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Characterizations of stability of first order linear Hahn difference equations

Alaa E. Hamza[†], A. S. Zaghrout[*1], S. M. Ahmed[*2]

[†] Mathematics Department, Faculty of Science, Cairo Univ. , Giza, Egypt.

hamzaaeg2003@yahoo.com

[*] Mathematics Department, Faculty of Science, Al-Azhar Univ. , Cairo, Egypt.

[1] afaf211@yahoo.com

[2] samah_moh_eg@yahoo.com

Abstract

Hahn introduced the difference operator $D_{q,\omega}f(t)=(f(qt+\omega)-f(t))/(t(q-1)+\omega)$ in 1949, where $0<q<1$ and $\omega>0$ are fixed real numbers. This operator extends the classical difference operator $\Delta_{\omega}f(t)=(f(t+\omega)-f(t))/\omega$ as well as Jackson $q-$difference operator $D_q f(t)=(f(qt)-f(t))/(t(q-1))$. In this paper, our objective is to establish characterizations of many types of stability, like (uniform, uniform exponential, ψ -) stability of linear Hahn difference equations of the form $D_{q,\omega}x(t)=p(t)x(t)+f(t)$. At the end, we give two illustrative examples.

Keywords: Hahn difference operator; Jackson $q-$difference operator.

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Introduction and Preliminaries

Hahn introduced his difference operator which is defined by

$$D_{q,\omega}f(t)=\frac{f(qt+\omega)-f(t)}{t(q-1)+\omega},\ t\neq\theta,$$

where $0<q<1$ and $\omega>0$ are fixed real numbers , $\theta=\omega/(1-q)$ [9, 10]. This operator unifies and generalizes two well-known difference operators. The first is Jackson $q-$difference operator defined by

$$D_{q}f(t)=\frac{f(qt)-f(t)}{t(q-1)},t\neq 0,$$

where q is fixed. Here f is supposed to be defined on a $q-$geometric set $A\subset \mathbb{R}$ for which $qt\in A$ whenever $t\in A$, see [1, 2, 4, 7, 8, 13, 15, 16]. The second operator is the forward difference operator

$$\Delta_{\omega}f(t)=\frac{f(t+\omega)-f(t)}{\omega},$$

where $\omega>0$ is fixed, see [5, 6, 14, 17]. Fine mathematicians applied Hahn's operator to construct families of orthogonal polynomials and to investigate some approximation problems, see [18, 19, 20]. Recently, Annaby *et al.* established a calculus based on this operator, see [3]. An essential function $h(t)=qt+\omega$, which is normally taken to be defined on an interval I which containing the number θ, plays an important role in this calculus. One can see that the $k-$th order iteration of $h(t)$ is given by

$$h^{k}(t)=q^{k}t+\omega[k]_{q},t\in I.$$

Here $[k]_q$ is defined by

$$[k]_{q}=\frac{1-q^{k}}{1-q}.$$

The sequence $h^k(t)$ is uniformly convergent to θ on I. They defined the $q,\omega-$integral of a function f from I to a Banach space X on an interval $[a,b]\subset I$ by

$$\int_{a}^{b}f(t)d_{q,\omega}t=\int_{\theta}^{b}f(t)d_{q,\omega}t-\int_{\theta}^{a}f(t)d_{q,\omega}t,$$

where

$$\int_{\theta}^{x}f(t)d_{q,\omega}t=(x(1-q)-\omega)\sum_{k=0}^{\infty}q^{k}f(h^{k}(x)),\quad x\in I,$$

provided that the series converges at $x=a$ and $x=b$. As indicated in [3], if $f:I\to X$ is continuous at θ, then the following statements are true.

(i) $\{f(h^k(s))\}_{k\in\mathbb{N}}$ converges uniformly to $f(\theta)$ on I.

(ii) $\sum_{k=0}^{\infty}q^k\left|f(h^k(s))\right|$ is uniformly convergent on I and consequently f is q,ω-integrable over I.

(iii) The function F defined by

$$F(x)=\int_{\theta}^{x}f(t)d_{q,\omega}t,\quad x\in I.$$

is continuous at θ. Furthermore, $D_{q,\omega}F(x)$ exists for every $x\in I$ and

$$D_{q,\omega}F(x)=f(x).$$

Conversely,

$$\int_a^b D_{q,\omega} f(t) d_{q,\omega} t = f(b) - f(a) \quad \textit{for all} \quad a,b \in I.$$

Throughout this paper I is any interval of $\mathbb{R}$ containing θ and X is a Banach space. The following theorem gives us the required conditions to insure the existence of solutions of linear Hahn difference equations, see [11].

Theorem 1.1. *Assume the functions* $a_j(t): I \to \mathbb{C}, 1 \le j \le n,$ *and* $b(t): I \to X$ *satisfy the following conditions:*

(i) $a_j(t), j = 1,\ldots,n,$ *and* $b(t)$ *are continuous at* θ *with* $a_0(t) \neq 0 \quad \forall\, t \in I$.

(ii) $a_j(t)/a_0(t)$ *is bounded on* $I,\ j \in \{1,\ldots,n\}$.

Then, for any elements $y_r \in X$ *, the equation*

$$\left.\begin{array}{l} a_0(t) D^n_{q,\omega} x(t) + a_1(t) D^{n-1}_{q,\omega} x(t) + \ldots + a_n(t) x(t) = b(t), \\ D^{i-1}_{q,\omega} x(\theta) = y_i, \quad i = 1,\ldots,n. \end{array}\right\} \tag{1.1}$$

has a unique solution on a subinterval $J \subset I$ *containing* θ.

Definition 1.2 [4] *The exponential functions* $e_p(t)$ *and* $E_p(t)$ *are defined by*

$$e_p(t) = \frac{1}{\prod_{k=0}^{\infty}(1 - p(h^k(t)) q^k (t(1-q) - \omega))} \tag{1.2}$$

and

$$E_p(t) = \prod_{k=0}^{\infty}(1 + p(h^k(t)) q^k (t(1-q) - \omega)), \tag{1.3}$$

whenever the first product is convergent to a nonzero number for every $t \in I$. *It is worth noting that the two products are convergent since* $\sum_{k=0}^{\infty} | p(h^k(t)) | q^k (t(1-q) - \omega)$ *is convergent.*

Now, we state various stability types that will be examined in Section 2. These concepts involve the boundedness of solutions of Hahn difference equation

$$D_{q,\omega} x(t) = F(t,x),\ x(\tau) = x_\tau \in X, t, \tau \in I \tag{1.4}$$

where F is assumed to satisfy all conditions such that (4) has a unique solution.

Definition 1.3. *Equation* (1.4) *is called stable if for every* $\tau \in I$ *and every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon, \tau) > 0$ *such that for any two solutions* $x(t) = x(t,\tau,x_\tau)$ *and* $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ *of Equation* (1.4), *we have for all*

$$\|x_\tau - \hat{x}_\tau\| < \delta \Rightarrow \|x(t) - \hat{x}(t)\| < \varepsilon \quad \textit{for all} \quad t \ge \tau, t, \tau \in I.$$

Definition 1.4. *Equation* (1.4) *is called uniformly stable if for every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon) > 0$ *such that for any two solutions* $x(t) = x(t,\tau,x_\tau)$ *and* $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ *of Equation* (1.4), *we have*

$$\|x_\tau - \hat{x}_\tau\| < \delta \Rightarrow \|x(t) - \hat{x}(t)\| < \varepsilon \quad \textit{for all}\ t \ge \tau, t, \tau \in I.$$

Definition 1.5. *Equation* (1.4) *is called asymptotically stable if it is stable and there exists* $\gamma = \gamma(\tau) > 0$ *such that for any solution* $x(t) = x(t,\tau,x_\tau)$ *of Equation* (1.4), *we have*

$$\|x_\tau\| < \gamma \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0.$$

Definition 1.6. *Equation* (1.4) *is called uniformly asymptotically stable if it is uniformly stable and there exists* $\gamma > 0$ *such that for any solution* $x(t) = x(t,\tau,x_\tau)$ *of Equation* (1.4), *we have*

$$\|x_\tau\| < \gamma \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0.$$

Definition 1.7. *Equation* (1.4) *is called globally asymptotically stable if it is stable and for any solution* $x(t) = x(t,\tau,x_\tau)$ *of Equation* (1.4), *we have*

$$\lim_{t\to\infty} \|x(t)\| = 0.$$

Definition 1.8. *Equation* (1.4) *is called exponentially stable if there exist finite constants* $\alpha > 0$ *and* $\gamma = \gamma(\tau) > 0$ *such that for any two solutions* $x(t) = x(t,\tau,x_\tau)$ *and* $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ *of* (1.4), *we have*

$$\|x(t) - \hat{x}(t)\| \le \gamma \|x_\tau - \hat{x}_\tau\| e_{-\alpha}(t,\tau) \text{ for all } t \ge \tau, t, \tau \in I.$$

where the exponential function $e_p(t,\tau)$ is given by $e_p(t,\tau) = e_p(t)/e_p(\tau).$

Definition 1.9. *Equation* (1.4) *is called uniformly exponentially stable if* γ *is independent on* $\tau \in I$.

Definition 1.10. *Equation* (1.4) *is called* $\psi-$*stable if there exists a finite constant* $\gamma = \gamma(\tau) > 0$ *such that for any two solutions* $x(t) = x(t,\tau,x_\tau)$ *and* $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ *of* (1.4), *we have*

$$\|\psi(t)(x(t) - \hat{x}(t))\| \le \gamma \|\psi(\tau)(x_\tau - \hat{x}_\tau)\| \text{ for all } t \ge \tau, t, \tau \in I.$$

Definition 1.11. *Equation* (1.4) *is called* $\psi-$*uniformly stable if* $\gamma > 0$ *is independent on* $\tau \in I$.

Remark 1.12. *If* $\psi(t) = e_\alpha(\tau,t)$, *then* $\psi-$*uniform stability coincides with uniform exponential stability. Thus,* $\psi-$*uniform stability is an extension of uniform exponential stability.*

Main results

In this section, we are concerned with obtaining many results about characterizations of stability of linear Hahn difference equations of the form

$$CP(0): D_{q,\omega}x(t) = p(t)x(t),\ x(\tau) = x_\tau \in X, t \ge \tau, t, \tau \in I,$$

and

$$CP(f): D_{q,\omega}x(t) = p(t)x(t) + f(t),\ x(\tau) = x_\tau \in X, t \ge \tau, t, \tau \in I,$$

where $p: I \to \mathbb{C}$, $f: I \to X$ are continuous at θ and $p(h^k(t))q^k(t(1-q)-\omega) \ne 1$ for all $k \in \mathbb{N}, t \in I$ Simple computations show that $CP(0)$ and $CP(f)$ have the unique solutions

$$x(t) = e_p(t,\tau)x_\tau \tag{2.1}$$

and

$$x(t) = e_p(t,\tau)(x_\tau + \int_\tau^t f(s)e_p(\tau,h(s))d_{q,\omega}s) \tag{2.2}$$

respectively.

Theorem 2.1. *The following statements are equivalent*

(i) $CP(0)$ *is stable.*

(ii) *For every* $\tau \in I$ *and every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon,\tau)$ *such that for any solution*

$x(t)=x(t,\tau,x_\tau)$ of $CP(0)$, we have $\|x_\tau\|<\delta \Rightarrow \|x(t)\|<\varepsilon$.

(iii) $CP(f)$ is stable.

(iv) For every $\tau \in I$, $\{|e_p(t,\tau)|\}_{t\ge\tau,t\in I}$ is bounded.

(v) For every $\tau \in I$, there exist $\gamma(\tau)>0$ such that for any solution $x(t)=x(t,\tau,x_\tau)$ of $CP(0)$ (resp. $CP(f)$), we have

$$\|x(t)\|\le\gamma(\tau)\|x_\tau\|, t\ge\tau, t\in I.$$

Proof. $(i)\Rightarrow(ii)$ Assume that $CP(0)$ is stable. Let $\varepsilon>0$. Then, there exists $\delta=\delta(\varepsilon,\tau)$ such that for any two solutions $x(t)=x(t,\tau,x_\tau)$ and $\hat{x}(t)=\hat{x}(t,\tau,\hat{x}_\tau)$ of $CP(0)$ with initial values $x_\tau,\hat{x}_\tau\in X$ respectively, we have

$$\|x_\tau-\hat{x}_\tau\|<\delta \Rightarrow \|x(t)-\hat{x}(t)\|<\varepsilon, t\ge\tau. \quad (2.3)$$

Now, let $x(t)=x(t,\tau,x_\tau)$ be any solution of $CP(0)$ such that $\|x_\tau\|<\delta$. Suppose that $\hat{x}(t)=\hat{x}(t,\tau,\hat{x}_\tau)$ be the zero solution subject to the initial value $x_\tau=0$. Hence, we obtain $\|x(t)\|<\varepsilon$ (from (3)).

$(ii)\Rightarrow(iii)$ Let $\varepsilon>0$ and $\tau\in I$. There exists $\delta=\delta(\varepsilon,\tau)$ such that for any solution $x(t)=x(t,\tau,x_\tau)$ of $CP(0)$, we have $\|x_\tau\|<\delta\Rightarrow\|e_p(t,\tau)x_\tau\|<\varepsilon$. Let $x_f(t)=x_f(t,\tau,x_\tau)$ and $\hat{x}_f(t)=\hat{x}_f(t,\tau,\hat{x}_\tau)$ be two solutions for $CP(f)$ with initial values $x_\tau,\hat{x}_\tau\in X$ respectively such that $\|x_\tau-\hat{x}_\tau\|<\delta$. Then, we have $\|x_f-\hat{x}_f\|=\|e_p(t,\tau)(x_\tau-\hat{x}_\tau)\|<\varepsilon$ (since $x_f-\hat{x}_f$ is a solution of $CP(0)$)

Therefore, $CP(f)$ is stable.

$(iii)\Rightarrow(iv)$ Assume that $CP(f)$ is stable. Let $\varepsilon=1$. There is $\delta>0$ such that for any two solutions $x_f(t)=x_f(t,\tau,x_\tau)$ and $\hat{x}_f(t)=\hat{x}_f(t,\tau,\hat{x}_\tau=0)$, we have

$$\|x_\tau\|<\delta\Rightarrow\|e_p(t,\tau)x_\tau\|<1 \quad \forall t\ge\tau, t\in I.$$

Let $0\ne x_0\in X$. Take $x_\tau=\delta x_0/(2\|x_0\|)$. Since $\|x_\tau\|<\delta$, then $\|e_p(t,\tau)\delta x_0/(2\|x_0\|)\|<1$ i.e. $\|e_p(t,\tau)x_0\|<2\|x_0\|/\delta, t\ge\tau, t\in I$. Thus, for all $x_0\in X,\{\|e_p(t,\tau)x_0\|\}_{t\ge\tau,t\in I}$ is bounded. By the Uniform Boundedness Theorem, $\{|e_p(t,\tau)|\}_{t\ge\tau,t\in I}$ is bounded.

$(iv)\Rightarrow(v)$ Suppose that $\{|e_p(t,\tau)|\}_{t\ge\tau,t\in I}$ is bounded for every $\tau\in I$. Then, there exists a positive constant γ which is dependent on τ such that $|e_p(t,\tau)|\le\gamma(\tau)$ for every $t\in I, t\ge\tau$. Consequently, for any solution $x(t)=x(t,\tau,x_\tau)$, we have

$$\|x(t)\|=\|e_p(t,\tau)x_\tau\|\le\gamma(\tau)\|x_\tau\|, t\ge\tau, t,\tau\in I.$$

$(v)\Rightarrow(i)$ Assume that for every $\tau\in I$ there exists $\gamma(\tau)>0$ such that $\|x(t)\|\le\gamma(\tau)\|x_\tau\|, t\in I$. Let $\varepsilon>0$, $\tau\in I$. Take $\delta=\dfrac{\varepsilon}{\gamma(\tau)}$. For any $x_\tau,\hat{x}_\tau\in X$ such that $\|x_\tau-\hat{x}_\tau\|<\delta$, we have

$$\|x(t)-\hat{x}(t)\|\le\gamma(\tau)\|x_\tau-\hat{x}_\tau\|=\frac{\varepsilon}{\delta}\|x_\tau-\hat{x}_\tau\|<\varepsilon, t\ge\tau, t,\tau\in I.$$

Similarly, we can prove the following theorem.

Theorem 2.2. *The following statements are equivalent*

$CP(0)$ *is uniformly stable,*

(i) *For every* $\varepsilon > 0$, *there exists* $\delta = \delta(\varepsilon)$ *such that for any solution* $x(t) = x(t, \tau, x_\tau)$ *of* $CP(0)$, *we have* $\|x_\tau\| < \delta \Rightarrow \|x(t)\| < \varepsilon$,

(ii) $CP(f)$ *is uniformly stable,*

(iii) $\{|e_p(t,\tau)| : t, \tau \in I, t \geq \tau\}$ *is bounded.*

(iv) *There is* $\gamma > 0$ *such that for any solution* $x(t) = x(t, \tau, x_\tau)$ *of* $CP(0)$ *(resp.* $CP(f)$*), we have*

$$\|x(t)\| \leq \gamma \|x_\tau\|, t \geq \tau, t \in I. \tag{2.4}$$

Theorem 2.3. *The following statements are equivalent*

(i) $CP(0)$ *is asymptotically stable.*

(ii) $\lim_{t\to\infty} Pe_p(t,\tau)xP = 0$ *for every* $x \in X$ *and every* $\tau \in I$.

(iii) $CP(0)$ *is globally asymptotically stable.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $CP(0)$ is asymptotically stable. Then, there exists $\gamma > 0$ such that for any solution $x(t) = x(t, \tau, x_\tau)$ of $CP(0)$, with initial value x_τ, we have

$$\|x_\tau\| < \gamma \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0.$$

Let $0 \neq x \in X$. Put $x_\tau = \gamma x / (2\|x\|)$. Then, $\lim_{t\to\infty} \|e_p(t,\tau)\gamma x / (2\|x\|)\| = 0$. Consequently, $\lim_{t\to\infty} \|e_p(t,\tau)x\| = 0$.

$(ii) \Rightarrow (iii)$ By condition (ii) and the Uniform Boundedness Theorem, we insure the boundedness of $\{|e_p(t,\tau)|\}_{t\geq\tau, t\in I}$. Consequently, $CP(0)$ is stable (by Theorem(2.1)). Thus by our assumption, $CP(0)$ is globally asymptotically stable.

$(iii) \Rightarrow (i)$ is clear.

Theorem 2.4. *Assume that*

$$F(t) = \int_\tau^t f(s) E_{-p}(h(s)) d_{q,\omega} s$$

is bounded for any $\tau \in I$. *Then,* $CP(0)$ *is globally asymptotically stable if and only if* $CP(f)$ *is globally asymptotically stable.*

Proof. Assume that $CP(0)$ is globally asymptotically stable. Then, $\lim_{t\to\infty} e_p(t,\tau) = 0$. Consequently, $\lim_{t\to\infty} e_p(t) = 0$. Now, let $x_f(t)$ be a solution of $CP(f)$. Hence, it has the form

$$\begin{aligned} x_f(t) &= e_p(t,\tau)(x_\tau + \int_\tau^t f(s) e_p(\tau, h(s)) d_{q,\omega} s) \\ &= e_p(t,\tau)x_\tau + (e_p(t)\int_\tau^t f(s) E_{-p}(h(s)) d_{q,\omega} s). \end{aligned}$$

Thus, $\lim_{t\to\infty} x_f(t) = 0$ which is the desired result.

For the converse, take $x_\tau = 0$, to obtain

$$\lim_{t\to\infty} e_p(t,\tau)\int_\tau^t f(s)e_p(\tau,h(s))d_{q,\omega}s = 0.$$

Consequently, $CP(0)$ is globally asymptotically stable.

Theorem 2.5. *The following statements are equivalent*

(i) $CP(0)$ *is exponentially stable.*

(ii) *There exist* $\alpha > 0$ *and* $\gamma = \gamma(\tau) > 0$ *such that for any solution* $x(t) = x(t,\tau,x_\tau)$ *of* $CP(0)$ *with initial value* $x_\tau \in X$ *, we have*

$$\|x(t)\| \le \gamma\|x_\tau\|e_{-\alpha}(t,\tau) \text{ for all } t \ge \tau.$$

(iii) *There exist* $\alpha > 0$ *and* $\gamma = \gamma(\tau) > 0$ *such that*

$$|e_p(t,\tau)| \le \gamma e_{-\alpha}(t,\tau) \text{ for all } t \ge \tau.$$

(iv) $CP(f)$ *is exponentially stable.*

Proof.

$(i) \Rightarrow (ii)$ Assume that $CP(0)$ is exponentially stable. There exists $\gamma = \gamma(\tau) > 0$ such that for any two solutions $x(t) = x(t,\tau,x_\tau)$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ of $CP(0)$ with initial values $x_\tau, \hat{x}_\tau \in X$ respectively, we have

$$\|x(t) - \hat{x}(t)\| = \|e_p(t,\tau)(x_\tau - \hat{x}_\tau)\| \le \gamma\|x_\tau - \hat{x}_\tau\|e_{-\alpha}(t,\tau) \text{ for all } t \ge \tau.$$

Let $x(t) = x(t,\tau,x_\tau)$ be any solution of $CP(0)$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ be the zero solution corresponding to the initial value $\hat{x}_\tau = 0$. Then, we obtain our desired result.

$(ii) \Rightarrow (iii)$ Let $x(t)$ be any nontrivial solution corresponding to the initial value $x_\tau \ne 0$. Then, we have $\|x(t)\| \le \gamma e_{-\alpha}(t,\tau)\|x_\tau\|$. Consequently, we have $\|e_p(t,\tau)x_\tau\| \le \gamma e_{-\alpha}(t,\tau)\|x_\tau\|$ and thereby $|e_p(t,\tau)| \le \gamma e_{-\alpha}(t,\tau)$.

$(iii) \Rightarrow (iv)$ Let $x_f(t) = x_f(t,\tau,x_\tau)$ and $\hat{x}_f(t) = \hat{x}_f(t,\tau,\hat{x}_\tau)$ be two solutions for $CP(f)$ with initial values $x_\tau, \hat{x}_\tau \in X$ respectively. We deduce that

$$\|x_f - \hat{x}_f\| = \|e_p(t,\tau)(x_\tau - \hat{x}_\tau)\| = |e_p(t,\tau)|\,\|x_\tau - \hat{x}_\tau\| \le \gamma e_{-\alpha}(t,\tau)\|x_\tau - \hat{x}_\tau\| \text{ for } t \ge \tau.$$

Hence, $CP(f)$ is exponentially stable.

$(iv) \Rightarrow (i)$ There exist $\alpha > 0$ and $\gamma = \gamma(\tau) > 0$ such that for two solutions $x_f(t) = x_f(t,\tau,x_\tau)$ and $\hat{x}_f(t) = \hat{x}_f(t,\tau,\hat{x}_\tau)$ of $CP(f)$, the inequality

$$\|x_f(t) - \hat{x}_f(t)\| \le \gamma e_{-\alpha}(t,\tau)\|x_\tau - \hat{x}_\tau\| \text{ for } t \ge \tau.$$

holds. For any two solutions $x(t) = x(t,\tau,x_\tau)$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ of $CP(0)$, we conclude that

$$\|x(t) - \hat{x}(t)\| = \|x_f(t) - \hat{x}_f(t)\| \le \gamma e_{-\alpha}(t,\tau)\|x_\tau - \hat{x}_\tau\| \text{ for } t \ge \tau.$$

Hence, $CP(0)$ is exponentially stable.

Similarly, we can prove the following theorem.

Theorem 2.6. *The following statements are equivalent*

(i) $CP(0)$ *is uniformly exponentially stable.*

(ii) *There exist* $\alpha > 0$ *and* $\gamma > 0$ *independent on* τ *such that for any solution* $x(t) = x(t,\tau,x_\tau)$ *of*

$CP(0)$ *with initial value* $x_\tau \in X$ *, we have*

$$\|x(t)\| \le \gamma \|x_\tau\| e_{-\alpha}(t,\tau) \text{ for all } t \ge \tau.$$

(iii) *There exist* $\alpha > 0$ *and* $\gamma > 0$ *independent on* τ *such that*

$$|e_p(t,\tau)| \le \gamma e_{-\alpha}(t,\tau) \text{ for all } t \ge \tau.$$

(iv) $CP(f)$ *is uniformly exponentially stable.*

The following results concerning $\psi-$stability and $\psi-$unifom stability are more general than Theorems 2.5, 2.6 replacing the exponential function $e_{-\alpha}$ by $|\psi|$. The proofs are similar, so they will be omitted.

Theorem 2.7. *The following statements are equivalent*

(i) $CP(0)$ *is* $\psi-$*stable.*

(ii) *There is* $\gamma(\tau) > 0$ *such that for any solution* $x(t) = x(t,\tau,x_\tau)$ *of* $CP(0)$ *with initial value* $x_\tau \in X$ *, we have*

$$\|\psi(t)x(t)\| \le \gamma \|\psi(\tau)x_\tau\| \text{ for all } t \ge \tau.$$

(iii) *There exist* $\gamma > 0$ *such that*

$$|\psi(t)e_p(t,\tau)| \le \gamma |\psi(\tau)| \text{ for all } t \ge \tau.$$

(iv) $CP(f)$ *is* $\psi-$*stable.*

Theorem 2.8. *The following statements are equivalent*

(i) $CP(0)$ *is* $\psi-$*uniformly stable.*

(ii) *There is* $\gamma > 0$ *independent on* τ *such that for any solution* $x(t) = x(t,\tau,x_\tau)$ *of* $CP(0)$ *with initial value* $x_\tau \in X$ *, we have*

$$\|\psi(t)x(t)\| \le \gamma \|\psi(\tau)x_\tau\| \text{ for all } t \ge \tau.$$

(iii) *There exist* $\gamma > 0$ *independent on* τ *such that*

$$|\psi(t)e_p(t,\tau)| \le \gamma |\psi(\tau)| \text{ for all } t \ge \tau.$$

(iv) $CP(f)$ *is* $\psi-$*uniformly stable.*

Numerical Examples

In this section we give some numerical examples about the various types of stability for solutions of Hahn difference equations.

Example 3.1. *The Hahn difference equation*

$$D_{q,\omega}x(t) = 0,\ x(\tau) = c \in X, t, \tau \in I, \tag{3.1}$$

where $c \ne 0$ *is uniformly stable but it is not asymptotically stable.*

Examaple 3.2. *Let* $I = [\theta, \infty[$ *. Consider the Hahn difference equations of the form*

$$D_{q,\omega}x(t) = -r(t)x(t) + f(t),\ x(\tau) = x_\tau \in X, t, \tau \in I, \tag{3.2}$$

where r *is a non-negative function and continuous at* θ *.*

Equation (3.2) is

(i) Stable since

$$
\begin{aligned}
|e_{-r}(t,\tau)| &= \frac{e_{-r}(t)}{e_{-r}(\tau)} \\
&= \frac{\prod_{k=0}^{\infty}(1+r(h^k(\tau))q^k(\tau(1-q)-\omega))}{\prod_{k=0}^{\infty}(1+r(h^k(t))q^k(t(1-q)-\omega))} \\
&\leq \frac{1}{e_{-r}(\tau)} = E_r(\tau) \ \ forallt \geq \tau.
\end{aligned}
$$

Consequently, for every $\tau \in I$, $\{|e_p(t,\tau)|\}_{t\geq\tau, t\in I}$ *is bounded.*

(ii) Exponentially stable when $\alpha = \inf_{t\in I} r(t) > 0$*. Indeed, if we take* $\gamma(\tau) \geq \frac{e_{-\alpha}(\tau)}{e_{-r}(\tau)}$*, then*

$$
\begin{aligned}
|e_{-r}(t,\tau)| &= \frac{e_{-r}(t)}{e_{-r}(\tau)} \\
&\leq \gamma(\tau)e_{-\alpha}(t,\tau) \ \ \text{for all } t \geq \tau.
\end{aligned}
$$

(iii) Globally asymptotically stable if f *has the form* $f(t) = e_p(t)g(t)$ *where* $\int_{\tau}^{t} g(t)d_{q,\omega}t$ *is bounded for every* $\tau \in I$ *and* $\alpha = \inf_{t\in I} r(t) > 0$*. In fact, the homogeneous equation associated with the above liner Hahn difference equation is globally asymptotically stable. Indeed, we have*

$$
\begin{aligned}
|e_{-r}(t,\tau)| &= \frac{1}{e_{-r}(\tau)} \times \frac{1}{\prod_{k=0}^{\infty}(1+r(h^k(t))q^k(t(1-q)-\omega))} \\
&< \frac{1}{e_{-r}(\tau)} \times \frac{1}{1+\alpha q^k(t(1-q)-\omega)}.
\end{aligned}
$$

Consequently, $\lim_{t\to\infty} |e_{-r}(t,\tau)| = 0$.

(iv) $\psi -$ *Stable for any function* $\psi(t)$ *such that* $|\psi(t)|$ *is decreasing function because*

$$
\begin{aligned}
|\psi(t)e_{-r}(t,\tau)| &= |\psi(t)e_{-r}(t)/e_{-r}(\tau)| \\
&\leq |\psi(t)/e_{-r}(\tau)| \\
&\leq \gamma(\tau)|\psi(\tau)| \qquad \text{for all } t \geq \tau.
\end{aligned}
$$

where $\gamma(\tau) = \frac{1}{e_{-r}(\tau)}$.

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