gamma(\beta+s-2)\alpha^{3}}+\cdots+\frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s}}\right\}\\=&\left\{(\beta+s-1-\alpha y)\frac{y^{s-1}}{\alpha}+(\beta+s-2-\alpha y)\frac{\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-1)\alpha^{2}}+(\beta+s-3-\alpha y)\frac{\Gamma(\beta+s)y^{s-3}}{\Gamma(\beta+s-2)\alpha^{3}}+\cdots\right.\\&\left.+(\beta-\alpha y)\frac{\Gamma(\beta+s)}{\Gamma(\beta+1)\alpha^{s}}\right\}.\end{aligned}$$

Using the identity $(\beta+s-j-1)\frac{\Gamma(\beta+s)}{\Gamma(\beta+s-j)}=\frac{\Gamma(\beta+s)}{\Gamma(\beta+s-j-1)}$, it can easily be verified that

$$\begin{aligned}B=&\left\{\frac{\Gamma(\beta+s)y^{s-1}}{\Gamma(\beta+s-1)\alpha}+\frac{\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-2)\alpha^{2}}+\frac{\Gamma(\beta+s)y^{s-3}}{\Gamma(\beta+s-3)\alpha^{3}}+\cdots+\frac{\Gamma(\beta+s)y}{\Gamma(\beta+1)\alpha^{s-1}}+\frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^{s}}\right\}\\&-\left\{y^{s}+\frac{\Gamma(\beta+s)y^{s-1}}{\Gamma(\beta+s-1)\alpha}+\frac{\Gamma(\beta+s)y^{s-2}}{\Gamma(\beta+s-2)\alpha^{2}}+\cdots+\frac{\Gamma(\beta+s)y}{\Gamma(\beta+1)\alpha^{s-1}}\right\}.\end{aligned}$$

Or equivalently

$$(\beta-1-\alpha y)\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}+\sum_{j=0}^{s-1}\frac{(s-j)\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}=\frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^{s}}-y^{s}.$$

Then

$$\sum_{j=0}^{s-1}\frac{(s-j)\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}=\frac{\Gamma(\beta+s)}{\Gamma(\beta)\alpha^{s}}-y^{s}-(\beta-1-\alpha y)\sum_{j=0}^{s-1}\frac{\Gamma(\beta+s)y^{s-j-1}}{\Gamma(\beta+s-j)\alpha^{j+1}}.$$