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Some Geometrical Calculations for the Spherical Indicatrices of Involutes of a Timelike Curve in Minkowski 3-Space

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ABSTRACT

In this paper, we introduce the spherical indicatrices of involutes of a given timelike curve. Then we give some important relationships between arc lengths and geodesic curvatures of the base curve and its involutes on E_1^3, S_1^2, H_0^2. Additionally, some important results concerning these curves are given.

Keywords:

Minkowski space; involute evolute curve couple; spherical indicatrices; geodesic curvature.

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INTRODUCTION

The notion of the involute of a curve was introduced by Christiaan Huygens in his work titled *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demonstrationes geometricae* (1673). In the theory of curves in Euclidean space, one of the important and interesting problems is the characterizations of a regular curve. In particular, the involute of a given curve is a well known concept in the classical differential geometry (see [5]).

At the beginning of the twentieth century, A. Einstein's theory opened a door of use of new geometries. One of them is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary Lorentzian manifold, was introduced and some of classical differential geometry topics have been treated by the researchers. According to reference ([1], [2], [3]), it has been investigated the involutes of the timelike curves in Minkowski 3-space. Also, Bükcü and Karacan studied on the involute and evolute curves of the spacelike curve with a spacelike binormal in Minkowski 3-space, [4].

In this study, we carry tangents of the spacelike involute β of a timelike curve α to the center of the pseudohyperbolic space H_0^2 and we obtain a spacelike curve $\left(T^*\right)$ with equation $\beta_{T^*} = T^*$ on H_0^2. This curve is called the first spherical indicatrix or tangent indicatrix of β. Similarly one consider the principal normal indicatrix $\left(N^*\right)$ and the binormal indicatrix $\left(B^*\right)$ on the pseudosphere S_1^2. Then we give some important relationships between arc lengths and geodesic curvatures of the base curve and its involutes on E_1^3, S_1^2, H_0^2. Additionally, some important results concerning these curves are given.

PRELIMNARIES

The Minkowski 3-space E_1^3 is the vector space E^3 provided with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2,$$

where $X = \left(x_1, x_2, x_3\right) \in E^3$. A vector X is said to be timelike if $g\left(X,X\right) < 0$, spacelike if $g\left(X,X\right) > 0$ and lightlike (or null) if $g\left(X,X\right) = 0$. Similarly, an arbitrary curve $\gamma = \gamma\left(s\right)$ in E_1^3 where s is a pseudo-arclenght parameter, can locally be timelike spacelike or null (lightlike), if all of its velocity vectors $\gamma\left(s\right)$ are respectively timelike, spacelike or null. The norm of a vector X is defined by

$$\|X\| = \sqrt{\left|g\left(X,X\right)\right|}$$

and two vectors $X = \left(x_1, x_2, x_3\right)$, $Y = \left(y_1, y_2, y_3\right) \in E_1^3$ are orthogonal if and only if $g\left(X,Y\right) = 0$.

Now let X and Y be two vectors in E_1^3, then the Lorentzian cross product is given by

$$X \times Y = \begin{vmatrix} -e_1 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left(x_3 y_2 - x_2 y_3, x_1 y_3 - x_3 y_1, x_1 y_2 - x_2 y_1\right), \text{[6]}.$$

Denote by $\left\{T\left(s\right), N\left(s\right), B\left(s\right)\right\}$ the moving Frenet frame along the curve α in the space E_1^3. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively. For these vectors, we can write

$$T \times N = -B \ , \ N \times B = T \ , \ B \times T = -N .$$

Depending on the causal character of the curve α, the following Frenet-Serret formulas are given in [7].

$$\begin{cases} T' = \kappa N, \quad N' = \kappa T - \tau B, \quad B' = \tau N \\ g\left(T,T\right) = -1,\ g\left(N,N\right) = g\left(B,B\right) = 1,\ g\left(T,N\right) = g\left(T,B\right) = g\left(N,B\right) = 0 \end{cases}.$$

If the curve α is non-unit speed curve in space E_1^3, then

$$\kappa = \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3}, \quad \tau = \frac{\det\left(\gamma', \gamma'', \gamma'''\right)}{\|\gamma' \times \gamma''\|^2}.$$

The Darboux vector for the timelike curve is defined by [8]

$$\omega = \tau T - \kappa B.$$

There are two cases corresponding to the causal characteristic of Darboux vector ω.

Case I.

If $|\kappa| \rangle |\tau|$, then ω is a spacelike vector. In this situation, we can write

$$\begin{cases} \kappa = \|\omega\| \cosh\varphi \\ \tau = \|\omega\| \sinh\varphi \end{cases}, \quad \|\omega\|^2 = g(\omega,\omega) = \kappa^2 - \tau^2$$

and the unit vector C of direction ω is

$$C = \frac{1}{\|\omega\|}\omega = \sinh\varphi\, T - \cosh\varphi\, B,$$

where φ is the lorentzian timelike angle between $-B$ and unit timelike vector C' that lorentz orthogonal to the normalisation of the Darboux vector C (see Figure 1).

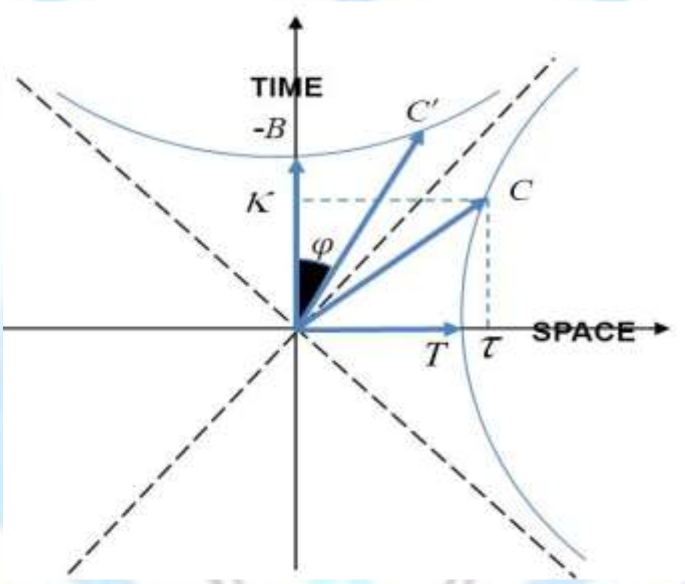

Fig 1: Lorentzian timelike angle φ

Case II.

If $|\kappa| \langle |\tau|$, then ω is a timelike vector. In this situation, we have

$$\begin{cases} \kappa = \|\omega\| \sinh\varphi \\ \tau = \|\omega\| \cosh\varphi \end{cases}, \quad \|\omega\|^2 = -g(\omega,\omega) = \tau^2 - \kappa^2$$

and the unit vector C of direction ω is

$$C = \frac{1}{\|\omega\|}\omega = \cosh\varphi\, T - \sinh\varphi\, B.$$

Remark 1. *We can easily see from equations of the section Case I and Case II that:* $\frac{\tau}{\kappa} = \tanh\varphi$ *or* $\frac{\tau}{\kappa} = \coth\varphi$, *if* $\varphi = \text{constant}$ *then* α *is a general helix.*

For the arc length of the spherical indicatrix of (C) we get

$$s_C = \int_0^s |\varphi'|\, ds\,.$$

After some calculations, we have for the arc lenghts of the spherical indicatrices $(T),(N),(B)$ measured from the points corresponding to $s=0$

$$s_T = \int_0^s |\kappa|\, ds,\; s_N = \int_0^s \|\omega\|\, ds,\; s_B = \int_0^s |\tau|\, ds$$

for their geodesic curvatures with respest to E_1^3

$$k_T = \frac{1}{\cosh\varphi} \left(k_T = \frac{1}{\sinh\varphi},\ \text{for timelike}\ \omega \right)$$
$$k_N = \frac{1}{\|\omega\|}\sqrt{\left|\varphi'^2 + \|\omega\|^2\right|} \left(k_N = \frac{1}{\|\omega\|}\sqrt{\left|-\varphi'^2 + \|\omega\|^2\right|},\ \text{for timelike}\ \omega \right)$$
$$k_B = \frac{1}{\sinh\varphi} \left(k_B = \frac{1}{\cosh\varphi},\ \text{for timelike}\ \omega \right)$$
$$k_C = \sqrt{\left|1 + \left(\frac{\|\omega\|}{\varphi'}\right)^2\right|} \left(k_C = \sqrt{\left|-1 + \left(\frac{\|\omega\|}{\varphi'}\right)^2\right|},\ \text{for timelike}\ \omega \right)$$

and for their geodesic curvatures with respest to S_1^2 or H_0^2

$$\begin{aligned}
\gamma_T &= \left\| \overline{\overline{\nabla}}_{t_T} t_T \right\| = \tanh\varphi\ (\gamma_T = \coth\varphi,\ \text{for timelike}\ \omega) \\
\gamma_N &= \left\| \overline{\nabla}_{t_N} t_N \right\| = \frac{\varphi'}{\|W\|} \\
\gamma_B &= \left\| \overline{\nabla}_{t_B} t_B \right\| = \coth\varphi\ (\gamma_B = \tanh\varphi,\ \text{for timelike}\ \omega) \\
\gamma_C &= \left\| \overline{\nabla}_{t_C} t_C \right\| = \frac{\|W\|}{\varphi'}
\end{aligned}\ ,\ [6].$$

Note that $\overline{\nabla}$ and $\overline{\overline{\nabla}}$ are Levi-Civita connections on S_1^2 and H_0^2 , respectively. Then Gauss equations are given by the followings.

$$\nabla_X Y = \overline{\nabla}_X Y + \varepsilon g(S(X),Y)\xi,\ \nabla_X Y = \overline{\overline{\nabla}}_X Y + \varepsilon g(S(X),Y)\xi,\ \varepsilon = g(\xi,\xi),$$

where ξ is a unit normal vector field and S is the shape operator of S_1^2 (or H_0^2).

The unit pseudosphere and pseudohyperbolic space of radius 1and center 0 in E_1^3 are given by

$$S_1^2 = \left\{ X = (x_1, x_2, x_3) \in E_1^3 : g(X,X) = 1 \right\}$$

and

$$H_0^2 = \left\{ X = (x_1, x_2, x_3) \in E_1^3 : g(X,X) = -1 \right\}$$

respectively, [4].

Definition 1. *Let $\alpha = \alpha(s)$, $\beta = \beta(s^*) \subset E_1^3$ be two curves. Let Frenet frames of α and β be $\{T,N,B\}$ and $\{T^*,N^*,B^*\}$, respectively. β is called the involute of α (α is called the evolute of β) if*

$$g(T,T^*) = 0\,.$$

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Remark 2. Let α be a timelike curve. In this situation its involute curve β must be a spacelike curve. (α,β) being the involute-evolute curve couple, the following lemma was presented by Bilici and Çalışkan (see [1], [3]).

Lemma 1. *Let (α,β) be the timelike-spacelike involute-evolute curve couple. The relations between the Frenet vectors of the curve couple as follow.*

(1) *If ω is a spacelike vector($|\kappa|\rangle|\tau|$), then*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ -\sinh\varphi & 0 & \cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

(2) *If ω is a timelike vector($|\kappa|\langle|\tau|$), then*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ -\cosh\varphi & 0 & \sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Example 1. Let the curve

$$\alpha(s) = \left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, s\right)$$

be a unit speed timelike curve such that

$$T(s) = \left(\frac{2}{\sqrt{3}}\cosh\left(\frac{s}{\sqrt{3}}\right), \frac{2}{\sqrt{3}}\sinh\left(\frac{s}{\sqrt{3}}\right), \frac{1}{\sqrt{3}}\right).$$

If α is a timelike curve then its involute curve is a spacelike curve. In this situation, involute curve β of the curve α can be given the equation

$$\beta(s) = \left(\frac{2}{\sqrt{3}}\sinh\left(\frac{s}{\sqrt{3}}\right) + |c-s|\frac{2}{\sqrt{3}}\cosh\left(\frac{s}{\sqrt{3}}\right), \frac{2}{\sqrt{3}}\cosh\left(\frac{s}{\sqrt{3}}\right) + |c-s|\frac{2}{\sqrt{3}}\sinh\left(\frac{s}{\sqrt{3}}\right), \frac{c}{\sqrt{3}}\right),$$

where c is an arbitrary constant. If we compute the tangent vector field of the curve β we have

$$T^*(s) = \left(\sinh\left(\frac{\theta}{\sqrt{3}}\right), \cosh\left(\frac{\theta}{\sqrt{3}}\right), 0\right)$$

and

$$g\left(T(s), T^*(s)\right) = 0 .$$

3. SOME GEOMETRICAL CALCULATIONS FOR THE SPHERICAL INDICATRICES OF INVOLUTES OF A TIMELIKE CURVE

In this section, we compute the arc-lengths of the spherical indicatrix curves $(T^*),(N^*),(B^*)$ and then we calculate the geodesic curvatures of these curves in E_1^3 and S_1^2 (or H_0^2).

Firstly, for the arc-length s_{T^*} of tagent indicatrix (T^*) of the involute curve β, we can write

$$s_{T^*} = \int_0^{S^*} \left\| \frac{dT^*}{ds^*} \right\| ds^* = \int_0^{S} \left\| \frac{dN}{ds} \right\| ds ,$$

$$s_{T^*} = \int_0^{S} \sqrt{\left|\kappa^2 - \tau^2\right|} ds$$

or from the definition of Darboux vector

$$s_{T^*} = \int_0^s \|\omega\| ds .$$

If the arc length for the principal normal indicatrix (N^*) is s_{N^*} it is

$$s_{N^*} = \int_0^s \left\| \frac{dN^*}{ds} \right\| ds = \int_0^s \left\| \frac{d(-\cosh\varphi\, T + \sinh\varphi\, B)}{ds} \right\| ds ,$$

$$s_{N^*} = \int_0^s \sqrt{\left|\varphi'^2 + \|\omega\|^2\right|} ds .$$

For timelike ω, from the Lemma1. (2) we have

$$s_{N^*} = \int_0^s \sqrt{\left|-\varphi'^2 + \|\omega\|^2\right|} ds .$$

If the arc length for the binormal indicatrix (B^*) is s_{B^*} it is

$$s_{B^*} = \int_0^s \left\| \frac{dB^*}{ds} \right\| ds = \int_0^s \left\| \frac{d(-\sinh\varphi\, T + \cosh\varphi\, B)}{ds} \right\| ds ,$$

$$s_{B^*} = \int_0^s |\varphi'|\, ds .$$

If ω is a timelike vector, then we have the same result. Thus we can give the following corollaries:

Corollary 3.1. *For the arc lenght of the tagent indicatrix (T^*) of the involute of a timelike curve, it is obvious that*

$$s_{T^*} = s_N .$$

Corollary 3.2. *If the evolute curve α is a helix then for the arc-length of the principal normal indicatrix (N^*), we can write*

$$s_{N^*} = s_N .$$

Corollary 3.3. *For the arc lenght of the binormal indicatrix (B^*) of the involute of a timelike curve, we have*

$$s_{B^*} = s_C .$$

Now let us compute the geodesic curvatures of the spherical indicatrices $(T^*),(N^*),(B^*)$ with respect to E_1^3. For the geodesic curvature k_{T^*} of the tangent indicatrix (T^*) of the curve β, we can write

$$k_{T^*} = \left\| \nabla_{t_{T^*}} t_{T^*} \right\| . \quad (1)$$

Differentiating the curve $\beta_{T^*}(s_{T^*}) = T^*(s)$ with the respect to s_{T^*} and normalizing, we obtain

$$t_{T^*} = \cosh\varphi T - \sinh\varphi B . \quad (2)$$

By taking derivative of the last equation we have

$$\nabla_{t_{T^*}} t_{T^*} = (\varphi' \sinh\varphi T + \|\omega\| N - \varphi' \cosh\varphi B) \frac{1}{\|\omega\|} . \quad (3)$$

By substituting (3) into the Eq. (1) we get

$$k_{T^*} = \frac{1}{\|\omega\|}\sqrt{\left|\varphi'^2 + \|\omega\|^2\right|}. \tag{4}$$

From $k_T = \frac{1}{\cosh\varphi}$ we have $\varphi' = -\frac{k_T'}{k_T\sqrt{1-k_T^2}}$. If we set φ' in the Eq. (4) then we have

$$k_{T^*} = \sqrt{\left|1 + \frac{k_T'^2}{k_T^2\left(1-k_T^2\right)\|\omega\|^2}\right|}. \tag{5}$$

In the case of (α,β) is the timelike-spacelike involute-evolute curve couple with timelike ω, similar results can be easily obtained as follow in following same procedure.

$$k_{T^*} = \frac{1}{\|\omega\|}\sqrt{\left|\|\omega\|^2 - \varphi'^2\right|}, \tag{6}$$

$$k_{T^*} = \sqrt{\left|1 - \frac{k_T'^2}{k_T^2\left(1+k_T^2\right)\|\omega\|^2}\right|}. \tag{7}$$

Corollary 3.4. *If the evolute curve α is a helix then we have for the geodesic curvature of the tangent indicatrix (T^*) of the involute curve β*

$$k_{T^*} = 1.$$

Similarly, by differentiating the curve $\beta_{N^*}(s_{N^*}) = N^*(s)$ with the respect to s_{N^*} and by normalizing we obtain

$$t_{N^*} = -\sigma\sinh\varphi T - \frac{1}{k_N}N + \sigma\cosh\varphi B, \left(\sigma = \frac{\gamma_N}{k_N}\right). \tag{8}$$

By taking derivative of the last equation and using the definition of geodesic curvature, we have

$$\nabla_{t_{N^*}} t_{N^*} = \left[\left(-\sigma'\sinh\varphi - \varphi'\sigma\cosh\varphi - \frac{\kappa}{k_N}\right)T + \left(\frac{k_N'}{k_N^2}\right)N + \left(\sigma'\cosh\varphi + \varphi'\sigma\sinh\varphi + \frac{\tau}{k_N}\right)B\right]\frac{1}{\|W\|k_N}. \tag{9}$$

$$k_{N^*} = \frac{1}{\|\omega\|k_N}\sqrt{\left|\sigma'^2 - \left(\varphi'\sigma + \frac{\|W\|}{k_N}\right)^2 + \frac{k_N'^2}{k_N^4}\right|}. \tag{10}$$

In the case of (α,β) is the timelike-spacelike involute-evolute curve couple with timelike ω, similar result can be easily obtained as follow in following same procedure.

$$k_{N^*} = \frac{1}{\|\omega\|k_N}\sqrt{\left|-\sigma'^2 + \left(\varphi'\sigma - \frac{\|\omega\|}{k_N}\right)^2 + \frac{k_N'^2}{k_N^4}\right|}. \tag{11}$$

Corollary 3.5. *If the evolute curve α is a helix then we have for the geodesic curvature of the principal normal indicatrix (N^*) of the involute curve β*

$$k_{N^*} = 1.$$

By differentiating the curve $\beta_{B^*}(s_{B^*}) = B^*(s)$ with the respect to s_{B^*} and by normalizing we obtain

$$t_{B^*} = -\cosh\theta\, T + \sinh\theta\, B\,. \tag{12}$$

By taking derivative of the last equation

$$\nabla_{t_{B^*}} t_{B^*} = -\sinh\varphi\, T - \frac{\|\omega\|}{\varphi'} N + \cosh\varphi\, B\,, \tag{13}$$

and by taking the norm of the last equation, we obtain

$$k_{B^*} = \sqrt{\left| 1 + \left(\frac{\|\omega\|}{\varphi'}\right)^2 \right|}\,. \tag{14}$$

From $k_B = \dfrac{1}{\sinh\varphi}$ we have $\varphi' = -\dfrac{k_B{}'}{k_B\sqrt{k_B{}^2 + 1}}$. If we set φ' in the Eq. (14) then we have

$$k_{B^*} = \sqrt{\left| 1 + \frac{\|\omega\|^2 k_B{}^2\left(k_B{}^2 + 1\right)}{k_B{}'^2} \right|}\,. \tag{15}$$

In the case of timelike ω, similar results can be easily obtained as follow,

$$k_{B^*} = \sqrt{\left| \left(\frac{\|\omega\|}{\theta'}\right)^2 - 1 \right|}\,, \tag{16}$$

$$k_{B^*} = \sqrt{\left| \frac{\|\omega\|^2 k_B{}^2\left(1 - k_B{}^2\right)}{k_B{}'^2} - 1 \right|}\,. \tag{17}$$

Corollary 3.6. *If the evolute curve α is a helix then the geodesic curvature k_{B^*} of the binormal indicatrix $\left(B^*\right)$ of the involute curve β is undefined.*

Now let us compute the geodesic curvatures of the spherical indicatrices $\left(T^*\right),\left(N^*\right),\left(B^*\right)$ with respect S_1^2 (or H_0^2).

For the geodesic curvature γ_{T^*} of the tangent indicatrix $\left(T^*\right)$ of the curve β with respect to S_1^2, we can write

$$\gamma_{T^*} = \left\| \bar{\nabla}_{t_{T^*}} t_{T^*} \right\|. \tag{18}$$

From the Gauss equation we can write

$$\nabla_{t_{T^*}} t_{T^*} = \bar{\nabla}_{t_{T^*}} t_{T^*} + \varepsilon\, g\left(S\left(t_{T^*}\right), t_{T^*}\right) T^*\,, \tag{19}$$

where $\varepsilon = g\left(T^*, T^*\right) = +1$, $S\left(t_{T^*}\right) = -t_{T^*}$ and $g\left(S\left(t_{T^*}\right), t_{T^*}\right) = +1$. From the Eq. (3) and (19), it follows that

$$\bar{\nabla}_{t_{T^*}} t_{T^*} = \frac{\varphi'}{\|\omega\|}\sinh\varphi\, T - \frac{\varphi'}{\|\omega\|}\cosh\varphi\, B\,. \tag{20}$$

Substituting (20) in the Eq. (18), we obtain

$$\gamma_{T^*} = \frac{\varphi'}{\|\omega\|}\,. \tag{21}$$

By using $\gamma_T = \tanh\varphi$, we obtain following relationship between γ_T and γ_{T^*}:

$$\gamma_{T^*} = \frac{1}{\|\omega\|}\left(\frac{\gamma_T{}'}{1 - \gamma_T{}^2}\right). \tag{22}$$

If ω is a timelike vector, then we have the same result. Thus we can give the following corollary:

Corollary 3.7. *If the evolute curve* α *is a helix then we have for the geodesic curvature of the tangent indicatrix* (T^*) *of the involute curve* β

$$\gamma_{T^*} = 0$$

For the geodesic curvature γ_{N^*} of the principal normal indicatrix (N^*) of the curve β with respect to H_0^2, we can write

$$\gamma_{N^*} = \left\| \overline{\overline{\nabla}}_{t_{N^*}} t_{N^*} \right\|. \tag{23}$$

If ω is a spacelike vector, by using the Gauss equation and the Eq. (9), we can write

$$\begin{aligned} \overline{\overline{\nabla}}_{t_{N^*}} t_{N^*} = \Bigg[& \left(-\sigma' \sinh\varphi - \varphi'\sigma \cosh\varphi - \frac{\kappa}{k_N} + \|\omega\| k_N \cosh\varphi \right) T + \left(\frac{k_N'}{k_N{}^2} \right) N \\ & + \left(\sigma' \cosh\varphi + \varphi'\sigma \sinh\varphi + \frac{\tau}{k_N} - \|\omega\| k_N \sinh\varphi \right) B \Bigg] \frac{1}{\|\omega\| k_N}. \end{aligned} \tag{24}$$

By taking the norm of the last equation we obtain

$$\gamma_{N^*} = \frac{1}{\|\omega\| k_N} \sqrt{\left| \sigma'^2 - \left[\frac{\|\omega\|}{k_N} + \left(\varphi'\sigma - \|W\| k_N \right) \right]^2 + \frac{k_N'^2}{k_N{}^4} \right|}. \tag{25}$$

By using $\gamma_N = \dfrac{\varphi'}{\|\omega\|}$, we get

$$\gamma_{N^*} = \frac{1}{\|\omega\| k_N} \sqrt{\left| \sigma'^2 - \left[\frac{\|\omega\|}{k_N} \left(1 + \gamma_N{}^2 - k_N{}^2 \right) \right]^2 + \frac{k_N'^2}{k_N{}^4} \right|}. \tag{26}$$

In the case of timelike ω, similar results can be easily obtained as follow,

$$\gamma_{N^*} = \frac{1}{\|\omega\| k_N} \sqrt{\left| -\sigma'^2 + \left[\frac{\|\omega\|}{k_N} - \left(\varphi'\sigma - \mu \|W\| k_N \right) \right]^2 + \frac{k_N'^2}{k_N{}^4} \right|}, \tag{27}$$

$$\gamma_{N^*} = \frac{1}{\|\omega\| k_N} \sqrt{\left| -\sigma'^2 + \left[\frac{\|\omega\|}{k_N} \left(1 - \gamma_N{}^2 - \mu k_N{}^2 \right) \right]^2 + \frac{k_N'^2}{k_N{}^4} \right|}, \tag{28}$$

where $g\left(S\left(t_{N^*} \right), t_{N^*} \right) = \mp 1 = \mu$.

Corollary 3.8. *If the evolute curve* α *is a helix then we have for the geodesic curvature of the principal normal indicatrix* (N^*) *of the involute curve* β

$$\gamma_{N^*} = 0.$$

For the geodesic curvature γ_{B^*} of the principal normal indicatrix (B^*) of the curve β with respect to S_1^2, we can write

$$\gamma_{B^*} = \left\| \overline{\nabla}_{t_{B^*}} t_{B^*} \right\|. \tag{29}$$

If ω is a spacelike vector, by using the Gauss equation and the Eq. (13), we can write

$$\overline{\nabla}_{t_{B^*}} t_{B^*} = -\sinh\varphi T - \frac{\|\omega\|}{\varphi'} N + \cosh\varphi B - B^*. \tag{30}$$

By using $B^{*}=-sinh\varphi T+cosh\varphi B$ given in the *Lemma 1. (i),* we obtain

$$\bar{\nabla}_{t_{B^*}}t_{B^*}=-\frac{\|\omega\|}{\varphi'}N\,. \tag{31}$$

By taking the norm of the last equation we obtain

$$\gamma_{B^*}=\frac{\|\omega\|}{\varphi'}. \tag{32}$$

By using $\gamma_B=\coth\varphi$, we obtain following relationship between γ_B and γ_{B^*} :

$$\gamma_{B^*}=\frac{\|\omega\|\left(1-\gamma_B{}^2\right)}{\gamma_B{}'}. \tag{33}$$

If ω is a timelike vector, then we have the same result. Thus we can give the following corollary:

Corollary 3.9. *If the evolute curve α is a helix then the geodesic curvature γ_{B^*} of the binormal indicatrix $\left(B^*\right)$ of the involute curve β is undefined.*

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