ISSN 2347-1921

Blowup of solution for a reaction diffusion equation with memory and multiple nonlinearities

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ABSTRACT

In this paper, the blow-up of solution for the initial boundary value problem of a class of reaction diffusion equation with memory and multiple nonlinearities is studied. Using a differential inequalities, we obtain sufficient conditions for the blow-up of solutions in a finite time interval under suitable conditions on memory and nonlinearities term and for vanishing initial energy.

Indexing terms/Keywords

reaction diffusion equation; blow-up; memory term; multiple nonlinearities.

Academic Discipline And Sub-Disciplines

Partial Differential Equation.

Mathematics Subject Classification: 35K57,35B44.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 6, No. 3
editor@cirworld.com
www.cirworld.com, member.cirworld.com

1. INTRODUCTION

In this paper, we study the following initial boundary value problem of a class of reaction diffusion equation with memory and multiple nonlinearities

$$u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + |u|^{m-2} u_t = |u|^{p-2} u, \tag{1.1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $m > 2, p > 2$ are real numbers and Ω is bounded domain in $R^n (n \geq 1)$ with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. Here, g represents the kernel of memory term and Δ denotes the Laplace operator in Ω.

This type of equation arises from a variety of mathematical models in engineering and physical sciences, it appears in the models of chemical reactions, heat transfer, population dynamics, and so on (see [1] and references therein).

When $g = 0$ and the nonlinear diffusion term $|u|^{m-2} u_t$ is absent in (1.1), it is well known that the nonlinear $|u|^{p-2} u$ reaction term drives the solution of (1.1)-(1.3) to blow up in finite time and the diffusion term is known to yield existence of global solution if the reaction term is removed from the equation [2]. The more general equation

$$u_t - div(|\nabla u|^{m-2} \nabla u) = f(u), \tag{1.4}$$

has attracted a great deal of people. The obtained results show that global existence and nonexistence depend roughly on m, the degree of nonlinearity in f, the dimension n, and the size of the initial data. See in this regard, the works of Levine [3], Kalantarov and Ladyzhenskaya [4], Levine et al. [5], Messaoudi [6], Liu et al. [7] and references therein. In the absence of the memory term, Pucci and Serrin [8] discuss the stability of the following equation

$$|u_t|^{l-2} u_t - div(|\nabla u|^{m-2} \nabla u) = f(u), \tag{1.5}$$

Levine et al. [5] got the global existence and nonexistence of solution for (1.5). Pang [9,10] and Berrimi [11] given the sufficient condition of blow-up result for certain solutions of (1.5) with positive or negative initial energy.

When $g \neq 0$, the equation (1.1) without term $|u|^{m-2} u_t$ arises from the study of heat conduction in materials with memory. In this case, Messaoudi [12,13] obtained a blow-up result for certain solutions with positive or negative initial energy and Giorgi [14] got the asymptotic behavior. Furthermore, for the quasilinear case

$$|u_t|^{m-1} u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds = |u|^{p-2} u, \tag{1.6}$$

Messaoudi et al. [15] established a general decay result from which the usual exponential and polynomial decay results are only special cases for (1.6) without reaction term $|u|^{p-2} u$. Liu et al. [17] obtained a general decay of the energy function for the global solution and a blow-up result for the solution with both positive and negative initial energy for (1.6).

Equation (1.1) can also be as a special case of doubly nonlinear parabolic-type equations (or the porous medium equation)[18,5] without memory,

$$\beta(u)_t - \Delta u = |u|^{p-2} u$$

if we take $\beta(u) = u + |u|^{m-2} u$. We only mention the work [18,19] for this class equation. The above equation can also describes an electric breakdown in crystalline semiconductors with allowance for the linear dissipation of bound- and free-charge sources [20]. We should also point out that Polat [21] established a blow up result for the solution with vanishing initial energy of the following initial boundary value problem

$$u_t - u_{xx} + |u|^{m-2} u_t = |u|^{p-2} u.$$

In this paper, we consider the blow-up of solution for the initial boundary value problem of a reaction diffusion equation with memory and multiple nonlinearities. Using a differential inequalities, we obtain sufficient conditions for the blow-up of solutions in a finite time interval under suitable conditions on memory and nonlinearities term and for vanishing initial energy. This article is organized as follows. Section 2 is concerned with some notations and statement of assumptions. In Section 3, we give and prove the main result.

2. PRELIMINARIES

In this section, we will give some notations and statement of assumptions for p,m,g. We denote $L^p(\Omega)$ by L^p, $H_0^1(\Omega)$ by H_0^1, the usual Soblev space. The norm and inner of $L^p(\Omega)$ are denoted by $\|.\|_p=\|.\|_{L^p(\Omega)}$ and $(u,v)=\int_\Omega u(x)v(x)dx$, respectively. Especially, $\|.\|=\|.\|_{L^2(\Omega)}$ for $p=2$.

For the relaxation function g and the number m and p, we assume that

(A1) $g:R^+\to R^+$ is a differentiable function satisfying

$$g(s)\geq 0,\ 1-\int_0^\infty g(s)ds=l>0;$$

(A2) There exists a nonincreasing function $\xi:R^+\to R^+$ such that

$$g'(s)\leq -\xi(s)g(s);$$

(A3) We also assume that

$$2<m<p\leq \frac{2(n-1)}{n-2},\ \text{if } n\geq 3;\ 2<m<p<+\infty,\ \text{if } n=1,2.$$

Similar to [21,17], we call $u(x,t)$ a weak solution of problem (1.1)-(1.3) on $\Omega\times[0,T)$, if

$$u\in C(0,T;H_0^1)\cap C^1(0,T;L^2),\ |u|^{m-2}u_t\in L^2((0,T)\times\Omega)$$

satisfying $u(0)=u_0$, and

$$\int_0^t\int_\Omega [\nabla u(s)\nabla v(s)-\int_0^s g(s-\tau)\nabla u(\tau)\nabla v(\tau)d\tau+v(s)u_t(s)$$

$$+|u|^{m-2}u_t v-|u|^{p-2}uv]dxds=0,\quad \forall v\in C(0,T;H_0^1),\ t\in[0,T).$$

In this paper, we always assume that the problem (1.1)-(1.3) exist a local solution.

Now, we introduce two functionals

$$E(t)=\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}(g\otimes\nabla u)(t)-\frac{1}{2}\int_0^t g(s)ds\|\nabla u(t)\|^2-\frac{1}{p}\|u\|_p^p, \tag{2.1}$$

$$E(0)=\frac{1}{2}\|\nabla u_0\|^2-\frac{1}{p}\|u_0\|_p^p, \tag{2.2}$$

where $u\in H_0^1$ and

$$(g\otimes\nabla v)(t)=\int_0^t g(t-s)\|\nabla v(s)-\nabla v(t)\|^2ds.$$

Multiplying Equation (1.1) by u_t and integrating over Ω, we have

$$E'(t)=-\|u_t\|^2+\frac{1}{2}(g'\otimes\nabla u)(t)-\int_\Omega|u|^{m-2}u_t^2dx-\frac{1}{2}g(t)\|\nabla u(t)\|^2<0. \tag{2.3}$$

3.BLOWUP OF SOLUTION

In this section, we consider the blow-up condition for the system (1.1)-(1.3) and prove the main result. Our technique is more complex than that in [21], [17] and [5] because of the presence of the nonlinear term $|u|^{m-2}u_t$ and the memory term.

Theorem 3.1 Suppose that the assumption (A1),(A2) and (A3) hold, $u_0 \in H_0^1$ and u is a local solution of the system (1.1)-(1.3), $E(0)<0$ is sufficient negative. Furthermore, g satisfies

$$\int_0^\infty g(s)ds < \frac{p-2}{p-\frac{3}{2}}, \tag{3.1}$$

Then the solution of the system (1.1)-(1.3) blows up in finite time.

Proof Let

$$F(t) = \tfrac{1}{2}\|u\|^2 + \frac{1}{m}\|u\|_m^m. \tag{3.2}$$

Noting that

$$\frac{d}{dt}\|u\|_m^m = m\int_\Omega |u|^{m-2}uu_t dx,$$

using (1.1), we get from (3.2)

$$\begin{aligned} F'(t) &= \int_\Omega uu_t dx + \int_\Omega |u|^{m-2}uu_t dx \\ &= -\int_\Omega [\nabla u(t)\nabla u(t) - \int_0^t g(t-s)\nabla u(s)\nabla u(t)ds + |u|^{m-2}uu_t - |u|^p]dx + \int_\Omega |u|^{m-2}uu_t dx \\ &= -\|\nabla u(t)\|^2 + \int_0^t g(t-s)ds\|\nabla u(t)\|^2 - \int_\Omega \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\nabla u(t)ds + \|u\|_p^p. \end{aligned}$$

By Young inequality, we get

$$F'(t) \ge \|u\|_p^p - (1-\tfrac{3}{4}\int_0^t g(s)ds)\|\nabla u(t)\|^2 - (g\otimes\nabla u)(t). \tag{3.3}$$

From the expression of $E(t)$,

$$\|\nabla u(t)\|^2 = \frac{2}{1-\int_0^t g(s)ds}[E(t) + \frac{1}{p}\|u\|_p^p - \frac{1}{2}(g\otimes\nabla u)],$$

then the equation (3.3) can be rewrite

$$\begin{aligned} F'(t) &\ge \|u\|_p^p - \frac{2(1-\frac{3}{4}\int_0^t g(s)ds)}{1-\int_0^t g(s)ds}[E(t) + \frac{1}{p}\|u\|_p^p - \frac{1}{2}(g\otimes\nabla u)] - (g\otimes\nabla u) \\ &= [1-\frac{2(1-\frac{3}{4}\int_0^t g(s)ds)}{p(1-\int_0^t g(s)ds)}]\|u\|_p^p + \frac{2(1-\frac{3}{4}\int_0^t g(s)ds)}{1-\int_0^t g(s)ds}(-E(t)) + [\frac{2(1-\frac{3}{4}\int_0^t g(s)ds)}{1-\int_0^t g(s)ds} - 1](g\otimes\nabla u)(t). \end{aligned} \tag{3.4}$$

Since $E(0)<0$ we have $-E(t) \ge -E(0) > 0$ by (2.3), then we deduce

$$F'(t) \ge r\|u\|_p^p, \tag{3.5}$$

where $r = 1-\frac{2(1-\frac{3}{4}\int_0^t g(s)ds)}{p(1-\int_0^t g(s)ds)} > 0$ by (3.1).

Since $E(t)\le E(0)<0$ is sufficient negative, by the expression of $E(t)$, we can suppose that $\|u\|_p^2\ge\|u\|^2>1$, then, by $p>m>2$ and embedding theorem,

$$\|u\|_p^p\ge C(\|u\|^2)^{\frac{p}{2}}\ge C(\|u\|^2)^{\frac{p}{m}},\quad \|u\|_p^p\ge C(\|u\|_m^m)^{\frac{p}{m}},$$

where $C>0$ is a positive constant from the embedding theorem. Therefore

$$\begin{aligned} F'(t) &\ge r\|u\|_p^p=\tfrac{r}{2}\|u\|_p^p+\tfrac{r}{2}\|u\|_p^p \\ &\ge \tfrac{r}{2}C(\|u\|^2)^{\frac{p}{m}}+\tfrac{r}{2}C(\|u\|_m^m)^{\frac{p}{m}} \\ &\ge C_1(\|u\|^2+\|u\|_m^m)^{\frac{p}{m}} \\ &=C_0F^{\frac{p}{m}}(t), \end{aligned} \tag{3.6}$$

where $C_1=2^{-\frac{p}{m}}Cr$, $C_0=\frac{C_1}{m}$, and using the inequality

$$(a_1+a_2)^k\le 2^{k-1}(a_1{}^k+a_2{}^k),\ a_1,a_2\ge 0,k>1.$$

A simple integration of the inequality (3.6) over $(0,t)$ yields

$$F^{\frac{p-m}{m}}(t)\ge\frac{1}{F^{\frac{p-m}{m}}(0)-\frac{p-m}{m}C_0t}.$$

Therefore there exists a positive constant given by $T=\dfrac{m}{C_0(p-m)F^{\frac{p-m}{m}}(0)}$ (where $F(0)>0, p>m$), such that $F(t)\to\infty, t\to T$. This completes the proof.

ACKNOWLEDGMENTS

This work is supported by National Natural Science Foundation of China (No.11171311) and by Natural Science Foundation of Henan Provinence (1323004100360).

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