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The Natural Lift Curve and Geodesic Sprays in Euclidean 4-Space

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ABSTRACT

In this work, we presented the natural lift curves of the spherical indicatrices of a curve in E^4. Also, some interesting results about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T(S^2)$.

Keywords:

Natural Lift; Geodesic Spray; Spherical indicatrix.

Mathematics Subject Classification:

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1. INTRODUCTION AND PRELIMINARIES

Thorpe gave the concepts of the natural lift curve and geodesic spray and proved the natural lift curve is an integral curve of the geodesic spray iff the original curve is an geodesic on M in [3]. Çalışkan et al. studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve and gave some interesting results about the original curve were obtaied, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T(S^2)$ in [5]. To meet the requirements in the next section, here, the basic elements of the theory of curves in the space E^4 are briefly presented (A more complete elemantary treatment can be found in [2]).

Let $\alpha: I \subset IR \to E^4$ be regular curve with arclength parameter s, $\langle , \rangle$ is the standard scalar product of E^4 given by

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

for each vectors $X = (x_1, x_2, x_3, x_4), Y = (y_1, y_2, y_3, y_4) \in E^4$. The norm of a vector $X = (x_1, x_2, x_3, x_4) \in E^4$ is defined by

$$\|X\| = \sqrt{|\langle X, X \rangle|}$$

The hypersphere of radius 1 and center 0 in E^4 is given by

$$S^3 = \{X = (x_1, x_2, x_3, x_4) \in E^4 : \langle X, X \rangle = 1\}.$$

We denote the moving Frenet frame along the curve α by $\{T(s), N(s), B_1(s), B_2(s)\}$ where T, N, B_1 and B_2 are the tangent, the principal normal, the first binormal vector and the second binormal vector of the curve α, respectively. The functions κ_1, κ_2 and κ_3 are called the first, second and third curvature of α. Then, Frenet formulas are given by

$$\dot{T} = \kappa_1 N,\ \dot{N} = -\kappa_1 T + \kappa_2 B_1,\ \dot{B}_1 = -\kappa_2 N + \kappa_3 B_2,\ \dot{B}_2 = -\kappa_3 B_1, \text{ [3]}.$$

The following properties characterize some special curves;

$\kappa_1 = 0$ if and only if α is a straight line,

$\kappa_2 = 0$ if and only if α is a planar curve,

$\kappa_3 = 0$ if and only if α is lying in a three dimensional supspace of E^4,

$\kappa_2 = 0$ and $\kappa_1 = \text{constant} > 0$ if and only if α is a circle,

$\kappa_3 = 0$ and $\kappa_2 = c_2$, $\kappa_1 = c_1$ $(c_1, c_2 \in IR_0)$ if and only if α is a circular helix,

$\kappa_3 = c_3$ and $\kappa_2 = c_2$, $\kappa_1 = c_1$ $(c_1, c_2, c_3 \in IR_0)$ if and only if

$$\alpha(s) = \frac{1}{\lambda_1}\sin(\lambda_1 s)V_1 - \frac{1}{\lambda_1}\cos(\lambda_1 s)V_2 + \frac{1}{\lambda_2}\sin(\lambda_2 s)V_3 \frac{1}{\lambda_2}\cos(\lambda_2 s)V_4$$

with $\lambda_1^2 = \dfrac{K - \sqrt{K^2 - 4c_1^2c_3^2}}{2}, \lambda_2^2 = \dfrac{K + \sqrt{K^2 - 4c_1^2c_3^2}}{2}$ and $K = c_1^2 + c_2^2 + c_3^2$. $V_i (1 \le i \le 4)$ are orthogonal, constant vectors satifying the following conditions $\langle V_1, V_1 \rangle = \langle V_2, V_2 \rangle$ and $\langle V_3, V_3 \rangle = \langle V_4, V_4 \rangle$. This curve α lies on a sphere with radius $\dfrac{1}{|c_3|}$,[3].

2. THE NATURAL LIFT CURVES AND GEODESIC SPRAYS

Definition 2.1

Let M be a hypersurface in E^4 and let $\alpha: I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{dt}\big(\alpha(t)\big) = X\big(\alpha(t)\big), \forall t \in I$$

where X is a smooth tangent vector field on M, [1]. Thus we can write

$$TM = \bigcup_{p \in M} T_pM = \chi(M),$$

where T_pM is the tangent space of M at P and $\chi(M)$ is the space of vector fields of M.

Definition 2.2

For any parametrized curve $\alpha: I \to M$, $\bar{\alpha}: I \to TM$ given by

$$\bar{\alpha}(t) = \big(\alpha(t), \dot{\alpha}(t)\big) = \dot{\alpha}(t)\big|_{\alpha(t)}$$

is called the natural lift of α on TM, [4]. Thus, we can write

$$\frac{d\bar{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}(t)\big|_{\alpha(t)}\right) = \nabla_{\dot{\alpha}(t)}\dot{\alpha}(t)$$

where ∇ is the Levi-Civita connection on E^4.

Definition 2.3

A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$,

$$X(V) = -\langle S(V), V\rangle N, \text{ [4].}$$

Theorem 2.1

The natural lift $\bar{\alpha}$ of the curve α is an integral curve of geodesic spray X if and only if α is a geodesic on M, [5].

Let ∇ and ∇ be Levi-Civita connections on E^4 and S^3 respectively and ξ be a unit normal vector field of S^3. Then the Gauss Equation are given by the following

$$\nabla_X Y = \bar{\nabla}_X Y - \langle S(X), Y\rangle \xi,$$

where S is the shape operator of S^3.

(i) The natural lift of the spherical indicatrix of the tangent vectors of α

Let α_T be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_T$ be the natural lift of the curve α_T. If $\bar{\alpha}_T$ is an integral curve of the geodesic spray, then from Theorem 2.1 we have

$$\bar{\nabla}_{\dot{\alpha}_T}\dot{\alpha}_T = 0,$$

that is

$$\left(\kappa_1^2 - \kappa_1\right)T + \frac{\dot{\kappa}_1}{\kappa_1}N + \kappa_2 B_1 = 0$$

Since T, N, B_1, B_2 are linearly independent, we have

$$\begin{cases} \kappa_1 = 1 \\ \kappa_2 = 0 \end{cases}$$

Thus we can give the following corollary:

Corollary 2.1

If the natural lift $\bar{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle T(S^3) , then α is a planar circle.

(ii) The natural lift of the spherical indicatrix of the principal normal vectors of α

Let α_N be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_N$ be the natural lift of the curve α_N. If $\bar{\alpha}_N$ is an integral curve of the geodesic spray, then because of Theorem 2.1. we have

$$\bar{\nabla}_{\dot{\alpha}_N} \dot{\alpha}_N = 0,$$

that is

$$\frac{\dot{\kappa}_1}{\sqrt{\kappa_1^2+\kappa_2^2}} T + \left(\left(\kappa_1^2+\kappa_2^2\right) - \sqrt{\kappa_1^2+\kappa_2^2}\right) N + \frac{\dot{\kappa}_2}{\sqrt{\kappa_1^2+\kappa_2^2}} B_1 + \frac{\kappa_2\kappa_3}{\sqrt{\kappa_1^2+\kappa_2^2}} B_2 = 0 .$$

Since T, N, B_1, B_2 are linearly independent, we obtain

$$\begin{cases} \kappa_1 = \text{constant} \\ \kappa_2 = \text{constant} \\ \kappa_2\kappa_3 = 0 \\ \kappa_1^2+\kappa_2^2 = 0, \left(\kappa_1^2+\kappa_2^2=1\right) \end{cases} .$$

Thus we can give the following corollary:

Corollary 2.2

If the natural lift $\bar{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle T(S^3), then α is lying in a three dimensional supspace of E^4 , a circular helix or a planer circle.

(iii) The natural lift of the spherical indicatrix of the first binormal vectors of α

Let α_{B_1} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_1}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_1}$ is an integral curve of the geodesic spray, then by using Theorem 2.1 we have

$$\bar{\nabla}_{\dot{\alpha}_{B_1}} \dot{\alpha}_{B_1} = 0,$$

that is

$$\frac{\dot{\kappa}_2}{\sqrt{\kappa_2^2+\kappa_3^2}} T + \left(\left(\kappa_2^2+\kappa_3^2\right) - \sqrt{\kappa_2^2+\kappa_3^2}\right) N + \frac{\dot{\kappa}_3}{\sqrt{\kappa_2^2+\kappa_3^2}} B_1 + \frac{\kappa_1\kappa_2}{\sqrt{\kappa_2^2+\kappa_3^2}} B_2 = 0.$$

Because T, N, B_1, B_2 are linearly independent, we have

$$\begin{cases} \kappa_2 = \text{constant} \\ \kappa_3 = \text{constant} \\ \kappa_1\kappa_2 = 0 \\ \kappa_2^2+\kappa_3^2 = 0, \left(\kappa_2^2+\kappa_3^2=1\right) \end{cases} .$$

Then the following corollary can be given:

Corollary 2.3

If the natural lift $\bar{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T(S^3)$ then we have $\kappa_1 = 0, \kappa_2 =$ constant, $\kappa_3 = 0$ or $\kappa_1 = 0, \kappa_2 =$ constant, $\kappa_3 =$ constant or $\kappa_1 = 0, \kappa_2 = 0, \kappa_3 =$ constant or $\kappa_1 =$ constant, $\kappa_2 = 0, \kappa_3 =$ constant. Therefore there is no such curve satisfying these conditions.

(iv) The natural lift of the spherical indicatrix of the second binormal vectors of α

Let α_{B_2} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_2}$ be the natural lift of the curve α_{B_2}. If $\bar{\alpha}_{B_2}$ is an integral curve of the geodesic spray, then by using Theorem 2.1 we have

$$\bar{\bar{\nabla}}_{\dot{\alpha}_{B_2}} \dot{\alpha}_{B_2} = 0,$$

that is

$$-\kappa_2 N - \frac{\dot{\kappa}_3}{\kappa_3} B_1 + \left(\kappa_3^{\,2} - \kappa_2^{\,2}\right) B_2 = 0.$$

Because T, N, B_1, B_2 are linearly independent,

$$\begin{cases} \kappa_2 = 0 \\ \kappa_3 = \text{constant} \\ \kappa_3 = 0, (\kappa_3 = 1) \end{cases}.$$

We have the following corollary:

Corollary 2.4

If the natural lift $\bar{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T(S^3)$, then we have $\kappa_2 = 0$ and $\kappa_3 = 1$. Therefore there is no such curve satisfying this condition.

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