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Distance Ratio Metric on the Unit Disk

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Abstract

We prove Lipschitz continuity of arbitrary analytic mapping $f : D \to D$ regarding the distance ratio metric with the Lipschitz constant $C = 2$. This represents a generalization for the unit disk domain of Gehring - Palka theorem on Möbius transformations.

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1. Introduction

For a subdomain $G \subset \mathbb{R}^n$ and for all $x, y \in G$ distance ratio metric j_G is defined as

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right),$$

where $d(x,\partial G)$ denotes the Euclidean distance from x to ∂G. The distance ratio metric was introduced by F.W. Gehring and B.P. Palka [3] and in the above simplified form by M. Vuorinen [9]. As the "first approximation" of the quasihyperbolic metric, it is frequently used in the study of hyperbolic type metrics ([1],[2],[6],[10]) and geometric theory of functions.

For an open continuous mapping $f : G \to G'$ we consider the following condition: there exists a constant $C \geq 1$ such that for all $x, y \in G$ we have

$$j_{G'}(f(x), f(y)) \leq C j_G(x,y),$$

or, equivalently, that the mapping

$$f : (G, j_G) \to (G', j_{G'})$$

between metric spaces is Lipschitz continuous with the Lipschitz constant C.

However, neither the quasihyperbolic metric k_G nor the distance ratio metric j_G are invariant under Möbius transformations. Therefore, it is natural to ask what the Lipschitz constants are for these metrics under conformal mappings or Möbius transformations in higher dimension. F. W. Gehring, B.P. Palka and B. G. Osgood proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [2], [3]:

Theorem A. If G and G' are proper subdomains of $\mathbb{R}^n$ and if h is a Möbius transformation of G onto G', then for all $x, y \in G$

$$m_{G'}(h(x), h(y)) \leq 2 m_G(x,y),$$

where $m \in \{j, k\}$.

On the other hand, the next theorem from [6] yields a sharp form of Theorem A for Möbius automorphisms of the unit ball.

Theorem B. A Möbius transformation $h : \mathbb{B}^n \to \mathbb{B}^n$, satisfies

$$j_{\mathbb{B}^n}(h(x), h(y)) \leq (1 + |h(0)|) j_{\mathbb{B}^n}(x,y)$$

for all $x, y \in \mathbb{B}^n$. The constant is best possible.

An interesting fact is that for lower dimension n much more can be said.

In this paper we show that for $n = 2$ an analogous to the above result is valid for arbitrary analytic mappings of the unit disk D into itself. Thereby Theorem A is substantially generalized in this case.

In addition, we determine a class of analytic functions with bounded l_1 norm of their Maclorain coefficients which maps are contractions.

2. Results

A generalization of Theorem A for the unit disk case is given by the following

Theorem 1 For any analytic mapping f, $f : D \to D$, and all $z, w \in D, z \neq w$, we have

$$j_D(f(z), f(w)) < 2 j_D(z,w). \qquad (1.1)$$

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Proof.

Let max $\{|z|,|w|\}=r$ and suppose that $|f(z)|\geq|f(w)|$.Then

$$j_D(z,w)=\log(1+\frac{|z-w|}{1-r});\ j_D(f(z),f(w))=\log(1+\frac{|f(z)-f(w)|}{1-|f(z)|}).$$

Our main tool in the proof will be the famous Schwarz-Pick lemma, stated in the following form

Theorem C. Let f be an analytic mapping of the unit disk D into itself. Then for any $z_1,z_2\in D,$ we have

$$|\frac{f(z_2)-f(z_1)}{1-\bar{f}(z_1)f(z_2)}|\leq|\frac{z_2-z_1}{1-\bar{z}_1z_2}|.$$

An application of this lemma gives

$$|\frac{f(z)-f(w)}{1-\bar{f}(z)f(w)}|^{-2}-1\geq|\frac{z-w}{1-\bar{z}w}|^{-2}-1,$$

that is,

$$\begin{aligned}|\frac{f(z)-f(w)}{z-w}|^2&\leq\frac{(1-|f(z)|^2)(1-|f(w)|^2)}{(1-|z|^2)(1-|w|^2)}\\&\leq\frac{(1-|f(z)|^2)(1-|f(w)|^2)}{(1-r^2)^2},\end{aligned}$$

where we used the well known identity for complex numbers $x,y,$

$$|1-\bar{x}y|^2-|x-y|^2=(1-|x|^2)(1-|y|^2).$$

Therefore,

$$\frac{|f(z)-f(w)|}{1-|f(z)|}\leq\frac{|z-w|}{1-r}\frac{\sqrt{(1+|f(z)|)(1+|f(w)|)}}{1+r}\sqrt{\frac{1-|f(w)|}{1-|f(z)|}},$$

i.e.,

$$\frac{|f(z)-f(w)|}{1-|f(z)|}\leq\frac{|z-w|}{1-r}\frac{1+|f(z)|}{1+r}\sqrt{1+\frac{|f(z)-f(w)|}{1-|f(z)|}},\tag{1.2}$$

since

$$\frac{1-|f(w)|}{1-|f(z)|}=1+\frac{|f(z)|-|f(w)|}{1-|f(z)|}\leq1+\frac{|f(z)-f(w)|}{1-|f(z)|}.$$

To obtain an estimation for $|f(z)|, z\in D,$ suppose that $f(0)=a\in D.$ Applying Theorem C with $z_1=0, z_2=z,$ we get

$$|z|\geq\frac{|f(z)-a|}{|1-\bar{a}f(z)|}\geq\frac{|f(z)|-|a|}{1-|a||f(z)|},$$

that is,

$$|f(z)|\leq\frac{|z|+|a|}{1+|a||z|}\leq\frac{r+|a|}{1+|a|r}\tag{1.3}$$

By (1.3) we get $\frac{1+|f(z)|}{1+r} \leq \frac{1+|a|}{1+|a|r} := 2c(a,r) = 2c$ and, denoting $\frac{|z-w|}{1-r} := X \in (0,+\infty)$, the inequality (1.2) gives

$$1 + \frac{|f(z) - f(w)|}{1 - |f(z)|} \leq (cX + \sqrt{1 + c^2X^2})^2.$$

Therefore

$$\frac{j_D(f(z), f(w))}{j_D(z,w)} \leq \frac{2\log(cX + \sqrt{1 + c^2X^2})}{\log(1+X)}.$$

Now, it is easy to see that the function g,

$$g(X) := cX + \sqrt{1 + c^2X^2} - (1+X)$$

is negative for $X \in (0,T)$, where $T = \frac{2(1-c)}{2c-1} = \frac{2|a|r+1-|a|}{|a|(1-r)}$.

Since $X = \frac{|z-w|}{1-r} \leq \frac{2r}{1-r}$, it follows that $X \in (0,T)$, i.e., $\log(cX + \sqrt{1+c^2X^2}) < \log(1+X)$ $X \neq 0.$

Also,

$$\lim_{X \to 0} \frac{2\log(cX + \sqrt{1 + c^2X^2})}{\log(1+X)} = 2c \leq 1 + |a| < 2.$$

Therefore the inequality sign in (1.1) is strict.

Remark. An interesting problem is to find the best possible universal constant C^* such that for all analytic mappings $f : D \to D$ we have

$$j_D(f(z), f(w)) \leq C^* j_D(z,w).$$

By the above result and Theorem B, we get

$$1 + |f(0)| \leq C^* < 2.$$

In addition we give a sufficient condition for an analytic mapping to be a contraction, that is to have the Lipschitz constant at most 1.

Theorem 2 Let $f : D \to D$ be a non-constant mapping given by $f(z) = \sum_{k=0}^{\infty} a_k z^k$, under the condition

$$\sum_{k=0}^{\infty} |a_k| \leq 1.$$

Then for all $x, y \in D$,

$$j_D(f(x), f(y)) \leq j_D(x,y),$$

and this inequality is sharp.

Proof. Suppose that $|f(x)| \geq |f(y)|$. Then

$$j_D(x,y) = \log(1 + \frac{|x-y|}{\min\{1-|x|, 1-|y|\}})$$

and

$$j_D(f(x), f(y)) = \log(1 + \frac{|f(x) - f(y)|}{1 - |f(x)|}).$$

We have

$$|f(x) - f(y)| = |x - y||\sum_{k=1}^{\infty} a_k(\sum_{i+j=k-1} x^i y^j)| \le |x - y|\sum_{k=1}^{\infty}|a_k|(\sum_{i=0}^{k-1}|x|^i)$$

and

$$1 - |f(x)| \ge \sum_{k=1}^{\infty}|a_k| - \sum_{k=1}^{\infty}|a_k||x|^k = (1 - |x|)\sum_{k=1}^{\infty}|a_k|(\sum_{i=0}^{k-1}|x|^i).$$

Hence

$$j_D(f(x), f(y)) \le \log(1 + \frac{|x - y|}{1 - |x|}) \le j_D(x, y).$$

For the sharpness of the inequality let $a_p = 1, a_i = 0, i \neq p$ i.e., $f(z) = z^p$ ($p \in \mathbb{N}$). For $s, t \in (0, 1)$ and s < t, we have

$$\frac{j_D(f(t), f(s))}{j_D(t, s)} = \frac{\log \frac{1-s^p}{1-t^p}}{\log \frac{1-s}{1-t}} = \frac{\log \frac{1-s}{1-t} + \log \frac{1+s+\cdots+s^{p-1}}{1+t+\cdots+t^{p-1}}}{\log \frac{1-s}{1-t}}.$$

Letting $t \to 1^-$ we obtain $C = 1$. Therefore this constant is sharp.

Note that the equality sign in (2.1) is possible. Take for example $f(z) = z$ or $f(z) = \frac{z+1}{2}$ with $z = -w = r, \ 0 < r < 1$.

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