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Integral Variants of Jensen's Inequality for Convex Functions of Several Variables

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ABSTRACT

The paper focuses on the derivation of the integral variants of Jensen's inequality for convex functions of several variables. The work is based on the integral method, using convex combinations as input, and set barycentres as output.

Keywords and phrases:

convex combination; barycenter; convex function; Jensen's inequality

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1. INTRODUCTION

1.1. Combinations of Scalars and Vectors

Convex sets are generally observed in a real vector space $\mathbb{X}$. Affiliation to some vector set is analytically expressed by combinations of vectors (points) $x_i \in \mathbb{X}$ and scalars (coefficients) $p_i \in \mathbb{R}$. The combination

$$\sum_{i=1}^{n} p_i x_i \tag{1}$$

belongs to the vector subspace $\text{lin}\{x_i\}$ as the smallest vector space that contains all x_i, and it is called the linear combination. If $\sum_{i=1}^{n} p_i = 1$, the combination in (1) belongs to the affine hull $\text{aff}\{x_i\}$ as the smallest translated vector space that contains all x_i, and it is called the affine combination. If $\sum_{i=1}^{n} p_i = 1$ and all $p_i \in [0,1]$, the combination in (1) belongs to the convex hull $\text{conv}\{x_i\}$ as the smallest convex vector set that contains all x_i, and it is called the convex combination.

1.2. Two Basic Forms of Jensen's Inequality

In the discrete case, Jensen's inequality is applied to convex function and convex combinations of vectors from the convex set.

Theorem A. 1 *Let $\mathcal{S} = \{x_1, \ldots, x_n\}$ be a set of vectors x_i in a real vector space $\mathbb{X}$. Let $p : \mathcal{S} \to \mathbb{R}$ be a coefficient function, and $g : \mathcal{S} \to \mathbb{X}$ be a vector mapping.*

Every convex function $f : \text{conv}\{g(\mathcal{S})\} \to \mathbb{R}$ verifies the inequality

$$f\left(\sum_{i=1}^{n} p(x_i) g(x_i)\right) \le \sum_{i=1}^{n} p(x_i) f(g(x_i)) \tag{2}$$

if the function p is non-negative with $\sum_{i=1}^{n} p(x_i) = 1$, and the inequality

$$f\left(\frac{\sum_{i=1}^{n} p(x_i) g(x_i)}{\sum_{i=1}^{n} p(x_i)}\right) \le \frac{\sum_{i=1}^{n} p(x_i) f(g(x_i))}{\sum_{i=1}^{n} p(x_i)} \tag{3}$$

if the function p is either a non-negative or non-positive with $\sum_{i=1}^{n} p(x_i) \ne 0$.

In the integral case, Jensen's inequality is applied to convex function and integral arithmetic means of real valued integrable functions on the measurable set.

Theorem B. 2 *Let μ be a measure on a set $\mathcal{S}$. Let $p : \mathcal{S} \to \mathbb{R}$ be a μ-integrable function, and $g : \mathcal{S} \to \mathbb{R}$ be a μ-measurable function.*

Every convex function $f : \text{conv}\{g(\mathcal{S})\} \to \mathbb{R}$ verifies the inequality

$$f\left(\int_{\mathcal{S}} p(x) g(x) d\mu(x)\right) \le \int_{\mathcal{S}} p(x) f(g(x)) d\mu(x) \tag{4}$$

if the function p is non-negative with $\int_{\mathcal{S}} p(x) d\mu(x) = 1$, and the inequality

$$f\left(\frac{\int_{\mathcal{S}} p(x) g(x) d\mu(x)}{\int_{\mathcal{S}} p(x) d\mu(x)}\right) \le \frac{\int_{\mathcal{S}} p(x) f(g(x)) d\mu(x)}{\int_{\mathcal{S}} p(x) d\mu(x)} \tag{5}$$

if the function p is either a non-negative or non-positive with $\int_{\mathcal{S}} p(x) \neq 0$, provided that the functions pg and $pf(g)$ are μ-integrable in both cases.

1.3. Recent Result

Recall the following result:

Theorem C. [9, Theorem 1] *Let* p *be a non-negative continuous function on* $[a,b]$ *such that* $\int_a^b p(x)dx > 0$. *If* g *and* h *are real-valued continuous functions on* $[a,b]$ *and*

$$m_1 \le g(x) \le M_1, m_2 \le h(x) \le M_2$$

for all $x \in [a,b]$, *and* f *is convex on*

$$[m_1, M_1] \times [m_2, M_2],$$

then

$$f\left(\frac{\int_a^b p(x)g(x)dx}{\int_a^b p(x)dx}, \frac{\int_a^b p(x)h(x)dx}{\int_a^b p(x)dx} \right) \le \frac{\int_a^b p(x) f(g(x), h(x))dx}{\int_a^b p(x)dx} \tag{6}$$

and

$$f\left(\frac{\int_a^b g(x)dx}{b-a}, \frac{\int_a^b h(x)dx}{b-a} \right) \le \frac{1}{b-a} \int_a^b f(g(x), h(x))dx. \tag{7}$$

The inequalities hold in reversed order if f *is concave on* $[m_1, M_1] \times [m_2, M_2]$.

2. BARYCENTERS AND INTEGRAL ARITHMETIC MEANS

In this section we emphasize the basic meaning and significance of convex combinations. The main result is Lemma 2.1 which deals with a barycenter of a closed or open convex set in $\mathbb{R}^2$. All that follows in this section can be easily generalized to convex sets in $\mathbb{R}^n$.

2.1. Set Barycenter as a Limit of Set Centers

Let μ be a measure on a set $\mathcal{S} \subseteq \mathbb{R}^2$ with $\mu(\mathcal{S}) > 0$. Given a positive integer n, let

$$\mathcal{S} = \bigcup_{i=1}^{n} \mathcal{S}_{ni} \tag{8}$$

be the partition of the set $\mathcal{S}$ with pairwise disjoint μ-measurable sets $\mathcal{S}_{ni}$, where every $\mathcal{S}_{ni}$ contracts to the point or vanishes in infinity as n approaches infinity. Take one point $P_{ni} \in \mathcal{S}_{ni}$ for every index $i = 1, \ldots, n$ and denote with $\vec{r}_{ni}$ the radius-vector of the point P_{ni}. Consider the center P_n of the convex combination

$$\vec{r}_n = \sum_{i=1}^{n} \frac{\mu(\mathcal{S}_{ni})}{\mu(\mathcal{S})} \vec{r}_{ni}. \tag{9}$$

If the sequence $(P_n)_n$ approaches the point P as n approaches infinity, then the point P is called the barycenter of the set $\mathcal{S}$ with respect to the measure μ, and it will be denoted with $B(\mathcal{S}, \mu)$. Using point coordinates $P_{ni}(x_{ni}, y_{ni})$, we obtain the integral presentation

$$B(\mathcal{S},\mu) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{\mu(\mathcal{S}_{ni})}{\mu(\mathcal{S})}x_{ni},\sum_{i=1}^{n}\frac{\mu(\mathcal{S}_{ni})}{\mu(\mathcal{S})}y_{ni}\right) = \left(\frac{\iint_{\mathcal{S}} x\,d\mu(x,y)}{\mu(\mathcal{S})},\frac{\iint_{\mathcal{S}} y\,d\mu(x,y)}{\mu(\mathcal{S})}\right). \quad (10)$$

According to the above formula, the barycenter $B(\mathcal{S},\mu)$ belongs to the closure of the convex hull of the set $\mathcal{S}$. In the next lemma, relying on separation theorem we prove that the barycenter of a closed or open convex set belongs to the set.

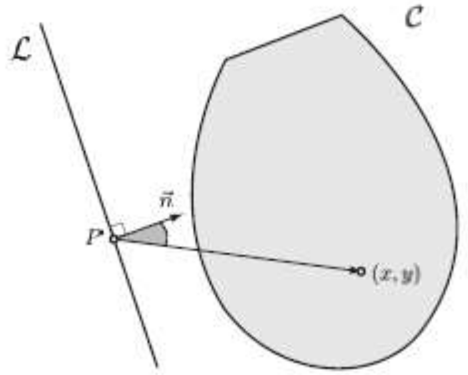

Figure 1. Barycenter of a closed or open convex set

Lemma 2.1. 3 *Let μ be a measure on a closed or open convex set $\mathcal{C}\subseteq\mathbb{R}^2$ with $\mu(\mathcal{C})>0$. If the barycenter of the set $\mathcal{C}$ with respect to the measure μ exists, then it belongs to to the set $\mathcal{C}$.*

Proof. Suppose that $P\notin\mathcal{C}$. Applying the separation theorem, we can choose the line $\mathcal{L}$ through the point P so that $\mathcal{L}\cap\mathcal{C}=\varnothing$, and the entire $\mathcal{C}$ is contained in one of the half-planes determined by $\mathcal{L}$. Let $\vec{n}$ be the vector normal to the line $\mathcal{L}$, oriented towards the set $\mathcal{C}$ as shown in Figure 1. Denote with $\vec{r}_{(x,y)} = x\vec{i}+y\vec{j}$ radius-vectors of the points $(x,y)\in\mathcal{C}$, and with $\vec{r}_P = x_P\vec{i}+y_P\vec{j}$ radius-vector of the barycenter P. The vector $\vec{n}$ makes acute angles with the vectors $\vec{r}_{(x,y)}-\vec{r}_P$. Applying the inner product, we find that the following holds for every pint $(x,y)\in\mathcal{C}$:

$$\begin{gathered}\left\langle \vec{n},\vec{r}_{(x,y)}-\vec{r}_P\right\rangle>0\\ \left\langle \vec{n},\vec{r}_{(x,y)}-\frac{1}{\mu(\mathcal{C})}\iint_{\mathcal{C}}\vec{r}_{(x,y)}\,d\mu(x,y)\right\rangle>0\\ \left\langle \vec{n},\vec{r}_{(x,y)}\right\rangle-\frac{1}{\mu(\mathcal{C})}\left\langle \vec{n},\iint_{\mathcal{C}}\vec{r}_{(x,y)}\,d\mu(x,y)\right\rangle>0\end{gathered} \quad (11)$$

Integrating the above inner products over $\mathcal{C}$, we get the contradiction

$$\left\langle \vec{n},\iint_{\mathcal{C}}\vec{r}_{(x,y)}\,d\mu(x,y)\right\rangle-\left\langle \vec{n},\iint_{\mathcal{C}}\vec{r}_{(x,y)}\,d\mu(x,y)\right\rangle>0. \quad (12)$$

So, it has to be $P\in\mathcal{C}$. □

The method of proving used in Lemma 2.1 can be applied to more dimensions. If the closed or open set $\mathcal{C}\subseteq\mathbb{R}^3$, then we use the plane $\mathcal{P}$ through the point P so that $\mathcal{P}\cap\mathcal{C}=\varnothing$. Similar procedure as in Lemma 2.1, without using integrals, was applied to convex polygon in [2] when determining the geometric position of the mass center.

2.2. Barycenter and Integral Arithmetic Mean of a Function

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We follow the ideas of the formula in (10). Let μ be a measure on a set $\mathcal{S}\subseteq\mathbb{R}^2$ with $\mu(\mathcal{S})>0$.

If $p:\mathcal{S}\to\mathbb{R}$ is either a non-negative or non-positive μ-integrable function so that $\iint_{\mathcal{S}} p(x,y)\,d\mu(x,y)\neq 0$, and the functions $p(x,y)x$ and $p(x,y)y$ are μ-integrable, then the point

$$B(p,\mathcal{S},\mu)=\left(\frac{\iint_{\mathcal{S}} p(x,y)x\,d\mu(x,y)}{\iint_{\mathcal{S}} p(x,y)\,d\mu(x,y)},\frac{\iint_{\mathcal{S}} p(x,y)y\,d\mu(x,y)}{\iint_{\mathcal{S}} p(x,y)\,d\mu(x,y)}\right) \tag{13}$$

could be called the barycenter of the function p on the set $\mathcal{S}$ with respect to the measure μ. This point belongs to the set $\operatorname{conv}\{\mathcal{S}\}$ by Lemma 2.1. For example, if $p=\rho$ is the mass density, then $B(\rho,\mathcal{S},\mu)$ represents the barycenter of the density ρ. Note that the invariant property

$$B(\alpha p,\mathcal{S},\mu)=B(p,\mathcal{S},\mu) \tag{14}$$

holds for every number $\alpha\in\mathbb{R}\setminus\{0\}$.

If we have a μ-integrable function $g:\mathcal{S}\to\mathbb{R}$, then we have the number

$$M(g,\mathcal{S},\mu)=\frac{\iint_{\mathcal{S}} g(x,y)\,d\mu(x,y)}{\mu(\mathcal{S})} \tag{15}$$

called the integral arithmetic mean of the function g on the set $\mathcal{S}$ with respect to the measure μ. Using convex combinations in the above formula would be seen that this number belongs to the interval $\operatorname{conv}\{g(\mathcal{S})\}$. Take as an example the random variable $g=X$. Then $M(X,\mathcal{S},\mu)=E(X)$ represents the expectation of the random variable X. At that the point $B(X,\mathcal{S},\mu)$ can be interpreted as the expected position of the variable X.

Connecting formulas between barycenters and means can be written using the projections $\mathrm{pr}_x(x,y)=x$ and $\mathrm{pr}_y(x,y)=y$:

$$B(\mathcal{S},\mu)=\left(M(\mathrm{pr}_x,\mathcal{S},\mu),M(\mathrm{pr}_y,\mathcal{S},\mu)\right) \tag{16}$$

$$B(p,\mathcal{S},\mu)=\left(\frac{M(p\cdot\mathrm{pr}_x,\mathcal{S},\mu)}{M(p,\mathcal{S},\mu)},\frac{M(p\cdot\mathrm{pr}_y,\mathcal{S},\mu)}{M(p,\mathcal{S},\mu)}\right) \tag{17}$$

Remark 2.2. 4 *A function barycenter can be reduced to a set barycenter. Let μ be a measure on a set $\mathcal{S}\subseteq\mathbb{R}^2$, and $p:\mathcal{S}\to\mathbb{R}$ be a non-negative μ-integrable function with $\iint_{\mathcal{S}} p(x,y)\,d\mu(x,y)>0$. We define the measure ν on $\mathcal{S}$ putting*

$$\nu(\mathcal{A})=\iint_{\mathcal{A}} p(x,y)\,d\mu(x,y) \tag{18}$$

for *every μ-measurable set $\mathcal{A}\subseteq\mathcal{S}$. Since $\nu(\mathcal{S})>0$ and $\mu(\mathcal{S})>0$, we have*

$$B(\mathcal{S},\nu)=B(p,\mathcal{S},\mu). \tag{19}$$

The properties of the measure ν defined in (18) can be seen in [8, Theorem 1.29]. More on quantity centers and barycenters of the Riemann integrable quantity functions can be found in [5, Sections 3-4].

The general concept of barycenter in the framework of Choquet's theory was presented in [4, Section 6]. In mentioned paper, barycenters of compact convex subsets of a locally convex Hausdorff space were observed with respect to Borel probability measures.

3. JENSEN'S INEQUALITY FOR CONVEX FUNCTIONS OF SEVERAL VARIABLES

In this section we emphasize the practical meaning and application of convex combinations. The main result is Theorem 3.4 which presents Jensen's inequality for convex functions of two variables, and it can be easily generalized to convex functions of more variables. The convex sets that will be used will be closed or open.

3.1. Integral Variants of Inequality

Let $\mathcal{C} \subseteq \mathbb{R}^2$ be a convex set, $\vec{r}_i = x_i\vec{i} + y_i\vec{j}$ be radius-vectors of the points $P_i(x_i, y_i) \in \mathcal{C}$, and $p_i \in \mathbb{R}$ be non-negative coefficients with $\sum_{i=1}^{n} p_i = 1$. If $f : \mathcal{C} \to \mathbb{R}$ is a convex function with two variables, then the inequality in (2) takes the form

$$f\left(\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i\right) \le \sum_{i=1}^{n} p_i f(x_i, y_i). \tag{20}$$

The above inequality can be applied to obtain Jensen's inequality for the set barycenter:

Lemma 3.1. 5 *Let μ be a measure on a convex set $\mathcal{C} \subseteq \mathbb{R}^2$ with $\mu(\mathcal{C}) > 0$.*

Every two variables μ-integrable convex function $f : \mathcal{C} \to \mathbb{R}$ verifies the inequality

$$f\left(\frac{\iint_{\mathcal{C}} x d\mu(x,y)}{\mu(\mathcal{C})}, \frac{\iint_{\mathcal{C}} y d\mu(x,y)}{\mu(\mathcal{C})}\right) \le \frac{\iint_{\mathcal{C}} f(x,y) d\mu(x,y)}{\mu(\mathcal{C})} \tag{21}$$

provided that the projections $\mathrm{pr}_x(x,y) = x$ and $\mathrm{pr}_y(x,y) = y$ are μ-integrable on $\mathcal{C}$.

Proof. Using a partition $\mathcal{C} = \cup_{i=1}^{n} \mathcal{C}_{ni}$ that satisfies the conditions of the partition in (8), we form the convex combination

$$\vec{r}_n = \sum_{i=1}^{n} \frac{\mu(\mathcal{C}_{ni})}{\mu(\mathcal{C})} \vec{r}_{ni} = \sum_{i=1}^{n} \frac{\mu(\mathcal{C}_{ni})}{\mu(\mathcal{C})} x_{ni}\vec{i} + \sum_{i=1}^{n} \frac{\mu(\mathcal{C}_{ni})}{\mu(\mathcal{C})} y_{ni}\vec{j}, \tag{22}$$

of the radius-vectors $\vec{r}_{ni} = x_{ni}\vec{i} + y_{ni}\vec{j}$ of the points $P_{ni}(x_{ni}, y_{ni}) \in \mathcal{C}_{ni}$ with the coefficients $p_{ni} = \mu(\mathcal{C}_{ni}) / \mu(\mathcal{C})$. Applying the inequality in (20) on its center $P_n \in \mathcal{C}$, it follows

$$f\left(\sum_{i=1}^{n} \frac{\mu(\mathcal{C}_{ni})}{\mu(\mathcal{C})} x_{ni}, \sum_{i=1}^{n} \frac{\mu(\mathcal{C}_{ni})}{\mu(\mathcal{C})} y_{ni}\right) \le \sum_{i=1}^{n} \frac{\mu(\mathcal{C}_{ni})}{\mu(\mathcal{C})} f(x_{ni}, y_{ni}). \tag{21}$$

The sequence $(P_n)_n$ converges to the barycenter P by assumption (the projections are μ-integrable). The convex function f is continuous on the interior of the set $\mathcal{C}$. The discrete inequality in (23) approaches the integral inequality in (21) as n approaches infinity, respecting the continuity rule

$$\lim_{n\to\infty} f(P_n) = f(\lim_{n\to\infty} P_n)$$

if the set $\mathcal{C}$ is open. If the set $\mathcal{C}$ is closed, and if the function f is not continuous on the whole $\mathcal{C}$, then the right-hand side of (21) can only be increased. □

Smoothness properties of convex functions are presented in [3, pages 20-26].

Remark 3.2. 6 *If $f : \mathcal{C} \to \mathbb{R}$ is the Riemann integrable convex function, then the inequality in (21) reads as follows:*

$$f\left(\frac{\iint_{\mathcal{C}} x dxdy}{\iint_{\mathcal{C}} dxdy}, \frac{\iint_{\mathcal{C}} y dxdy}{\iint_{\mathcal{C}} dxdy}\right) \le \frac{\iint_{\mathcal{C}} f(x,y) dxdy}{\iint_{\mathcal{C}} dxdy}. \tag{24}$$

Using notations for the barycenter and integral arithmetic mean the inequality in (21) can be written as

$$f\left(B(\mathcal{C},\mu)\right)\le M(f,\mathcal{C},\mu). \tag{25}$$

Corollary 3.3. 7 *Let* μ *be a measure on a convex set* $\mathcal{C}\subseteq\mathbb{R}^2$*. Let* $p:\mathcal{C}\to\mathbb{R}$ *be either a non-negative or non-positive* μ *-integrable function with* $\iint_{\mathcal{C}} p(x,y)d\mu(x,y)\neq 0$.

Every two variables convex function $f:\mathcal{C}\to\mathbb{R}$ *verifies the inequality*

$$f\left(\frac{\iint_{\mathcal{C}} p(x,y)x\,d\mu(x,y)}{\iint_{\mathcal{C}} p(x,y)d\mu(x,y)},\frac{\iint_{\mathcal{C}} p(x,y)y\,d\mu(x,y)}{\iint_{\mathcal{C}} p(x,y)d\mu(x,y)}\right)\le\frac{\iint_{\mathcal{C}} p(x,y)f(x,y)d\mu(x,y)}{\iint_{\mathcal{C}} p(x,y)d\mu(x,y)} \tag{26}$$

provided that the functions px*,* py *and* pf *are* μ *-integrable.*

Proof. Should start from the equality in (22) with the coefficients

$$p_{ni}=\frac{\mu(\mathcal{C}_{ni})p(x_{ni},y_{ni})}{\sum_{i=1}^{n}\mu(\mathcal{C}_{ni})p(x_{ni},y_{ni})}, \tag{27}$$

and then follow the proof of Lemma 3.1. □

The inequality in (26), after dividing both numerator and denominator of the quotient of the right-hand side with $\mu(\mathcal{C})$, takes the form

$$f\left(B(p,\mathcal{C},\mu)\right)\le\frac{M(pf,\mathcal{C},\mu)}{M(p,\mathcal{C},\mu)}. \tag{28}$$

Corollary 3.3 can be generalized by introducing the mapping g consisting of two functions of two variables:

Theorem 3.4. 8 *Let* μ *be a measure on a set* $\mathcal{S}\subseteq\mathbb{R}^2$*. Let* $p:\mathcal{S}\to\mathbb{R}$ *be either a non-negative or non-positive* μ *-integrable function with* $\iint_{\mathcal{S}} p(x,y)d\mu(x,y)\neq 0$*, and* $g=(g_1,g_2):\mathcal{S}\to\mathbb{R}^2$ *be a mapping consisting of* μ *-measurable functions* g_1 *and* g_2*.*

Every two variables convex function $f:\mathrm{conv}\{g(\mathcal{S})\}\to\mathbb{R}$ *verifies the inequality*

$$\begin{aligned} f&\left(\frac{\iint_{\mathcal{S}} p(x,y)g_1(x,y)d\mu(x,y)}{\iint_{\mathcal{S}} p(x,y)d\mu(x,y)},\frac{\iint_{\mathcal{S}} p(x,y)g_2(x,y)d\mu(x,y)}{\iint_{\mathcal{S}} p(x,y)d\mu(x,y)}\right)\\ &\le\frac{\iint_{\mathcal{S}} p(x,y)f(g_1(x,y),g_2(x,y))d\mu(x,y)}{\iint_{\mathcal{S}} p(x,y)d\mu(x,y)}\end{aligned} \tag{29}$$

provided that the functions pg_1*,* pg_2 *and* $pf(g_1,g_2)$ *are* μ *-integrable.*

Proof. Applying the proof of Lemma 3.1 with the radius-vectors

$$\vec{r}_{ni}=g_1(x_{ni},y_{ni})\vec{i}+g_2(x_{ni},y_{ni})\vec{j} \tag{30}$$

and the coefficients

$$p_{ni}=\frac{\mu(\mathcal{S}_{ni})p(x_{ni},y_{ni})}{\sum_{i=1}^{n}\mu(\mathcal{S}_{ni})p(x_{ni},y_{ni})}, \tag{31}$$

we get the integral inequality in (29). □

Applying the notation of the integral arithmetic means the inequality in (29) can also be presented with

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$$f\left(\frac{M(pg_1,\mathcal{S},\mu)}{M(p,\mathcal{S},\mu)},\frac{M(pg_2,\mathcal{S},\mu)}{M(p,\mathcal{S},\mu)}\right)\le\frac{M(pf(g),\mathcal{S},\mu)}{M(p,\mathcal{S},\mu)}. \tag{32}$$

3.2. Discrete-Integral Variant of Inequality

Combining and connecting the inequalities in (20) and (29) as the inequalities for convex function with n variables, it follows:

Corollary 3.5. 9 *Let μ_j be measures on a set $\mathcal{S}\subseteq\mathbb{R}^n$. Let $p_j:\mathcal{S}\to\mathbb{R}$ be either non-negative or non-positive μ_j-integrable functions with the integral values $\int_{\mathcal{S}}p_j(x_1,\dots,x_n)d\mu_j(x_1,\dots,x_n)\ne 0$, and $g_j=(g_{1j},\dots,g_{nj}):\mathcal{S}\to\mathbb{R}^n$ be mappings consisting of μ_j-measurable functions g_{ij}, and $\alpha_j\in[0,1]$ be coefficients of the sum $\sum_{j=1}^m\alpha_j=1$.*

Every n-variables convex function $f:\mathrm{conv}\{\cup_{j=1}^m g_j(\mathcal{S})\}\to\mathbb{R}$ verifies the inequality

$$f\left(\sum_{j=1}^m\alpha_j\frac{\int_{\mathcal{S}}p_jg_{1j}\,d\mu_j}{\int_{\mathcal{S}}p_j\,d\mu_j},\dots,\sum_{j=1}^m\alpha_j\frac{\int_{\mathcal{S}}p_jg_{nj}\,d\mu_j}{\int_{\mathcal{S}}p_j\,d\mu_j}\right)\le\sum_{j=1}^m\alpha_j\frac{\int_{\mathcal{S}}p_j\,f(g_{1j},\dots,g_{nj})d\mu_j}{\int_{\mathcal{S}}p_j\,d\mu_j} \tag{33}$$

provided all the functions $p_jg_{1j},\dots,p_jg_{nj}$ and $p_j\,f(g_{1j},\dots,g_{nj})$ are μ_j-integrable.

Proof. If $\mathcal{C}=\mathrm{conv}\{\cup_{j=1}^m g_j(\mathcal{S})\}$, then all the points $P_j(x_{1j},\dots,x_{nj})$ with the coordinates

$$x_{1j}=\frac{\iint_{\mathcal{S}}p_jg_{1j}\,d\mu_j}{\iint_{\mathcal{S}}p_j\,d\mu_j},\dots,x_{nj}=\frac{\iint_{\mathcal{S}}p_jg_{nj}\,d\mu_j}{\iint_{\mathcal{S}}p_j\,d\mu_j} \tag{34}$$

belong to $\mathcal{C}$. Applying first the discrete inequality in (20) with the points P_j, then the integral inequality in (29), we get the inequality in (33).

Inequalities with convex combinations and barycenters for convex functions of one variable were observed in [7]. Some inequalities that include barycenters expressed with the Riemann integrals were obtained in [1, Chapter 4], in studying the Hermite-Hadamard inequality.

4. INEQUALITY WITH HYPERPLANE

Take the three planar points $A(x_A,y_A)$, $B(x_B,y_B)$ and $C(x_C,y_C)$ that do not belong to one line. If $\vec{r}_A$, $\vec{r}_B$ and $\vec{r}_C$ are its radius-vectors, the radius-vector $\vec{r}_P$ of any point $P(x,y)\in\mathbb{R}^2$ is presented by the unique affine combination

$$\vec{r}_P=\alpha_P\vec{r}_A+\beta_P\vec{r}_B+\gamma_P\vec{r}_C, \tag{35}$$

where

$$\alpha_P=\frac{\begin{vmatrix}x & y & 1\\ x_B & y_B & 1\\ x_C & y_C & 1\end{vmatrix}}{\begin{vmatrix}x_A & y_A & 1\\ x_B & y_B & 1\\ x_C & y_C & 1\end{vmatrix}},\beta_P=-\frac{\begin{vmatrix}x & y & 1\\ x_A & y_A & 1\\ x_C & y_C & 1\end{vmatrix}}{\begin{vmatrix}x_A & y_A & 1\\ x_B & y_B & 1\\ x_C & y_C & 1\end{vmatrix}},\gamma_P=\frac{\begin{vmatrix}x & y & 1\\ x_A & y_A & 1\\ x_B & y_B & 1\end{vmatrix}}{\begin{vmatrix}x_A & y_A & 1\\ x_B & y_B & 1\\ x_C & y_C & 1\end{vmatrix}}. \tag{36}$$

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If $f:\mathbb{R}^2 \to \mathbb{R}$ is a two variables function, and $z = f^{\text{pla}}_{\{A,B,C\}}(x,y)$ is the equation of the plane passing through the points $(A, f(A))$, $(B, f(B))$ and $(C, f(C))$ of the graph of f, then

$$f^{\text{pla}}_{\{A,B,C\}}(x,y) = \alpha_P f(A) + \beta_P f(B) + \gamma_P f(C). \tag{37}$$

If the function f is convex, the plane inequality

$$f(P) \le \alpha_P f(A) + \beta_P f(B) + \gamma_P f(C) = f^{\text{pla}}_{\{A,B,C\}}(P) \tag{38}$$

holds for every point $P \in \operatorname{conv}\{A,B,C\}$, because for these points the combination in (35) is convex. If $P \notin \operatorname{conv}\{A,B,C\}$, the reverse inequality is not necessarily valid in (38).

Let $A_1,\dots,A_{n+1} \in \mathbb{R}^n$ be points so that vectors $\vec{r}_{A_1} - \vec{r}_{A_{n+1}},\dots,\vec{r}_{A_n} - \vec{r}_{A_{n+1}}$ are linearly independent. The convex hull $\operatorname{conv}\{A_1,\dots,A_{n+1}\}$ is called the n-simplex in $\mathbb{R}^n$ with the vertices $A_1,\dots,A_{n+1}$. Geometrically speaking, all the simplex vertices can not belong to the same hyperplane in $\mathbb{R}^n$. If $A_k = A_k(x_{k1},\dots,x_{kn})$, the radius-vector $\vec{r}_P$ of any point $P(x_1,\dots,x_n) \in \mathbb{R}^n$ is presented by the unique affine combination

$$\vec{r}_P = \sum_{k=1}^{n+1} \alpha_k \vec{r}_{A_k}, \tag{39}$$

where the numerator and denominator of the coefficients α_k are

$$\alpha_k^{\text{num}} = (-1)^{k+1}\begin{vmatrix} x_1 & \dots & x_n & 1 \\ x_{11} & \dots & x_{1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{k-11} & \dots & x_{k-1n} & 1 \\ x_{k+11} & \dots & x_{k+1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n+11} & \dots & x_{n+1n} & 1 \end{vmatrix}, \qquad \alpha_k^{\text{den}} = \begin{vmatrix} x_{11} & \dots & x_{1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{k-11} & \dots & x_{k-1n} & 1 \\ x_{k1} & \dots & x_{kn} & 1 \\ x_{k+11} & \dots & x_{k+1n} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n+11} & \dots & x_{n+1n} & 1 \end{vmatrix}. \tag{40}$$

If $f:\mathbb{R}^n \to \mathbb{R}$ is an n-variables function, and $x_{n+1} = f^{\text{hyp}}_{\{A_k\}}(x_1,\dots,x_n)$ is the equation of the hyperplane (in $\mathbb{R}^{n+1}$) passing through the points $(A_k, f(A_k))$ of the graph of f, then

$$f^{\text{hyp}}_{\{A_k\}}(x_1,\dots,x_n) = \sum_{k=1}^{n+1} \alpha_k f(A_k). \tag{41}$$

If f is convex, the hyperplane inequality

$$f(P) \le \sum_{k=1}^{n+1} \alpha_k f(A_k) = f^{\text{hyp}}_{\{A_k\}}(P) \tag{42}$$

holds for every point $P \in \operatorname{conv}\{A_1,\dots,A_{n+1}\}$. If $P \notin \operatorname{conv}\{A_1,\dots,A_{n+1}\}$, the reverse inequality is not necessarily valid in (42).

We finish the paper by extending the inequality in (33) to double inequality which includes n-simplex and hyperplane. That version stands as follows:

Theorem 4.1. 10 *Let* $\mathcal{C} = \mathrm{conv}\{A_1, \ldots, A_{n+1}\}$ *be an* n*-simplex in* $\mathbb{R}^n$*,* $f : \mathcal{C} \to \mathbb{R}$ *be a convex function, and* $x_{n+1} = f_{\{A_k\}}^{\mathrm{hyp}}(x_1, \ldots, x_n)$ *be the hyperplane passing through all the points* $(A_k, f(A_k))$ *of the graph of* f*. Let* μ_j *be measures on a set* $\mathcal{S} \subseteq \mathbb{R}^n$*,* $p_j : \mathcal{S} \to \mathbb{R}$ *be either non-negative or non-positive* μ_j*-integrable functions with the integral values* $\int_{\mathcal{S}} p_j(x_1, \ldots, x_n) d\mu_j(x_1, \ldots, x_n) \neq 0$*, and* $g_j = (g_{1j}, \ldots, g_{nj}) : \mathcal{S} \to \mathcal{C}$ *be mappings consisting of* μ_j*-measurable functions* g_{ij}*, and* $\alpha_j \in [0,1]$ *be coefficients of the sum* $\sum_{j=1}^{m} \alpha_j = 1$*.*

Then the double inequality

$$f\left(\sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j g_{1j} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j}, \ldots, \sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j g_{nj} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j} \right) \leq \sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j f(g_{1j}, \ldots, g_{nj}) d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j} \leq f_{\{A_k\}}^{\mathrm{hyp}}\left(\sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j g_{1j} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j}, \ldots, \sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j g_{nj} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j} \right) \tag{43}$$

holds if provided that all the functions $p_j g_{1j}, \ldots, p_j g_{nj}$ *and* $p_j f(g_{1j}, \ldots, g_{nj})$ *are* μ_j*-integrable.*

Proof. The left-hand side of the inequality in (43) follows from Corollary 3.5. Let us prove the right-hand side. Using the inequality in (42) and the hyperplane affine presentation

$$f_{\{A_k\}}^{\mathrm{hyp}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i + a_{n+1}, \tag{44}$$

we get

$$\sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j f(g_{1j}, \ldots, g_{nj}) d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j} \leq \sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j f_{\{A_k\}}^{\mathrm{hyp}}(g_{1j}, \ldots, g_{nj}) d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j} = \sum_{j=1}^{m} \alpha_j f_{\{A_k\}}^{\mathrm{hyp}}\left(\frac{\int_{\mathcal{S}} p_j g_{1j} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j}, \ldots, \frac{\int_{\mathcal{S}} p_j g_{nj} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j} \right) = f_{\{A_k\}}^{\mathrm{hyp}}\left(\sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j g_{1j} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j}, \ldots, \sum_{j=1}^{m} \alpha_j \frac{\int_{\mathcal{S}} p_j g_{nj} d\mu_j}{\int_{\mathcal{S}} p_j d\mu_j} \right) \tag{45}$$

which ends the proof. □

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