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On Statistically Convergent and Statistically Cauchy Sequences in Non-Archimedean Fields

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ABSTRACT

In this paper, K denotes a complete, non-trivially valued non-archimedean field. In the present paper, statistical convergence of sequences and statistically Cauchy sequences are defined and a few theorems on statistically convergent sequences are proved in such fields K.

Keywords:

Statistically convergent sequences; Statistically Cauchy sequences; Statistical limit superior; Statistical limit inferior.

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INTRODUCTION

The concept of statistical convergence was introduced by Fast in 1951. It was further studied, in detail by Kolk, Maddox, Bulut and Huseyin Cakalli [5]. The purpose of this paper is to give characterizations of statistical convergence of sequences and statistical Cauchy sequences in Non-Archimedean fields, which are analogous to the work of D. Rath and B.C. Tripathy [7], in classical case.

Definition 1.

Let K be a complete, non-trivially valued non-archimedean field.

A sequence $x = \{x_k\}$ in K is said to be statistically convergent to a limit 'l' if for any ϵ>o ,

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: \|x_n - l\| \ge \epsilon\}| = 0$$

Symbolically we write stat-$\lim\limits_{n\to\infty} x_k = l$ (or) $x_k \xrightarrow{stat} l$.

Definition 2.

$x = \{x_k\}$ is a statistically Cauchy sequence if for any $\epsilon > o$, there exists an $n \in \mathbb{N}$ *such that*

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: \|x_{n+1} - x_n\| \ge \epsilon\}| = 0$$

Theorem 1.

A sequence $\{x_n\}$ is statistically convergent if and only if the following condition is satisfied:

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)}\| \ge \epsilon\right\}\right| = 0$$

where $\{x_{k'(r)}\}$ *is a subsequence of* $\{x_n\}$ *such that* $\lim\limits_{r\to\infty} x_{k'(r)} = l$

Proof:

Let a sequence $\{x_n\}$ be statistically convergent .

To prove $\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)}\| \ge \epsilon\right\}\right| = 0$ …. (1)

is satisfied.

By definition of statistical convergence of a sequence $\{x_n\}$ to limit l ,we have,

$$\lim_{n\to\infty}\frac{1}{n}|\{k \le n: \|x_k - l\| \ge \epsilon\}| = 0 \quad \text{…. (2)}$$

Now, $\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)}\| \ge \epsilon\right\}\right|$

$$= \lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)} + l - l\| \ge \epsilon\right\}\right|$$

$$= \lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n, k' \le n: \|(x_k - l) + (l - x_{k'(r)})\| \ge \epsilon\right\}\right|$$

$$\le \lim_{n\to\infty}\frac{1}{n}|\{k \le n: \|x_k - l\| \ge \epsilon\}| + \lim_{n\to\infty}\frac{1}{n}\left|\left\{k' \le n: \|l - x_{k'(r)}\| \ge \epsilon\right\}\right|$$

$$\le max\left[\lim_{n\to\infty}\frac{1}{n}|\{k \le n: \|x_k - l\| \ge \epsilon\}|, \lim_{n\to\infty}\frac{1}{n}\left|\left\{k' \le n: \|x_{k'(r)} - l\| \ge \epsilon\right\}\right|\right]$$

$$\le max\left[0, \lim_{n\to\infty}\frac{1}{n}\left|\left\{k' \le n: \|x_{k'(r)} - l\| \ge \epsilon\right\}\right|\right] \quad \text{…. (3) (using (2))}$$

It is given that $\lim\limits_{r\to\infty} x_{k'(r)} = l.$

Since it is convergent, it is also statistically convergent.

Therefore we can write

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k' \le n: \|x_{k'(r)} - l\| \ge \epsilon\right\}\right| = 0 \quad \text{…. (4)}$$

In view of (3) & (4)

$$\therefore \lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)}\| \ge \epsilon\right\}\right| = 0$$

Conversely, let

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)}\| \ge \epsilon\right\}\right| = 0 \quad \text{.... (5)}$$

To prove that the sequence $\{x_n\}$ is statistically convergent.

To this end, consider

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n: \|x_k - l\| \ge \epsilon\}|$$

$$= \lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)} + x_{k'(r)} - l\| \ge \epsilon\right\}\right|$$

$$\le \lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)}\| \ge \epsilon\right\}\right|$$

$$+ \lim_{n\to\infty} \frac{1}{n}\left|\left\{k' \le n: \|x_{k'(r)} - l\| \ge \epsilon\right\}\right|$$

$$\le \max[\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n, k' \le n: \|x_k - x_{k'(r)}\| \ge \epsilon\right\}\right| ,$$

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k' \le n: \|x_{k'(r)} - l\| \ge \epsilon\right\}\right|]$$

$$\le \max [0 , \lim_{n\to\infty} \frac{1}{n}\left|\left\{k' \le n: \|x_{k'(r)} - l\| \ge \epsilon\right\}\right|] \text{ , (using (5))}$$

implies that $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: \|x_k - l\| \ge \epsilon\}| = 0$, (using (4))

This implies that the sequence $\{x_n\}$ is statistically convergent.

Theorem 2.

If $\lim_{k\to\infty} x_k = l$ and $stat - \lim_{k\to\infty} y_k = 0$, then

$$stat - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k \text{ .}$$

Proof.

Given $\lim_{k\to\infty} x_k = l$

(i.e) $\|x_k - l\| = 0$, as $k \to \infty$ .... (6)

Also , $stat - \lim_{k\to\infty} y_k = 0$

That is $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|y_n - 0\| \ge \epsilon\}| = 0$ .... (7)

Now,

Let $stat - \lim_{k\to\infty}(x_k + y_k) = l'$ .... (8)

Therefore,

$stat - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|(x_n + y_n) - l'\| \ge \epsilon\}| = 0$ ....(9)

This implies that

$$\left|\lim_{k\to\infty}\|x_k - l'\| + \lim_{k\to\infty}\frac{1}{k}|\{n \le k: \|y_n - 0\| \ge \epsilon\}|\right| = 0$$

That is $max\left[\lim_{k\to\infty} \|x_k - l'\|, \quad \lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|y_n - 0\| \ge \epsilon\}|\right] = 0$

That is $max\left[\lim_{k\to\infty} \|x_k - l'\|, \ 0\right] = 0$, (using (2))

which implies that $\lim_{k\to\infty} \|x_k - l'\| = 0$

That is $\lim_{k\to\infty} x_k = l'$. But $\lim_{k\to\infty} x_k = l$. This $\Rightarrow l' = l$ ....(10)

From (8) & (10) , it is proved that

$stat - \lim_{k\to\infty}(x_k + y_k) = \lim_{k\to\infty} x_k$.

Theorem 3.

If a sequence $x = \{x_k\}$ is statistically convergent to l, then there are sequences

$y = \{y_k\}$ and $z = \{z_k\}$ such that $\lim_{k\to\infty} y_k = l, x = y + z$ and

$\lim_{n\to\infty} \frac{1}{n}|\{k \le n: x_k \ne y_k\}| = 0$ and $z = \{z_k\}$ is a statistically null sequence.

Proof.

Given a sequence $x = \{x_k\}$ is statistically convergent to l ,

$$That\ is\ \lim_{n\to\infty} \frac{1}{n}|\{k \le n: \|x_k - l\| \ge \epsilon\}| = 0 \qquad \text{.... (11)}$$

To prove

(i) there exist sequences $y = \{y_k\}$ and $z = \{z_k\}$ such that

$\|y_k - l\| \to 0, k \to \infty$ where $x = y + z; x = \{x_k\}$

(ii) $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: x_k \ne y_k\}| = 0$ and

(iii) $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: \|z_k - 0\| \ge \epsilon\}| = 0$

(i) $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: \|x_k - l\| \ge \epsilon\}| = 0$

which we write as

$$stat - \lim_{k\to\infty} x_k = l \qquad \text{.... (12)}$$

Now,

$stat - \lim_{k\to\infty} x_k$ means that $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|x_n - l\| \ge \epsilon\}| = 0$

That is $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|x_n - l + y_n - y_n\| \ge \epsilon\}| = 0$

That is $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|(x_n - y_n) + (y_n - l)\| \ge \epsilon\}| = 0$

Therefore $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|x_n - y_n\| \ge \epsilon\}| + \lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|y_n - l\| \ge \epsilon\}| = 0$

.... (13)

implies that $\max\left[\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|x_n - y_n\| \ge \epsilon\}|, \lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|y_n - l\| \ge \epsilon\}|\right] = 0$

That is $\max\left[\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|x_n - y_n\| \ge \epsilon\}| ,0 \right] = 0 \quad (since\ \lim_{n\to\infty} y_n = l)$

Hence $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|x_n - y_n\| \ge \epsilon\}| = 0,$

which implies that $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: \|x_n - y_n\| \nrightarrow 0\}| = 0$

That is $\lim_{k\to\infty} \frac{1}{k}|\{n \le k: x_n \ne y_n\}| = 0$

(or) $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: x_k \ne y_k\}| = 0$

Since $\lim_{k\to\infty} y_k = l$ and since $x = y + z$

and $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: x_k \ne y_k\}| = 0$, in view of previous theorem,

we have that

$stat - \lim_{k\to\infty} (y_k + z_k) = \lim_{k\to\infty} y_k \quad (= l)$

implies that $stat - \lim_{k\to\infty} z_k$ must be $=0$

(ie) $z = \{z_k\}$ is a statistically null sequence.

Definition3.

For a sequence $x = \{x_k\}$, let B_x denote the set

$B_x = \{b \in K / x_k > b\}$

Similarly ,

$A_x = \{a \in K / x_k < a\}$.

Definition 4.

If $x = \{x_k\}$ is a sequence, then statistical limit superior of x is given by

$$stat - limsupx = \begin{cases} supB_x & , if \ B_x \neq \emptyset \\ -\infty & , if \ B_x = \emptyset \end{cases}$$

Definition 5.

If $x = \{x_k\}$ is a sequence, then statistical limit superior of x is given by

$$stat - liminfx = \begin{cases} supA_x & , if \ A_x \neq \emptyset \\ +\infty & , if \ A_x = \emptyset \end{cases}$$

Theorem 4.

If $\beta = stat - limsupx$ is finite, then for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: \beta + \epsilon < x_k < \beta - \epsilon\}| = 0 \qquad \text{....(14)}$$

Conversly, if (14) holds, then $\beta = stat - limsupx$.

Proof.

Given $\beta = stat - limsupx$ is finite .... (15)

To prove $\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: \beta + \epsilon < x_k < \beta - \epsilon\}| = 0$

Let us consider the case of statistical limit superior as

$stat - limsupx = -\infty$, if $B_x = \emptyset$

$(i.e)$ if $|B_x| = |\{b \in k: x_k > b\}| = 0$ ....(16)

$\Rightarrow$ $\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: \beta + \epsilon < x_k < \beta - \epsilon\}| = 0$ ....(17)

Conversly,

let us suppose that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: \beta + \epsilon < x_k < \beta - \epsilon\}| = 0$$

To prove

$$\beta = stat - limsupx$$

$We\ know\ that\ |B_x| = |\{b \in k: x_k > b\}| = 0$, by the definition

That is, $B_x = \emptyset$,

$Which\ implies\ that,$ $lubB_x = -\infty$

$Therefore, \beta = stat - limsupx$

In view of (17) we have that

$$|x_k < \beta - \epsilon| \neq 0$$

$That\ is, B_x \neq \emptyset$

$Which\ implies\ that,\ \beta = lubB_x = supB_x$

$Therefore, \beta = stat - limsupx$

In any case, $\beta = stat - limsupx$.

Theorem 5.

If $\alpha = stat - liminfx$ is finite, then for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: \alpha + \epsilon < x_k < \alpha - \epsilon\}| = 0 \qquad \text{....(18)}$$

Conversly, if (18) holds, then $\alpha = stat - liminfx$.

Proof.

Given $\alpha = stat - liminf x$ is finite ....(19)

To prove $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: \alpha + \epsilon < x_k < \alpha - \epsilon\}| = 0$

Let us consider the case of statistical limit inferior as

$stat - liminf x = +\infty$, if $A_x = \phi$

That is, if $|A_x| = |\{a \in k: x_k < a\}| = 0$ .... (20)

$\Rightarrow$ $\lim_{n\to\infty} \frac{1}{n}|\{k \le n: \alpha + \epsilon < x_k < \alpha - \epsilon\}| = 0$ ....(21)

Conversly,

let us suppose that

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n: \alpha + \epsilon < x_k < \alpha - \epsilon\}| = 0$$

To prove

$$\alpha = stat - liminf x$$

We know that $|A_x| = |\{a \in k: x_k < a\}| = 0$, by the definition

That is $A_x = \emptyset$,

$which\ implies\ that,$ $\alpha = lubA_x = +\infty$

Therefore, $\alpha = stat - liminf x$

In view of (21) we have that

$|x_k > \alpha + \epsilon| \neq 0$

That is, $A_x \neq \emptyset$

which implies that, $lubA_x = inf x$.

Therefore, $\alpha = stat - liminf x$

In any case, $\alpha = stat - liminf x$.

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