ISSN 2347-1921

Differential subordination and superordination for certain subclasses of p-valent functions

Jamal M. Shenan
Department of mathematics, Alazhar University-Gaza, P. O. Box 1277, Gaza , Palestine.
shenanjm @yahoo.com
Ahmed S. Galiz
Department of Mathematics, Alquds Open University-Gaza Branch, P. O. Box 124, Gaza, Palestine.
Ahmad2911971@hotmail.com

ABSTRACT

In this paper, we study applications of the differential subordination and superordination of analytic p-valent functions in the open unit disc associated with certain operator defined by the Wright generalized hypergeometric function. Sandwich-type result involving this operator is also derived.

Keywords:. Analytic function; p-valent function; the Wright generalized hypergeometric function; differential subordination and superordination.

2010 Mathematics Subject Classification: 30C80 and 30C45

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics

Vol 6, No. 3
editor@cirworld.com
www.cirworld.com, member.cirworld.com

1 Introduction

Let $H(U)$ be the class of functions analytic in $U=\{z: z\in C \text{ and } |z|<1\}$ and $H[a,k]$ be the subclass of $H(U)$ consisting of functions of the form

$$f(z)=a+a_p z^k+a_{p+1}z^{p+1}+\ldots \quad (a\in\mathbb{C},\ p\in\mathbb{N}=\{1,2,\ldots\}).$$

Let A_p denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_k z^k \quad (p\in\mathbb{N},\ z\in U), \tag{1}$$

which are analytic in the open unit disk U, and set $A\equiv A_1$.

For two functions $f(z)$ given by (1) and

$$g(z)=z^p+\sum_{k=1+p}^{\infty}b_k z^k, \tag{2}$$

the hadmard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_k b_k z^k=(g*f)(z) \quad (p\in\mathbb{N},\ z\in U). \tag{3}$$

Let f and F be members of $H(U)$, the function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $w(z)$ analytic in U with $w(0)=0$ and $|w(z)|<1\ (z\in U)$, such that $f(z)=F(w(z))$. In such a case we write $f(z)\prec F(z)$. In particular, if F is univalent, then $f(z)\prec F(z)$ if and only if $f(0)=F(0)$ and $f(U)\subset F(U)$ (see [1, 2]).

Suppose that p and h are two functions in U, let

$$\phi(r,s,t;z): C^3\times U\to C.$$

If p and $\phi(p(z),zp'(z),z^2p''(z);z)$ are univalent in U and if p is analytic in U and satisfies the first order differential superordination

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z);z) \qquad (z\in U), \tag{4}$$

then p is called a solution of the differential superordination (4).

The univalent function q is called a subordinant solutions of (4) if $q\prec p$ for all p satisfying (4). A subordinant $\tilde{q}$ that satisfies $q\prec\tilde{q}$ for all subordinant q of (4) is said to be the best subordinant. (see the monograph by Miller and Mocanu [14], and [15]).

Recently, Miller and Mocanu [15] obtained sufficient conditions on the functions h,q and ϕ for which the following implication holds:

$$h(z)\prec\phi(p(z),zp'(z),z^2p''(z);z) \to q(z)\prec p(z)$$

Using these results, the second author considered certain classes of first-order differential superordinations [7], as well as superordination-preserving integral operators [6]. Ali et al. [1], using the results from [7], obtained sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z)\prec\frac{zf'(z)}{f(z)}\prec q_2(z), \tag{5}$$

where q_1 and q_2 are given univalent normalized functions in U.

Very recently, Shanmugam et al. [20–22] obtained the such called sandwich results for certain classes of analytic functions. Further subordination results can be found in [16, 23, 24 and 28].

Let $\alpha_1,A_1,\ldots,\alpha_q,A_q$ and $\beta_1,B_1,\ldots,\beta_s,B_s$ $(q,s\in\mathbb{N}=\{1,2\ldots\})$ be positive real parameters such that

$$1+\sum_{k=1}^{s}B_k-\sum_{k=1}^{q}A_k>0. \tag{6}$$

The Wright generalized hypergeometric function (see [25], [26] and [27])

${}_q\Psi_s\left[(\alpha_1,A_1,...,\alpha_q,A_q);(\beta_1,B_1,...,\beta_s,B_s);z\right]={}_q\Psi_s\left[(\alpha_n,A_n)_{1,q};(\beta_n,B_n)_{1,s};z\right]$ is defined by

$${}_q\Psi_s\left[(\alpha_n,A_n)_{1,q};(\beta_n,B_n)_{1,s};z\right]=\sum_{k=0}^{\infty}\left\{\prod_{n=1}^{q}\Gamma(\alpha_n+kA_n)\right\}\left\{\prod_{n=1}^{s}\Gamma(\beta_n+kB_n)\right\}^{-1}\frac{z^k}{k!}\ (z\in U). \tag{1.7}$$

If $A_i=1\ (i=1,\ldots,q)$ and $B_j=1\ (j=1,\ldots,s)$ we have

$$\Omega\,{}_q\Psi_s\left[(\alpha_n,1)_{1,q};(\beta_n,1)_{1,s};z\right]={}_qF_s(\alpha_1,...\alpha_q,\beta_1,...\beta_s,z), \tag{8}$$

which is the generalized hypergemetric function where

$$\Omega=\left(\prod_{n=1}^{q}\Gamma(\alpha_n)\right)^{-1}\left(\prod_{n=1}^{s}\Gamma(\beta_n)\right). \tag{9}$$

Let

$$\theta_{p,q,s}[\alpha_1,\beta_1;A_1,B_1;z]=\Omega z^p\,{}_q\Psi_s\left[(\alpha_n,A_n)_{1,q};(\beta_n,B_n)_{1,s};z\right]$$

$$=z^p+\sum_{k=1}^{\infty}\frac{\prod_{n=1}^{s}\Gamma(\beta_n)\prod_{n=1}^{q}\Gamma(\alpha_n+kA_n)}{\prod_{n=1}^{q}\Gamma(\alpha_n)\prod_{n=1}^{s}\Gamma(\beta_n+kB_n)k!}z^{k+p}. \tag{10}$$

Using the Wright hypergeomtric function, we introduce the following linear operator

$$\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f:A_p\to A_p$$

which is defined by the following convolution

$$\phi_{p,q,s}^{0,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)=f(z)*\theta_{p,q,s}[\alpha_1,\beta_1;A_1,B_1;z];$$

$$\phi_{p,q,s}^{1,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)=(1-\lambda)\left(f(z)*\theta_{p,q,s}[\alpha_1,\beta_1;A_1,B_1;z]\right)$$

$$+\frac{\lambda}{(p+l)^{l-1}}\left(z^l f(z)*\theta_{p,q,s}[\alpha_1,\beta_1;A_1,B_1;z]\right)';$$

and

$$\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)=\phi_{p,q,s}^{m,l,\lambda}\left(\phi_{p,q,s}^{m-1,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right). \tag{11}$$

If $f\in A_p$, then from (1) and (11), we can easily see that

$$\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)=z^p+\sum_{k=p+1}^{\infty}\left[\frac{p+l+\lambda(k-p)}{p+l}\right]^m\frac{\prod_{n=1}^{s}\Gamma(\beta_n)\prod_{n=1}^{q}\Gamma(\alpha_n+(k-p)A_n)}{\prod_{n=1}^{q}\Gamma(\alpha_n)\prod_{n=1}^{s}\Gamma(\beta_n+(k-p)B_n)(k-p)!}a_kz^k, \tag{12}$$

where $m\in N_0=N\bigcup\{0\},l\geq 0,\lambda\geq 0$, and $p\in N$.

We note that when $A_i=1\ (i=1,\ldots,q)$ and $B_j=1\ (j=1,\ldots,s)$, the operator $\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;1,1]f(z)=L_{p,q,s\lambda}^{m,l}(\alpha_1,\beta_1)f(z)$ was studied by El-Ashwah_ and Aouf [11], also when $A_i=1\ (i=1,\ldots,q)$, $B_j=1\ (j=1,\ldots,s)$, $p=1$, and $l=0$, the operator $\phi_{1,q,s}^{m,0,\lambda}[\alpha_1,\beta_1;1,1]f(z)=D_\lambda^m(\alpha_1,\beta_1)f(z)$ was studied by Selvaraj and Karthikeyan [19], and for $A_i=1\ (i=1,\ldots,q)$ and $B_j=1\ (j=1,\ldots,s)$, and $m=0$, the operator $\phi_{p,q,s}^{0,l,\lambda}[\alpha_1,\beta_1;1,1]f(z)=H_p^{p,q}[\alpha_1]f(z)$ is the Dziok–Srivastava operator [10]. Moreover by specializing the parameters $m,l,\lambda,p,q,s,\ \alpha_i,\ A_i\ (i=1,...,q)$ and $\beta_j,\ B_j\ (j=1,...,s)$, we obtain various new operators from the operator $\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)$ studied by several authors such as Catas [9], Kamali, and Orhan [12], Kumar et al. [13], Salagean [18], Al-Oboudi [2] and others.

It is easily verified from (12) that

$$z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)' = \frac{\alpha_1}{A_1}\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z) - \left(\frac{\alpha_1}{A_1}-p\right)\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)$$
$$(A_1>0), \quad (13)$$

$$\lambda z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)' = (p+l)\phi_{p,q,s}^{m+,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z) - [p(1-\lambda)+l]\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)$$
$$(\lambda>0). \quad (14)$$

To prove our results, we need the following definitions and lemmas.

Definition 1 ([14]). Denote by Q the set of all functions $q(z)$ that are analytic and injective on $\overline{U}/E(q)$ where

$$E(q)=\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\},$$

and are such that $q'(\zeta)\neq 0$ for $\zeta\in\partial U/E(q)$. Further let the subclass of Q for which $q(0)=a$ be denoted by $Q(a)$, $Q(0)\equiv Q_0$ and $Q(1)\equiv Q_1$.

Lemma 1 ([14]). Let $q(z)$ be univalent function in the unit disc U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w)\neq 0$ when $w\in q(U)$. Set $q(z)=zq'(z)\varphi(q(z)),\ h(z)=\theta(q(z))+Q(z)$ and suppose that

i) Q is a starlike function in U,

ii) $\operatorname{Re} zh'(z)/Q(z)>0,\ z\in U.$

If p is analytic in U with $p(0)=q(0),\ p(U)\subseteq D$ and

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(q(z))+zq'(z)\varphi(q(z)), \quad (15)$$

then $p(z)\prec q(z)$, and q is the best dominant of (15).

Lemma 2 ([21]). Let $q(z)$ be a convex univalent function in U and let $\alpha\in\mathbb{C},\ \eta\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ with

$$\Re\left\{1+\frac{z\,q''(z)}{q'(z)}\right\}>\max\left\{0,-\Re\left(\frac{\sigma}{\eta}\right)\right\}.$$

If the function $g(z)$ is analytic in U and

$$\sigma g(z)+\eta zg'(z)\prec\sigma q'(z)+\eta zq'(z),$$

then $g(z)\prec q(z)$ and $q(z)$ is the best dominant.

Lemma 3 ([8]). Let $q(z)$ be univalent function in the unit disc U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

i) $\operatorname{Re}\theta(q(z))/\phi(q(z))>0\ \ z\in U,$

ii) $h(z)=zq'(z)\varphi(q(z))$ is starlike in U.

If $p\in H[q(0),1]\cap Q$ with $p(U)\subseteq D$, $\theta(p(z))+zp'(z)\varphi(p(z))$ is univalent U, and

$$\theta(q(z))+zq'(z)\varphi(q(z))\prec\theta(p(z))+zp'(z)\varphi(p(z)), \quad (16)$$

then $q(z)\prec p(z)$, and q is the best dominant of (16).

Lemma 4 ([15]). Let $q(z)$ be convex function in U and let $\gamma\in\mathbb{C}$, with $\operatorname{Re}\gamma>0.$ If $p\in H[q(0),1]\cap Q$ and $p(z)+\gamma zp'(z)$ is univalent in U, then

$$q(z)+\gamma zq'(z)\prec p(z)+\gamma zp'(z),$$

implies $q(z)\prec p(z)$, and q is the best dominant.

Lemma 5 ([17]). The function $q(z)=(1-z)^{-2ab}$ is univalent in U if and only if

$\left|2ab-1\right|\le 1$ or $\left|2ab+1\right|\le 1$.

Unless otherwise mentioned, we assume throughout the following sections that $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1, B_1, \ldots, \beta_s, B_s$ $(q, s \in \mathbb{N} = \{1, 2 \ldots\})$ are positive real parameters such that $1+\sum_{k=1}^{s} B_k - \sum_{k=1}^{q} A_k > 0,$ $m \in N_0,\ l \ge 0,$ and $\lambda > 0$.

2. Subordination results for analytic functions.

Theorem 1 Let $\alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $q(z)$ be a univalent function in U, with $q(0)=1$, and suppose that

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0; -\frac{p\alpha_1}{A_1}\operatorname{Re}\frac{1}{\alpha}\right\}, \quad (z \in U; p \in \mathbb{N}) \tag{17}$$

If $f \in A_p$ satisfies the subordination

$$\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right) \prec q(z)+\frac{\alpha A_1 zq'(z)}{p\alpha_1}, \tag{18}$$

then

$$\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p} \prec q(z),$$

and the function q is the best dominant of (18).

Proof. If we consider the analytic function

$$h(z)=\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p},$$

by differentiating logarithmically with respect to z, we deduce that

$$\frac{zh'(z)}{h(z)}=\frac{z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p. \tag{19}$$

From (19), by using the identity (13), a simple computation shows that

$$\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right)=h(z)+\frac{\alpha A_1 zh'(z)}{p\alpha_1}$$

hence the subordination (18) is equivalent to

$$h(z)+\frac{\alpha A_1 zh'(z)}{p\alpha_1} \prec q(z)+\frac{\alpha A_1 zq'(z)}{p\alpha_1}.$$

Combining the last relation together with Lemma 2 for the special case $\eta=\alpha A_1/p\alpha_1$ and $\sigma=1$, we obtain our result.

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 1, where $-1\le B<A\le 1$, the condition (17) becomes

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0; -\frac{p\alpha_1}{A_{1\ 1}}\operatorname{Re}\frac{1}{\alpha}\right\}, \quad z \in U. \tag{20}$$

It is easy to check that the function $\phi(\zeta)=\frac{1-\zeta}{1+\zeta}$, $|\zeta|<|B|$, is convex in U and since $\phi(\overline{\zeta})=\overline{\phi(\zeta)}$ for all $|\zeta|<|B|$, it follows that the image $\varphi(U)$ is a convex domain symmetric with respect to the real axis, hence

$$\inf\left\{\Re\left(\frac{1-Bz}{1+Bz}\right); z \in U\right\}=\frac{1-|B|}{1+|B|}>0. \tag{21}$$

Then, the inequality (20) is equivalent to

$$\frac{p\alpha_1}{A_1}\operatorname{Re}\frac{1}{\alpha}\geq\frac{1-|B|}{1+|B|},$$

hence we obtain the following result:

Corollary 1 Let $\alpha\in\mathbb{C}^*$ and $-1\leq B<A\leq 1$ with

$$\max\left\{0;-\frac{p\alpha_1}{A_1}\operatorname{Re}\frac{1}{\alpha}\right\}\leq\frac{1-|B|}{1+|B|}.$$

If $f\in A_p$ satisfies the subordination

$$\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right)\prec\frac{1+Az}{1+Bz}+\frac{A_1\alpha(A-B)z}{p\alpha_1(1+Bz)^2}, \tag{22}$$

then

$$\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\prec\frac{1+Az}{1+Bz},$$

and the function $1+Az/1+Bz$ is the best dominant of (22).

For $p=1, A=1$ and $B=-1$, the above corollary reduces to:

Corollary 2 Let $\alpha\in\mathbb{C}^*$ such that

$$\frac{\alpha_1}{A_1}\operatorname{Re}\frac{1}{\alpha}\geq 0.$$

If $f\in A_p$ satisfies the subordination

$$\alpha\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z}\right)+(1-\alpha)\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z}\right)\prec\frac{1+Az}{1+Bz}+\frac{A_1\alpha(A-B)z}{\alpha_1(1+Bz)^2}, \tag{23}$$

then

$$\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z}\prec\frac{1+z}{1-z},$$

and the function $1+z/1-z$ is the best dominant of (23).

Theorem 2 Let $q(z)$ be a univalent function in U, with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$, and let $\delta,\mu\in\mathbb{C}^*$ and $\nu,\eta\in\mathbb{C}$ with $\nu+\eta\neq 0$, and suppose that $f\in A_p$ and q satisfy the conditions:

$$\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\neq 0 \quad z\in U, \tag{24}$$

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0 \quad z\in U. \tag{25}$$

If

$$1+\delta\mu\left[\frac{\nu z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)\right)'+\eta z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p\right]$$

$$\prec 1+\delta\frac{zq'(z)}{q(z)}, \qquad (26)$$

then

$$\left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q(z),$$

and the function q is the best dominant of (26) . (the power is the principal one).

Proof. Let

$$h(z)=\left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu},\ z\in U. \qquad (27)$$

According to (24) the function h is analytic in U , differentiating (27) logarithmically with respect to z we get

$$\frac{zh'(z)}{h(z)}=\mu\left[\frac{\nu z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)\right)'+\eta z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p\right]. \qquad (28)$$

In order to prove our result we will use Lemma 1. Considering in this lemma

$$\theta(w)=1 \text{ and } \phi(w)=\frac{\delta}{w},$$

then θ is analytic in $\mathbb{C}$ and $\phi(w)\neq 0$ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z)=zq'(z)=\varphi(q(z))=\delta\frac{zq'(z)}{q(z)},$$

and

$$g(z)=\theta(q(z))+Q(z)=1+\delta\frac{zq'(z)}{q(z)},$$

then, since $Q(0)=1$ and $Q'(0)\neq 0$, the assumption (2.9) yields that Q is a starlike function in U . From (25) we also have

$$\Re\frac{zq'(z)}{Q(z)}=\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0\ z\in U,$$

and then, by using Lemma 1 we deduce that the subordination (26) implies $h(z)\prec q(z)$ and the function q is the best dominant of (26).

Taking $\nu=0,\ \eta=\delta=1$ and $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 2 , it is easy to check that the assumption (25) holds whenever $-1\le B<A\le 1$, hence we obtain the next results.

Corollary 3 Let $-1\le B<A\le 1$ and $\mu\in\mathbb{C}^*$. Let $f\in A_p$ and suppose that

$$\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\neq 0 \qquad z\in U.$$

If

$$1+\mu\left[\frac{z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p\right]\prec 1+\frac{(A-B)z}{(1+Az)(1+Bz)}, \qquad (29)$$

then

$$\left[\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right]^{\mu} \prec \frac{1+Az}{1+Bz},$$

and the function $1+Az/1+Bz$ is the best dominant of (29). (the power is the principal one).

Remarks

1) Putting $A_i=1\,(i=1,\dots,q)$ and $B_j=1\,(j=1,\dots,s)$ in Theorem 2 we obtain the corresponding result due to El-Ashwah and Aouf [11, Theorem 2].
2) Putting $\nu=0,\ \eta=p=1,\ m=0,\ q=s+1,\ \alpha_i=A_i=1\,(i=1,\dots,s+1),\ \beta_j=B_j=1\ (j=1,\dots,s)$, $\delta=1/ab\ \left(a,b\in\mathbb{C}^*\right),\mu=a$, and $q(z)=(1-z)^{-2ab}$ in Theorem 2, then combining this together with Lemma 5 we obtain the corresponding result due to Obradović et al. [16, Theorem 1], see also Aouf and Bulboacă [4, Corollary 3.3].
3) For $\nu=0,\ \eta=p=1,\ m=0,\ q=s+1,\ \alpha_i=A_i=1\,(i=1,\dots,s+1),\ \beta_j=B_j=1\ (j=1,\dots,s)$, $\delta=1/b\ \left(b\in\mathbb{C}^*\right),\mu=1$ and $q(z)=(1-z)^{-2b}$, Theorem 2 reduces to the recent result of Srivastava and Lashin [24].
4) Putting $\nu=0,\ \eta=p=\delta=1,\ m=0,\ q=s+1,\ \alpha_i=A_i=1\,(i=1,\dots,s+1),\ \beta_j=B_j=1$ $(j=1,\dots,s)$ and $q(z)=(1+Bz)^{\mu(A-B)/B}\ \ (-1\le A<B\le 1,\ B\neq 0)$ in Theorem 2, and using Lemma 5 we get the corresponding result due to Aouf and Bulboacă [4, Corollary 3.4].
5) Putting $\nu=0,\ \eta=p=1,\ m=0,\ q=s+1,\ \alpha_i=A_i=1\,(i=1,\dots,s+1),\ \beta_j=B_j=1\ (j=1,\dots,s)$, $\delta=e^{i\lambda}/ab\cos\gamma\left(a,b\in\mathbb{C}^*;|\gamma|<\pi/2\right),\mu=a$ and $q(z)=(1-z)^{-2a\cos\gamma e^{-i\gamma}}$ in Theorem 2, we obtain the corresponding result due to Aouf et al. [5, Theorem 1], see also Aouf and Bulboacă [4, Corollary 3.5].

Theorem 3 Let $q(z)$ be a univalent function in U, with $q(0)=1$, and Let $\gamma,\mu\in\mathbb{C}^*$ and $\nu,\eta,\delta,\Omega\in\mathbb{C}$ with $\nu+\eta\neq 0$, and suppose that $f\in A_p$ and q satisfy the conditions:

$$\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\neq 0\quad z\in U, \tag{30}$$

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0;-\mathrm{Re}\frac{\delta}{\gamma}\right\},\ z\in U, \tag{31}$$

If

$$\psi(z)=\left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu}$$

$$\times\left[\delta+\mu\gamma\left(\frac{\nu z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)\right)'+\eta z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p\right)\right]+\Omega, \tag{32}$$

and

$$\psi(z)\prec\delta q(z)+\gamma zq'(z)+\Omega, \tag{33}$$

then

$$\left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu}\prec q(z),$$

and the function q is the best dominant of (33) (all the power are the principal ones).

Proof. Let $h(z)$ be defined by (27), the we have from (28)

ISSN 2347-1921

$$zh'(z)=\mu h(z)\left[\frac{\nu z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)\right)'+\eta z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p\right].$$

Let us consider the following functions:

$$\theta(w)=\delta w+\Omega, \text{ and } \phi(w)=\gamma, w\in\mathbb{C},$$

$$Q(z)=zq'(z)=\varphi(q(z))=\gamma\frac{zq'(z)}{q(z)}, \ z\in U,$$

and

$$g(z)=\theta(q(z))+Q(z)=\delta q(z)+\gamma zq'(z)+\Omega, \ z\in U.,$$

From the assumption we see that Q is starlike in U and, that

$$\Re\frac{zg'(z)}{Q(z)}=\Re\left\{\frac{\delta}{\gamma}+1+\frac{zq''(z)}{q'(z)}\right\}>0, \ z\in U,$$

thus, by applying Lemma 1 the proof is completed.

Taking $q(z)=\dfrac{1+Az}{1+Bz}$ in Theorem 3, where $-1\le B<A\le 1$, and according to (2.5), the condition (2.15) becomes

$$\max\left\{0;-\operatorname{Re}\frac{\delta}{\gamma}\right\}\le\frac{1-|B|}{1+|B|}.$$

Hence, for the special case $\nu=\gamma=0, \ \eta=0$, we obtain the following result:

Corollary 4 Let $-1\le B<A\le 1$, $\mu\in\mathbb{C}^*$ and $\delta\in\mathbb{C}$ with

$$\max\{0;-\operatorname{Re}(\delta)\}\le\frac{1-|B|}{1+|B|}.$$

Let $f\in A_p$ and suppose that

$$\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\neq 0, \qquad z\in U,$$

and

$$\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)^{\mu}\left[\delta+\mu\left(\frac{z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p\right)\right]+\Omega$$

$$\prec\delta\frac{1+Az}{1+Bz}+\Omega+\frac{(A-B)z}{(1+Bz)^2}, \qquad (34)$$

then

$$\left[\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right]^{\mu}\prec\frac{1+Az}{1+Bz},$$

and the function $1+Az/1+Bz$ is the best dominant of (32) (all the powers are the principal ones).

Remark: Taking $\nu=0, \ \eta=\gamma=p=1, \ \alpha=\beta=0$ and $q(z)=\dfrac{1+z}{1-z}$ in Theorem 3 we obtain the corresponding result due to Aouf and Bulboacă [4, Corollary 3.7].

3 Superordination and sandwich results

Theorem 4. Let $q(z)$ be convex function in U with $q(0)=1$ and let $\alpha \in \mathbb{C}^*$ with $\frac{\alpha_1}{A_1}\operatorname{Re}\alpha > 0$. Let $f \in A_p$ and suppose that $\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)/z^p \in H[q(0),1]\cap Q$. If the function

$$\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right),$$

is univalent in U, and

$$q(z)+\frac{\alpha A_1 zq'(z)}{p\alpha_1} \prec \frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right), \tag{35}$$

then

$$q(z)\prec \frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p},$$

and q is the best subordinate of (34).

Proof. Let us define the function g by

$$g(z)=\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}, \quad z\in U.$$

From the assumption of the theorem, the function g is analytic in U, by differentiating logarithmically with respect to z the function g, we deduce that

$$\frac{zg'(z)}{g(z)}=\frac{z\left(\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)\right)'}{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}-p. \tag{36}$$

After some computations, and using the identity (1.13), from (3.2) we get

$$g(z)+\frac{\alpha A_1 zg'(z)}{\alpha_1 p(p-\beta)}=\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right),$$

and now, by using Lemma 4 we get the desired result.

Taking $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 4 , where $-1\le B<A\le 1,$ hence we obtain the next results.

Corollary 5 Let $-1\le B<A\le 1$ and $f\in A_p$. Suppose that $\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)/z^p \in H[q(0),1]$. If the function

$$\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right),$$

is univalent in U, and

$$\frac{1+Az}{1+Bz}+\frac{\alpha A_1(A-B)z}{p\alpha_1(1+Bz)^2}\prec \frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right), \tag{37}$$

then

$$\frac{1+Az}{1+Bz}\prec \frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p},$$

and $1+Az/1+Bz$ is the best subordinate of (37).

Using arguments similar to those of the proof of Theorem 3, and then by applying Lemma 3 we obtain the following result.

Theorem 5 Let $q(z)$ be convex function in U, with $q(0)=1$. Let $\gamma,\mu\in\mathbb{C}^*$ and $\nu,\eta,\delta,\Omega\in\mathbb{C}$ with $\nu+\eta\neq 0$ $\mathrm{Re}(\delta/\gamma)>0$. Let $f\in A_p$ and suppose that f satisfies the conditions:

$$\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\neq 0,\quad z\in U,$$

and

$$\left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu}\in H[q(0),1]\cap Q$$

If the function ψ given by (2.16) is univalent in U, and

$$\delta q(z)+\gamma zq'(z)+\Omega\prec\psi(z), \tag{38}$$

then

$$q(z)\prec\left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu},$$

and the function q is the best subordinate of (38). (all the power are the principal ones).

Combining Theorem 1 with Theorem 4 and Theorem 3 with Theorem 5, we obtain, respectively, the following two sandwich results:

Theorem 6 Let q_1 and q_2 be two convex function in U, with $q_1(0)=q_2(0)=1$. Let $\alpha\in\mathbb{C}^*$ with $\frac{\alpha_1}{A_1}\mathrm{Re}\,\alpha>0$.

Let $f\in A_p$ and suppose that $\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)/z^p\in H[q(0),1]\cap Q$. If the function

$$\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right),$$

is univalent in U, and

$$q_1(z)+\frac{\alpha A_1 zq_1'(z)}{p\alpha_1}\prec\frac{\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)}{z^p}\right)+\frac{p-\alpha}{p}\left(\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\right)$$

$$\prec q_2(z)+\frac{\alpha A_1 zq_2'(z)}{p\alpha_1}, \tag{39}$$

then

$$q_1(z)\prec\frac{\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{z^p}\prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinate and the best dominant of (39).

Theorem 7 Let q_1 and q_2 be two convex function in U, with $q_1(0)=q_2(0)=1$. Let $\gamma,\mu\in\mathbb{C}^*$ and $\nu,\eta,\delta,\Omega\in\mathbb{C}$ with $\nu+\eta\neq 0$ $\mathrm{Re}(\delta/\gamma)>0$. Let $f\in A_p$ and suppose that f satisfies the conditions:

$$\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\neq 0,\quad z\in U,$$

and

$$\left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu}\in H[q(0),1]\cap Q$$

If the function ψ given by (2.16) is univalent in U, and

$$\delta q_1(z)+\gamma zq_1'(z)+\Omega\prec\psi(z)\prec\delta q_2(z)+\gamma zq_2'(z)+\Omega, \tag{40}$$

then

$$q_1(z) \prec \left[\frac{\nu\phi_{p,q,s}^{m,l,\lambda}[\alpha_1+1,\beta_1;A_1,B_1]f(z)+\eta\phi_{p,q,s}^{m,l,\lambda}[\alpha_1,\beta_1;A_1,B_1]f(z)}{(\nu+\eta)z^p}\right]^{\mu} \prec q_2(z),$$

and q_1 and q_2 are, respectively, the best subordinate and the best dominant of (40).
(all the power are the principal ones).

References

[1] Ali, R. Ravichandran, M. V. Khan, M. H. and Subramaniam, K. G. Subramaniam, Differential sandwich theorems for certain analytic functions, Far East J. Math. Sci. 15 (1) (2004) 87–94.

[2] Al-Oboudi, F. M. On univalent functions defined by a generalized Salagean operator, Internat. J. Math. Math. Sci. 27 (2004) 1429-1436.

[3] Aghalary, R. R. Ali, M. Joshi ,S. B. and Ravichandran, V. Ravichandran, Inequalities for analytic functions defined by certain liner operator, Internat. J. Math. Sci, 4 (2005), 267-274.

[4] Aouf, M. K. and Bulboacă, T. Subordination and superordination properties of multivalent functions defined by certain integral operator, J. Franklin Inst. 347 (2010 641-653).

[5] M. K. Aouf, F. M. Al-Oboudi, M. M. Haidan, On some results for λ-spirallike and λ-Robertson functions of complex order, Publ. Inst. Math. Belgrade 77 (91) (2005) 93–98.

[6] Bulboacă,T., A class of superordination preserving integral operators, Indag-Math. (New Ser.) 13 (3) (2002) 301–311.

[7] Bulboacă, T. Classes of first-order differential subordinations, Demonstratio Math. 35 (2) (2002) 287–392.

[8] Bulboacă, T. Differential Subordinations and Superordinations. Recent Results, House of Scientific Book Publ, Cluj-Napoca, 2005.

[9] Catas, A. On certain classes of p-valent functions defined by multiplier transformations, in: Proc. Book of the International Symposium on Geometric Function Theory and Applications, Istanbul, Turkey, August 2007, pp. 241-250.

[10] Dziok J. and Srivastava, H. M. Classes of analytic functions associated with the generalized hypergeometric function. Applied Mathematics and Computation, 103(1999), 1–13.

[11] El-Ashwah, R. M. and Aouf M. K., Differential subordination and superordination for certain subclasses of p-valent functions, Math. Comp. Mode. 51 (2010) 349-360.

[12] Kamali, M. and Orhan, H. On a subclass of certain starlike functions with negative coefficients, Bull. Korean Math. Soc. 41 (1) (2004) 53-71.

[13] Kumar, S. S. Taneja, H.C. and Ravichandran, V. , Classes multivalent functions defined by Dziok_Srivastava linear operator and multiplier transformations, Kyungpook Math. J. (46) (2006) 97109.

[14] Miller, S. S. and Mocanu, P. T., Differential Subordinations: Theory and Applications, in: Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, Basel, 2000.

[15] Miller, S. S. and Mocanu, P. T., Subordinants of differential superordinations, Complex Var. Theory Appl. 48 (10) (2003) 815-826.

[16] Obradović, M. and Aouf, M. K. S. Owa, On some results for starlike functions of complex order, Publ. Inst. Math. Belgrade 46 (60) (1989)79–85.

[17] Royster, W. C. , On the univalence of a certain integral, Michigan Math. J. 12 (1965) 385–387.

[18] Salagean, G. S., Subclasses of univalent functions, in: Lecture Notes in Math., vol. 1013, Springer-Verlag, 1983, pp. 362-372.

[19] Selvaraj, C. K. and Karthikeyan, R. Differential subordinant and superordinations for certain subclasses of analytic functions, Far East J. Math. Sci. 29 (2)(2008) 419-430.

[20] Shanmugam, T. N. Ravichandran, and Sivasubramanian, V. S., Differential sandwich theorems for some subclasses of analytic functions, Austral. J. Math. Anal. Appl. 3 (1) (2006) (e-journal) article 8.

[21] Shanmugam, T. N., Sivasubramanian, S. and Silverman, H., On sandwich theorems for some classes of analytic functions, Int. J. Math. Math. Sci. 2006, Article ID 29684, 1–13.

[22] Shanmugam, T. N. Ramachandran, C. Darus, M. and Sivasubbramanian, S., Differential sandwich theorems for some subclasses of analytic functions involving a linear operator, Acta Math. Univ. Comenianae 74 (2) (2007) 287–294.

[23] Singh, V. On some criteria for univalence and starlikeness, Indian J. Pure Appl. Math. 34 (4) (2003) 569–577.

[24] Srivastava, H. M. Lashin, A. Y. , Some applications of the Briot–Bouquet differential subordination, J. Inequalities Pure Appl. Math. 6 (2) (2005) 1–7 article 41.

[25] Srivastava, H. M. Gupta, K. C. and Goyal S. P., The H-Functions of One and Two Variables with Applications. South Asian Publishers, New Delhi and Madras, 1982.

[26] Srivastava, H. M. Karlsson P. W., Multiple Gaussian Hypergeometric Series. Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[27] Srivastava, H. M. and Manocha, H. L., A Treatise on Generating Functions. Halsted Press ,Ellis Horwood, Limited, Chichester, 1984.

[28] Wang, Z. Gao, C. and Liao, M., On certain generalized class of non-Bazilević functions, Acta Math. Acad. Paed. Nyireg. New Ser. 21 (2) (2005) 147–154.