



Partial Orders in BCL^+ -Algebra

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ABSTRACT

In this paper, we introduce a partially ordered BCL^+ -algebra, and we show that some of its fundamental properties.

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BCL -algebras; BCL^+ -algebras; Partial Order; Lattice

Academic Discipline And Sub-Disciplines:

Mathematics; Algebra.

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Type (Method/Approach)

The paper is proof that main theorem in BCL^+ -algebra.



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1 INTRODUCTION

Yonghong Liu introduced a new class of abstract algebras: BCL-algebras (see [1]). Recently, Yonghong Liu introduced a wide class of abstract algebras: BCL⁺-algebras (see [2]). In [3], Deena Al-Kadi and Rodyna Hosny introduced the deformation of BCL-algebra. They continue to investigate the relation between BCL-algebra and d/BCH/BCI and BCK-algebra.

We know that a partial order is an extremely important class of binary relation and useful in mathematics. In this paper we want to achieve it through this present paper can go on studying the partially ordered BCL⁺-algebra, which we will need the following definitions and theorems:

Definition 1.1: [2, Definition 2.1] An algebra $(X; *, 1)$ is called a BCL⁺-algebra if it satisfies the following laws hold: for any $x, y, z \in X$

- 1) BCL⁺-1: $x * x = 1$.
- 2) BCL⁺-2: $x * y = 1$ and $y * x = 1$ imply $x = y$.
- 3) BCL⁺-3: $((x * y) * z) * ((x * z) * y) = (z * y) * x$.

Theorem 1.1: [2, Theorem 2.2] Assume that $(X; *, 1)$ is any a BCL⁺-algebra. Then the following hold: for any $x, y, z \in X$

- 1) $(x * (x * y)) * y = 1$.
- 2) $x * 1 = x$ imply $x = 1$.
- 3) $((x * y) * (x * z)) * (z * y) = 1$.
- 4) BCL⁺-2: $x * y = 1$ and $y * x = 1$ imply $x = y$.

Theorem 1.2: [2, Theorem 2.3] An algebra $(X; *, 1)$ is a BCL⁺-algebra if and only if it satisfies the following conditions: for $\forall x, y, z \in X$

- 1) BCL⁺-1: $x * x = 1$.
- 2) BCL⁺-2: $x * y = 1$ and $y * x = 1$ imply $x = y$.
- 3) $((x * y) * z) * ((x * z) * y) * ((z * y) * x) = 1$.
- 4) $x * (1 * y) = x$.

Definition 1.2: [2, Definition 2.2] Suppose that $(X; *, 1)$ is a BCL⁺-algebra, the ordered relation if $x \leq y$ if and only if $x * y = 1$, for all $x, y \in X$, then $(X; \leq)$ is partially ordered set and $(X; *, 1)$ is an algebra of partially ordered relation.

Corollary 1.1: [2, Corollary 2.1] Let every $x \in X$. Then 1(one) is maximal element in a BCL⁺-algebra $(X; *, 1)$ such that integral $1 \leq x$ imply $x = 1$.

2 MAIN RESULTS

If a BCL⁺-algebra $(X; *, 1)$ is a partially ordered relation \leq on X , then we have the following definitions:

Definition 2.1: Let $(X; *, 1)$ be a BCL⁺-algebra with 1 (one) the unit element. If there exists a partial ordering \leq on X such that the isotonic property of the multiplication $*$ on X holds, we have

$$x \leq y \text{ implies } zx \leq zy \text{ for any } x, y, z \in X.$$

$(X; *, \leq, 1)$ is said to be a partially ordered BCL⁺-algebra ("BCL_≤⁺-algebra" forshort). Especially, it is called integral if 1 (one) is the maximal element of X , with respect to \leq provided.



Definition 2.2: Let $(X; *, 1)$ be a BCL_{\leq}^+ -algebra, for any nonempty subset Y of a BCL^+ -algebra X , if $\forall x, y \in Y$, we have $x * y \in Y$, then we call $(Y; *, 1)$ be a subalgebra of $(X; *, 1)$. Apparently, $(Y; *, 1)$ should also be a BCL_{\leq}^+ -algebra.

Definition 2.3: Let $(X; *, 1)$ be a BCL_{\leq}^+ -algebra, if $\forall x, y, z \in X$, then

- 1) BCL_{\leq}^+ -1: $x \leq x$.
- 2) BCL_{\leq}^+ -2: If $x \leq y$ and $y \leq x$, then $x = y$.
- 3) BCL_{\leq}^+ -3: $((x * y) * z) * ((x * z) * y) \leq (z * y) * x$.

Theorem 2.1: Let $(X; *, 1)$ be a BCL_{\leq}^+ -algebra. Then binary relation \leq is a partial order on X .

Proof: If $x \leq y$ and $y \leq z$, then

$$x * y = 1 \text{ and } y * z = 1.$$

Since

$$x = y \text{ and } y = z.$$

(By Theorem 1.1. 3), we have

$$\begin{aligned} x * z &= ((x * z) * 1) * 1 \\ &= ((x * z) * (x * y)) * (y * z) \\ &= ((x * y) * (x * z)) * (z * y), \\ &= 1 \end{aligned}$$

and thus $x \leq z$. Finally, by Definition 2.3. 1) and 2). Then \leq is proved.

Theorem 2.2: Suppose that $(X; *, 1)$ be a BCL^+ -algebra, we have that

$$((x * y) * (x * z)) * (z * y) = 1, \text{ for } \forall x, y, z \in X,$$

where $(X; \leq)$ is partially ordered set. Then BCL_{\leq}^+ -algebra is $(X; *, 1)$.

Proof: Let BCL^+ -algebra of $(X; \leq)$ be the $(X; *, \leq, 1)$, for $\forall x, y, z \in X$. If $x \leq y$, $x \leq z$ and $z \leq y$, then

$$x * y = 1, x * z = 1 \text{ and } z * y = 1,$$

we see that

$$x *^{\leq} y = 1, x *^{\leq} z = 1 \text{ and } z *^{\leq} y = 1.$$

We can write

$$x * y = x *^{\leq} y, x * z = x *^{\leq} z \text{ and } z * y = z *^{\leq} y.$$

In another aspect, if $x \not\leq y$, $x \not\leq z$ and $z \not\leq y$, then

$$x *^{\leq} y = x, x *^{\leq} z = x \text{ and } z *^{\leq} y = z,$$

we have

$$x * y \neq 1, x * z \neq 1 \text{ and } z * y \neq 1.$$

Since

$$((x * y) * (x * z)) * (z * y) = 1.$$



We conclude that

$$(x * y) * (x * z) \leq z * y,$$

but we see that

$$\begin{aligned} (z * y) * ((x * y) * (x * z)) &= (x * (x * 1)) * ((x * (x * 1)) * (x * x)) \\ &= (x * (x * 1)) * ((x * (x * 1)) * 1) \\ &= (x * (x * 1)) * (x * (x * 1)) \\ &= 1 \end{aligned}$$

(Substituting $x * 1$ for y and x for z), therefore, we have

$$x * y = x, x * z = x \text{ and } z * y = z,$$

i.e.,

$$x * y = x \leq y, x * z = x \leq z \text{ and } z * y = z \leq y.$$

The proof is now complete.

Corollary 2.1: Let $(X; *, 1)$ be a BCL_{\leq}^+ -algebra. Then

$$(x * y) * (x * z) \leq z * y \text{ for } \forall x, y, z \in X.$$

Theorem 2.3: Let $(X; *, 1)$ be a BCL^+ -algebra. Then 1(one) is maximal element of partial order \leq of a BCL^+ -algebra.

Proof: Let $1 \leq x$ for $\forall x \in X$, then $x * 1 = x = 1$, and the proof is complete.

Corollary 2.2: Let p be a maximal element of BCL^+ -algebra, $p \in X$. Then $p < x$.

Definition 2.4: Suppose that L is a lower semi-lattice, which is $(X; \leq)$ together with subset of S from homomorphism. Then we call lower semi-lattice L is according to the S representation.

Definition 2.5: Let partial order of lower semi-lattice is denoted by \leq^{\wedge} and suppose that lower semi-lattice L be a BCL_{\leq}^+ -algebra. Then

$$x \leq^{\wedge} y, x \leq^{\wedge} z \text{ and } z \leq^{\wedge} y,$$

i.e.,

$$x * y = x, x * z = x \text{ and } z * y = z,$$

and the lower semi-lattice partial order set is denoted by $(X; \leq^{\wedge})$.

Theorem 2.4: Let $(X; *, 1)$ be a BCL_{\leq}^+ -algebra and let $(X; \leq^{\wedge})$ be a nonempty set. Then $(X; \leq^{\wedge})$ is a lower semi-lattice and $x * y$ is greatest lower bound in $\{x, y\}$.

Proof: The proof of such \leq^{\wedge} is not intricate plot or to be more precise that satisfies the three relations: reflexive, antisymmetric and transitive. Next, we prove that the back of the conclusion.

Suppose w is the lower bound of the set $\{x, y\}$. Then

$$w \leq^{\wedge} x \text{ and } w \leq^{\wedge} y.$$

Apply Definition 2.5, we have

$$w * x = w \text{ and } w * y = w.$$

Then



$$w * (x * y) = w,$$

we can write

$$w \leq^{\wedge} x * y,$$

as desired.

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