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Mathematica Module for Singularity Free Computations of Euler Parameters

M.A. Sharaf[1] , Z.A. Mominkhan[2]
[1]Department of Astronomy, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Sharaf_adel@hotmail.com
[2]Department of Mathematics, College of Science for Girls King Abdulaziz University, Jeddah, Saudi Arabia
Zammomin@hotmail.com

Abstract

In this paper, a *Mathematica* module for singularity free computations of Euler parameters was established. The basic idea that we follow in developing these computations is to make the values of the denominators of the fractions used to compute the parameters always maximum, so by this artifice we avoid the divisions by small quantities that causes singularities.

Keywords : Euler parameters; rigid body dynamics; quaternions.

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1. Introduction

Euler's parameters play important rule in rigid body dynamics which in turn could be applied to the rotational motion of satellites and other space vehicles (Wiesel 2010).Moreover there are marvelous connections between Euler's parameters and orbital motion which lead to very accurate motion predication algorithms (Sharaf.&Goharji 1990,Sharaf et al 1991a,1991b,1992).

The Euler parameters can be viewed as the coefficients of a quaternion; the scalar parameter α is the real part, the vector parameters β, γ, δ are the imaginary parts. Thus we have the quaternion

$$\mathbf{q} = \alpha + \mathbf{i}\beta + \mathbf{j}\gamma + \mathbf{k}\delta$$

which is a quaternion of unit length since

$$|\mathbf{q}| = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$$

In fact quaternions have been found to be the ideal tool for describing and developing the theory of spatial regularization in celestial mechanics (Waldvogel 2006). Now, quaternions become a standard topic in higher analysis, today, they are in use in computer graphics, control theory, signal processing, orbital mechanics etc., for representing rotations and orientations in 3-space(Kuipers1999).The Space Mechanics Group of the University of Zaragoza(Spain) took advantage of the elegance of the quaternions language in various applications in orbital and rigid -body dynamics, see,e.g Arribas,Elipe and Palacios(2006).

Due to the above mentioned importance of the Euler parameters, and their roles in many applications, consequently they should be computed accurately. To this goal, the present paper is devoted to establish a *Mathematica* module for singularity free computations of Euler's parameters

2. Rotation Matrix

2.1 Formulations

Let $\mathbf{i}_x, \mathbf{i}_y, \mathbf{i}_z$ and $\mathbf{i}_\xi, \mathbf{i}_\eta, \mathbf{i}_\varsigma$ be two sets of orthogonal unit vectors parallel to their respective coordinate axes for some particular configuration of the x,y,z system. Then the transformations from x, y, z coordinates to ξ, η, ς is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R} \begin{bmatrix} \xi \\ \eta \\ \varsigma \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \xi \\ \eta \\ \varsigma \end{bmatrix} = \mathbf{R}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The matrix **R**, the rotation matrix , is then

$$\mathbf{R} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix},$$

where $\ell_1, m_1, \cdots, n_3$ are the direction –cosines. They are cosines of the nine angles determined by the axes of one triad and the other. Since **R** is orthogonal matrix then,

$$l_1^2 + l_2^2 + l_3^2 = 1 \qquad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$$

$$m_1^2 + m_2^2 + m_3^2 = 1 \qquad l_1 n_1 + l_2 n_2 + l_3 n_3 = 0$$

$$n_1^2 + n_2^2 + n_3^2 = 1 \qquad m_1 n_1 + m_2 n_2 + m_3 n_3 = 0$$

$$\ell_1^2 + m_1^2 + n_1^2 = 1 \qquad \ell_1 \ell_2 + m_1 m_2 + n_1 n_2 = 0$$

$$l_2^2 + m_2^2 + n_2^2 = 1 \qquad l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

$$l_3^2 + m_3^2 + n_3^2 = 1 \qquad l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$$

Furthermore, it is useful to note that each element of **R** is its own cofactor; that is:

$$l_1 = m_2n_3 - m_3n_2 \quad ; \quad l_2 = m_3n_1 - m_1n_3 \quad ; \quad l_3 = m_1n_2 - m_2n_1 ,$$

$$m_1 = l_3n_2 - l_2n_3 \quad ; \quad m_2 = l_1n_3 - l_3n_1 \quad ; \quad m_3 = l_2n_1 - l_1n_2 ,$$

$$n_1 = l_2m_3 - l_3m_2 \quad ; \quad n_2 = l_3m_1 - l_1m_3 \quad ; \quad n_3 = l_1m_2 - l_2m_1 .$$

The direction cosine elements of the rotation matrix **R** are obtained in terms of the three Euler angles Ω, i, ω as:

$$\begin{aligned}
\ell_1 &= \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i ,\\
\ell_2 &= -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i ,\\
\ell_3 &= \sin\Omega\sin i ,\\
m_1 &= \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i ,\\
m_2 &= -\sin\omega\sin\Omega + \cos\omega\cos\Omega\ \cos i\\
m_3 &= -\cos\Omega\sin i ,\\
n_1 &= \sin\omega\ \sin i \ ;\ n_2 = \cos\omega\ \sin i \ ;\ n_3 = \cos i .
\end{aligned} \tag{1}$$

In space dynamics Euler angles Ω, i and ω (e.g. Battin 1999), define the orientation of the orbit in space and known respectively as, the longitude of the ascending node ,the orbital inclination ,and the argument of latitude.

The characteristic equation of the rotation matrix **R** is

$$|\,\mathbf{R} - \lambda\mathbf{I}\,| = -\lambda^3 + (\mathrm{tr}\,\mathbf{R})\lambda^2 - (\mathrm{tr}\,\mathbf{R})\lambda + 1 = 0 \ , \tag{2}$$

where tr **R** is the trace of **R**. Thus the roots or the eigen values of the rotation matrix **R** are:

$$\lambda = 1 \quad , \quad \lambda = e^{i\psi} \quad , \quad \lambda = e^{-i\psi} \ .$$

It should be noted that**:**

- If $\mathrm{tr}\,\mathbf{R} = 3$,then Equation (2) yields

$$(1-\lambda)\{\lambda^2 - 2\lambda + 1\} = (1-\lambda)^3 = 0 ,$$

i.e , the root $\lambda = 1$ is thrice repeated.

- If $\mathrm{tr}\,\mathbf{R} = -1$,then Equation (2) yields

$$(1-\lambda)\{\lambda^2 + 2\lambda + 1\} = (1-\lambda)(1+\lambda)^2 = 0 ,$$

i.e. there is a double root at $\lambda = -1$

2.2 Kinematic form of the rotation matrix

Assume a constant magnitude for the position vector $\mathbf{r}$ and considering only rotation of the x, y, z, frame, as shown in Fig.1 The definition of $\dot{\mathbf{r}}$ is

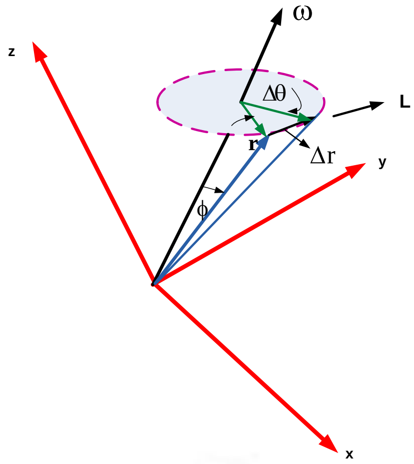

Fig.1: Rotation effects on derivatives

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{(\mathbf{r}+\Delta\mathbf{r})-\mathbf{r}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t}.$$

Here $\Delta\mathbf{r} = \Delta\theta\,(r\sin\phi)\mathbf{L}$ with $\mathbf{L}$ parallel to $\Delta\mathbf{r}$. Take

$$\frac{\Delta\mathbf{r}}{\Delta t} = .(r\sin\phi)\mathbf{L}\frac{\Delta\theta}{\Delta t},$$

Since r is constant here. This expression can taken to the limit,

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0}(r\sin\phi)\mathbf{L}\frac{\Delta\theta}{\Delta t} = \omega(r\sin\phi)\mathbf{L}$$

which is

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega}\times\mathbf{r}$$

Let ℓ, m, n be the direction cosines of $\boldsymbol{\omega}$ and $\frac{d\psi}{dt}$, the constant angular speed. Then

$$\boldsymbol{\omega}\times\mathbf{r} = \frac{d\Psi}{dt}\mathbf{i}_\omega \times \mathbf{r}.$$

Let us write the vector product $\mathbf{i}_\omega \times \mathbf{r}$ in a form of product of a matrix **S** and the position vector **r,** i.e.

$$\frac{d\mathbf{r}}{d\psi} = \mathbf{S}\,\mathbf{r}, \tag{3}$$

where the *skew-symmetric* matrix **S** is given as

$$\mathbf{S}=\begin{bmatrix}0 & -n & m\\ n & 0 & -\ell\\ -m & \ell & 0\end{bmatrix}. \tag{4}$$

Equation (3) is a linear vector differential equation for $\mathbf{r}$ with constant coefficients, the solution, as a function of ψ is

$$\mathbf{r}(\psi)=\mathbf{r}_0 e^{\mathbf{S}\psi},$$

where at $\psi=0$ $\mathbf{r}(0)=\mathbf{r}_0$,expand we get

$$\mathbf{r}(\psi)=\left(\mathbf{I}+\psi\mathbf{S}+\frac{1}{2!}\psi^2\mathbf{S}^2+\frac{1}{3!}\psi^3\mathbf{S}^3+\cdots\right)\mathbf{r}_0.$$

The matrix coefficient of $\mathbf{r}_0$ is the rotation matrix **R** (since $\mathbf{r}=\mathbf{R}\,\mathbf{r}_0$) expressed as an infinite matrix.

$$\mathbf{R}=\mathbf{I}+\psi\mathbf{S}+\frac{1}{2!}\psi^2\mathbf{S}^2+\frac{1}{3!}\psi^3\mathbf{S}^3+\cdots$$

which could be written as:

$$\mathbf{R}=\mathbf{I}+\sum_{j-0}^{\infty}\frac{(-1)^j\Psi^{2j+1}}{(2j+1)!}\mathbf{S}+\sum_{j-1}^{\infty}\frac{(-1)^{j-1}\Psi^{2j}}{(2j)!}\mathbf{S}^2,$$

that is to say

$$\mathbf{R}=\mathbf{I}+\sin\psi\,\mathbf{S}+(1-\cos\psi)\mathbf{S}^2\;. \tag{5}$$

This form of the rotation matrix shows the explicit dependence on the kinematical quantities-the direction cosines ℓ, m, n of the rotation axis (as the elements of **S**) and the rotation angle ψ .

3. Euler parameters

Suppose that we express the trigonometric relation (5) in terms of half angles ,i.e.,

$$\sin\psi=2\sin\frac{1}{2}\psi\ \cos\frac{1}{2}\psi \qquad ; \qquad 1-\cos\psi=2\sin^2\frac{1}{2}\psi.$$

That it is reasonable to combine the term $\sin\frac{1}{2}\psi$ with the direction cosines of the rotation axis ℓ, m, n by defining the matrix :

$$\mathbf{E}=\sin\frac{1}{2}\psi\,\mathbf{S}=\begin{bmatrix}0 & -n\sin\frac{1}{2}\psi & m\sin\frac{1}{2}\psi\\ n\sin\frac{1}{2}\psi & 0 & -\ell\sin\frac{1}{2}\psi\\ -m\sin\frac{1}{2}\psi & \ell\sin\frac{1}{2}\psi & 0\end{bmatrix},$$

so that Equation (5) may be written as :

$$\mathbf{R}=\mathbf{I}+2\cos\frac{1}{2}\psi\,\mathbf{E}+2\mathbf{E}^2. \tag{6}$$

Now , for convenience, define:

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$$\alpha = \ell \sin\frac{1}{2}\psi \quad ; \quad \beta = m \sin\frac{1}{2}\psi$$
$$\gamma = n \sin\frac{1}{2}\psi \quad ; \quad \delta = \cos\frac{1}{2}\psi \tag{7}$$

$$\mathbf{E} = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} \tag{8}$$

These quantities are called the *Euler parameters.*

Clearly they satisfy the identity:

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1. \tag{9}$$

By expanding the new expression for **R** and using this identity, we can express the rotation matrix in terms of the Euler parameters as :

$$\mathbf{R} = \begin{bmatrix} 1-2(\beta^2+\gamma^2) & 2(\alpha\beta-\gamma\delta) & 2(\alpha\gamma+\beta\delta) \\ 2(\alpha\beta+\gamma\delta) & 1-2(\alpha^2+\gamma^2) & 2(\gamma\beta-\alpha\delta) \\ 2(\alpha\gamma-\beta\delta) & 2(\gamma\beta+\alpha\delta) & 1-2(\alpha^2+\beta^2) \end{bmatrix}. \tag{10}$$

The above equations are not appropriate for either vanishingly small values of ψ or values near 180^o, i.e., tr **R** near 3 or -1.Stanley [1978] developed the following set of relations (cited in Battin 1999)

Let the element in the i^{th} row and j^{th} column of the rotation matrix **R** be defined as r_{ij} and introduce the notation

$$p_0 = 2\delta \quad , \quad p_1 = 2\alpha \quad , \quad p_2 = 2\beta \quad , \quad p_3 = 2\gamma \, , \tag{11}$$

$$r_{00} = \mathrm{tr}\mathbf{R} = r_{11} + r_{22} + r_{33} \quad , \tag{12}$$

then it could be shown that (Battin 1999) that:

a-The p's may be calculated from

$$p_j^2 = 1 + 2\, r_{jj} - \mathrm{tr}\mathbf{R} \quad \text{for} \quad j = 0,1,2,3 \tag{13}$$

b- The largest value of the p_j^2's lies in the closed interval [1,4] and the lower bound is attained only if all diagonal elements of **R** are zero.

c- The products $p_i p_j ; i = 0,1,2,3 \; ; \; j \neq i$ are given as :

$$p_o p_1 = r_{32} - r_{23} \; ; \; p_2 p_3 = r_{32} + r_{23} , \tag{14}$$

$$p_o p_2 = r_{13} - r_{31} \; ; \; p_1 p_3 = r_{13} + r_{31} , \tag{15}$$

$$p_o p_3 = r_{21} - r_{12} \; ; \; p_1 p_2 = r_{21} + r_{12} \tag{16}$$

4. Mathematica module for singularity free computation of Euler parameters

In what follows we develop a Mathematica module for singularity free computation of Euler parameters based on Equations (11) to (16)

4.1 Module Sin Free Euler

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♦ Purpose

To compute Euler parameters which are free from singularity

♦Input

The rotation matrix R

♦Output

The Euler parameters $\alpha, \beta, \gamma, \delta$ which are free from singularity

♦Required program

None

♦List of the module

```
Sin Free Euler [R_List :=Module[{  }, Q=Tr[R];
C1=Sqrt[1+Q]/2 ; C2=Sqrt [1+2*R[[1,1]]-Q]/2;
C3=Sqrt[1+2*R[[2,2]]-Q]/2;
C4=Sqrt[1+2*R[[3,3]]-Q]/2;
y={C1,C2,C3,C4};x=Max[y];z=Position [y,z];
Which [z[[1,1]] = 1,
{ δ = x , α = (R[[3,2]] – R[[2,3]]) /(4 * x),
β = (R[[1,3]] – R[[3,1]]) /(4 * x),
γ = (R[[2,1]] – R[[1,2]]) /(4 * x)},
z[[1,1]] =2,
{ α = x , δ = (R[[3,2]] – R[[2,3]]) /(4 * x),
β = (R[[2,1]] + R[[1,2]]) /(4 * x),
γ = (R[[1,3]] + R[[3,1]]) /(4 * x)},
z[[1,1]] =3,
{ β = x , δ = (R[[1,3]] – R[[3,1]]) /(4 * x),
α = (R[[2,1]] + R[[1,2]]) /(4 * x),
γ = (R[[2,3]] + R[[3,2]]) /(4 * x)},
z[[1,1]] =4,
{ γ = x , δ = (R[[2,1]] – R[[1,2]]) /(4 * x),
α = (R[[1,3]] + R[[3,1]]) /(4 * x),
β = (R[[2,3]] + R[[3,2]]) /(4 * x)};]]
```

The basic idea that we follow in developing singularity free computations of Euler's parameters is to make the values of the denominators of the fractions used to compute the parameters always maximum, so by this artifice we avoid the divisions by small quantities that causes singularities.

♦Example

Let us use the above program to find the Euler parameters $\alpha, \beta, \gamma, \delta$ which are free from singularity for the rotation matrix

$$\begin{pmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.23457 \\ 0.357073 & 0.325773 & 0.875426 \end{pmatrix}$$

and we get

$$\alpha = 0.145651 \ ; \ \beta = \text{-}0.202665 \quad ; \ \gamma = \text{-}0.112505 \ ; \ \delta = 0.961798.$$

The accuracy of these computed values could be checked by the condition that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

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