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# ON upper and lower SP-θ (semi-pre-θ)-Continuous Multifunctions

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## ABSTRACT

In this paper we introduce and investigate a new types of multifunctions namely upper(lower) sp-θ-continuous multifunctions and upper(lower) semi-pre-θ-continuous multifunctions by using the notions in [1]. The concept of upper(lower) sp-θ-continuous multifunctions is stronger than both upper(lower) sp-continuous multifunctions [1] and upper(lower) semi-pre-θ-continuous multifunctions. And the concept of upper(lower) semi-pre-θ-continuous multifunctions is a generalization of upper(lower) sp-θ-continuous multifunctions and stronger than upper(lower) semi-pre-continuous multifunctions. Furthermore, the relationships among these notions and other of well-known types of multifunctions are also discussed.

**Keywords:** b-θ-open sets; β-θ-open sets; uppe(lower) b-θ-continuous multifunctions; upper(lower) β-θ-continuous multifunctions.

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## 1. INTRODUCTION

It is well known that various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunction. This implies that both functions and multifunction are important tools for studying properties of spaces and for constructing new spaces from previously existing ones, also Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc, in 1996, El- Atik [2] introduced the concept of γ-continuous functions as a generalization of semi-continuous functions due to Levine [3] and pre-continuous functions due to Mashhour etal. [4]. Most of these weaker forms of continuity in ordinary topology such as α-continuity, continuous, pre-continuity, qusai-continuity and β-continuity have been extended to multifunction (cf. [5, 6-9]) In this paper we introduce two new classes of Multifunctions, namely upper(lower) sp-θ-Continuous Multifunctions and upper(lower) semi-pre-θ-Continuous Multifunctions, the class of upper(lower) sp-θ-Continuous Multifunctions is stronger than upper(lower) sp-Continuous Multifunctions, and the class of upper(lower) semi-pre-θ-Continuous Multifunctions is a generalization of upper(lower) sp-θ-Continuous Multifunctions and to obtain several characterizations of these multifunctions and present several of their properties and we discussion the relationships among upper(lower) sp-θ (resp. semi-pre-θ)-Continuous Multifunctions and some other known types of Multifunctions.

## 2. PRELIMINARIES

Throughout this paper, (X, T) and (Y, $T^*$) (or simply X and Y) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of X, The closure and interior of A are denoted by Cl(A) and Int(A), respectively. We recall the following definitions, which will be used often throughout this paper.

**Definition 2.1:** Let (X, T) be a topological space. A subset A of X is said to be:

a) α-open [10] if $A \subset Int(Cl(Int(A)))$.

b) Semi-open [11] if $A \subset Cl(Int(A))$.

c) Pre-open [4] if $A \subset Int(Cl(A))$.

d) β-open [12] or semi-pre-open [13] if $A \subset Cl(Int(Cl(A)))$.

e) b-open [14] or γ-open[2]or sp-open [15]) if $A \subset Int(Cl(A)) \cup Cl(Int(A)))$.

**Remark 2.2:** The complement of a semi-open (resp. α-open, preopen, β-open, b-open) set is said to be semi-closed [16], (resp. α-closed [17], preclosed [18], β-closed [4], b-closed [2]. The intersection of all b-closed (resp. semi-closed, α-closed, preclosed, β-closed) sets of X containing A is called the b-closure [2] (resp. s-closure[16], α-closure[10], pre-closure[18], β-closure [12])of A and are denoted by bCl(A) (resp. SCl(A),αCl(A),PCl(A),βCl(A)).The union of all b-open sets of X contained in A is called the b-interior[14] of A and is denoted by bInt(A).

**Definition 2.3:** A point x of X is called a b-θ-cluster point of A if $bCl(U) \cap A \neq \emptyset$ for every b-open set U containing x. The set of all b-θ-cluster points of A is called the b-θ-closure of A and is denoted by $bCl_\theta(A)$ [19]. A sub set A is said to be b-θ-closed if $A = bCl_\theta(A)$.The complement of a b-θ-closed set is called b-θ-open set. The union of all b-θ-open sets of X contained in A is called the bθ-interior of A and is denoted by $bInt_\theta(A)$.

**Definition 2.4:** The β-θ-closure of A [20], denoted by $\beta Cl_\theta(A)$, is defined to be the set of all $x \in X$ such that $\beta Cl(V) \cap (A) \neq \emptyset$ for every $V \in \beta\Sigma(X, x)$ with $x \in V$. A subset A is said to be β-θ-closed [20] if $A = \beta Cl_\theta(A)$. The complement of a β-θ-closed set is said to be β-θ-open. The union of all β-θ-open sets of X contained in A is called the βθ-interior of A and is denoted by $\beta Int_\theta(A)$.

**Definition 2.5:** Let x be a point of X and V a subset of X. The set V is called a b-θ-neighborhood [21] of x in X if there exists a b-θ-open set A of X such that $x \in A \subset V$.

**Remark 2.6:** The family of all b-open (resp. β-open, α-open, semi-open, preopen) subsets of X containing a point $x \in X$ is denoted by BΣ(X, x) (resp. βΣ(X, x), αΣ(X, x), SΣ(X, x), pΣ(X, x)), The family of all b-open (resp. β-open, α-open, semi-open, preopen) sets in X are denoted by BΣ(X, T) (resp. βΣ(X, T), αΣ(X, T), SΣ(X, T), pΣ(X, T)). The family of all b-θ-open (resp. β-θ-open) subsets of X containing a point $x \in X$ is denoted by BθΣ(X, x) (resp. βθΣ(X, x)) the family of all b-θ-open (resp. β-θ-open) sets in X is denoted by BθΣ(X, T) (resp. βθΣ(X, T)).

**Definition 2.7:** A subset A is said to be b-regular [19] (resp. β-regular [20]) if it is both b-open and b-closed (resp. β-open and β-closed). The family of all b-regular (resp. β-regular) sets of X is denoted by BR(X) (resp. βR(X)).

**Definition 2.8:** A point $x \in X$ is called a θ-cluster point of A if $Cl(V) \cap A \neq \emptyset$ for every open subset V of X containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $Cl_\theta(A)$. If $A = Cl_\theta(A)$, then A is said to be θ-closed [22]. The complement of a θ-closed set is said to be θ-open.

By a multifunction $F: (X, T) \rightarrow (Y, T^*)$, we mean a point to set correspondence from X into Y, also we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F: (X, T) \rightarrow (Y, T^*)$, the upper and lower inverse of any subset A of Y are denoted by $F^+(A)$ and $F^-(A)$,respectively, Where $F^+(A) = \{x \in X: F(x) \subset A\}$ and $F^-(A) = \{x \in X: F(x) \cap A \neq \emptyset\}$. In particular,

$F^{-}(y) = \{x \in X: y \in F(x)\}$ for each point $y \in Y$. A multifunction $F: (X, T) \rightarrow (Y, T^{*})$ is said to be a surjection if $F(X) = Y$. a multifunction $F: (X, T) \rightarrow (Y, T^{*})$ is called upper semi-continuous (rename upper continuous) (resp. lower semi-continuous (rename lower continuous) if $F^{+}(V)$ (resp. $F^{-}(V)$) is open in X for every open set V of Y [1].

**Remark 2.9:**

a) Since the notion of b-open (resp. β-open) sets and the notion of sp-open (resp. semi-preopen) sets are same, we will use the term b-θ-open (resp. β-θ-open) sets instead of sp-θ-open (resp. semi-pre-θ-open) sets.

b) Since the notion of b-open sets and the notion of γ-open sets are same, we will use The term b-open sets instead of γ-open sets.

**Lemma 2.10:** For a subset A [19] of a topological space X, the following properties hold:

a) If $A \in B\Sigma(X, T)$, then $bcl(A) = bCl_{\theta}(A)$.

b) $A \in BR(X)$, if and only if A is b-θ-open and b-θ-closed.

**Lemma 2.11:** For a subset A [20] of a topological space X, the following properties hold:

a) If $A \in \beta\Sigma(X, T)$, then $\beta cl(A) = \beta Cl_{\theta}(A)$.

b) $A \in \beta R(X)$, if and only if A is β-θ-open and β-θ-closed.

**Remark 2.12:**

a) It is obvious that b-regular ⇒ b-θ-open ⇒ b-open. But the Converses are not necessarily true as shown by the examples in [19].

b) It is obvious that β-regular ⇒ β-θ-open ⇒ β-open. But the Converses are not necessarily true as shown by the examples in [20].

**Remark 2.13:** For modifications of open sets defined in Definitions (2.1), (2.3) and (2.4) the following relationships are known:

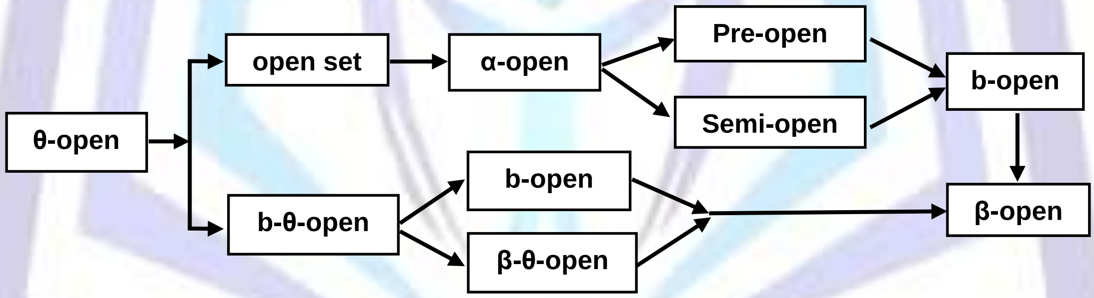


**Fig 1: The relationships among some well-known generalized open sets in topological spaces.**

**Definition 2.14:** A multifunction $F: (X, T) \rightarrow (Y, T^{*})$ is said to be:

a) Upper b-continuous [1] (resp. upper almost continuous [23,7,24] or upper precontinuous [7], upper quasi-continuous [6], upper α-continuous [5], upper β-continuous [8,9]) if for each $x \in X$, and each open set V of Y containing F(x),there exists $U \in B\Sigma(X, x)$ (resp. $U \in p\Sigma(X, x)$,$U \in s\Sigma(X, x)$, $U \in \alpha\Sigma(X, x)$, $U \in \beta\Sigma(X, x)$) such that, $F(U) \subset V$.

b) Lower b-continuous [1] (resp. lower almost continuous [23,7,24] or lower precontinuous [7], lower quasi-continuous [6], lower α-continuous [5], lower β-continuous [8,9]) at a point $x \in X$, if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in B\Sigma(X, x)$, (resp. $U \in p\Sigma(X, x)$, $U \in s\Sigma(X, x)$, $U \in \alpha\Sigma(X, x)$, $U \in \beta\Sigma(X, x)$) such that $F(u) \cap V \neq \emptyset$, for every $u \in U$.

c) Upper (lower) b-continuous (resp. upper (lower) precontinuous, upper (lower) Quasi-continuous, upper (lower) α-continuous, upper (lower) β-continuous) if F has this property at each point of X.

## 3. CHARACTERIZATIONS OF UPPER AND LOWER b-θ(β-θ)-CONTINUOUS MULTIFUNCTIONS

**Definition 3.1:** A multifunction $F: (X, T) \to (Y, T^*)$ is said to be:

a) Upper b-θ-continuous (resp. upper β-θ-continuous) if for each $x \in X$, and each open set V of Y such that, $F(x) \subset V$. there exists $U \in B\theta\Sigma(X, x)$ (resp. $U \in \beta\theta\Sigma(X, x)$) such that $F(U) \subset V$.

b) Lower b-θ-continuous (resp. lower β-θ-continuous) at a point $x \in X$, if for each open set V of a space Y such that, $F(x) \cap V \neq \emptyset$ there exists $U \in B\theta\Sigma(X, x)$ (resp. $U \in \beta\theta\Sigma(X, x)$) such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

c) Upper (lower) b-θ-continuous (resp. upper (lower) β-θ-continuous) if F upper (lower) b-θ-continuous (resp. upper (lower) β-θ-continuous) at each point of X.

**Theorem 3.2:** For a multifunction $F: (X, T) \to (Y, T^*)$, the following statements are equivalent:

a) F is Upper b-θ-continuous;

b) $F^+(V) \in B\theta\Sigma(X, T)$ for every open set V of Y;

c) $F^-(V)$ is b-θ-closed in X for every closed set V of Y;

d) $bCl_\theta(F^-(B)) \subset F^-(Cl(B))$ for every $B \subset Y$;

e) For each point $x \in X$ and each neighborhood V of F(x), $F^+(V)$ is an b-θ-Neighborhood of x;

f) For each point $x \in X$ and each neighborhood V of F(x), there exists an b-θ-Neighborhood U of x such that, $F(U) \subset V$;

g) $bCl_\theta(bInt_\theta(F^-(B))) \cap bInt_\theta(bCl_\theta(F^-(B))) \subset F^-(Cl(B))$ for every subset B of Y.

h) $F^+(Int(B)) \subset bInt_\theta(bCl_\theta(F^+(B))) \cup bCl_\theta(bInt_\theta(F^+(B)))$ for every subset B of Y.

**Proof:** (a) ⇒ (b): Let V be any open set of Y and $x \in F^+(V)$, there exists $U \in B\theta\Sigma(X, x)$ such that, $F(U) \subset V$.Therefore, we obtain, $x \in U \subset bCl_\theta(bInt_\theta(U)) \cup bInt_\theta(bCl_\theta((U)) \subset bCl_\theta(bInt_\theta(F^+(V))) \cup bInt_\theta(bCl_\theta(F^+(V)))$.

Then we have, $F^+(V) \subset bCl_\theta(bInt_\theta(F^+(V))) \cup bInt_\theta(bCl_\theta(F^+(V)))$ and hence $F^+(V) \in B\theta\Sigma(X, T)$.

(b) ⇒ (c): This Proof follows immediately from the fact that, $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(c) ⇒ (d): for any subset B of Y, Cl(B) is closed in Y and $F^-(Cl(B))$ is b-θ-closed in X. Therefore, we obtain, $bCl_\theta(F^-(B)) \subset F^-(Cl(B))$.

(d) ⇒ (c): Let V be any closed set of Y. Then we have, $bCl_\theta(F^-(V)) \subset F^-(Cl(V)) = F^-(V)$. This shows that, $F^-(V)$ is b-θ-closed in X.

(b) ⇒ (e): Let $x \in X$ and V be a neighborhood of F(x). Then there exists an open set A of Y such that $F(x) \subset A \subset V$. Therefore, we obtain $x \in F^+(A) \subset F^+(V)$. Since $F^+(A) \in B\theta\Sigma(X, T)$ Then, $F^+(V)$ is an b-θ-neighborhood of x.

(e) ⇒ (f): Let $x \in X$ and V be a neighborhood of F(x). Put $U = F^+(V)$, Then U is an b-θ-neighborhood of x and $F(U) \subset V$.

(f) ⇒ (a): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighborhood of F(x). There exists a b-θ-neighborhood U of x such that $F(U) \subset V$. Therefore there exists, $A \in B\theta\Sigma(X, T)$ such that $x \in A \subset U$, hence $F(A) \subset V$.

(c) ⇒ (g): For any subset B of Y, Cl(B) is closed in Y and by (c), we have $F^-(Cl(B))$ is b-θ-close in X. This means, $bCl_\theta(bInt_\theta(F^-(B))) \cap bInt_\theta(bCl_\theta(F^-(B))) \subset bCl_\theta(bInt_\theta(F^-(Cl(B)))) \cap bInt_\theta(bCl_\theta(F^-(Cl(B)))) \subset F^-(Cl(B))$.

(g) ⇒ (h): By replacing Y–B instead of B in part (g), we have, $bCl_\theta(bInt_\theta(F^+(Y-B))) \cap bInt_\theta(bCl_\theta(F^+(Y-B))) \subset F^+(Cl(Y-B))$, and therefore, $F^+(Int(B)) \subset bInt_\theta(bCl_\theta(F^+(B))) \cup bCl_\theta(bInt_\theta(F^+(B)))$.

(h) ⇒ (b): Let V be any open set of Y. Then, by using (h) we have $F^+(V) \in B\theta\Sigma(X, T)$ and this completes the proof.

**Theorem 3.3:** For a multifunction $F: (X, T) \to (Y, T^*)$, the following statements are equivalent:

a) F is lower b-θ-continuous;

b) $F^-(V) \in B\theta\Sigma(X, T)$ for every open set V of Y;

c) $F^+(V)$ is b-θ-closed in X for every closed set V of Y;

d) $bCl_\theta(F^+(B)) \subset F^+(Cl(B))$ for every $B \subset Y$;

e) $F(bCl_\theta(A) \subset Cl(F(A))$ for every $A \subset X$;

f) $bCl_\theta(bInt_\theta(F^+(B))) \cap bInt_\theta(bCl_\theta(F^+(B))) \subset F^+(Cl(B))$ for every subset B of Y;

g) $F^-(Int(B)) \subset bInt_\theta(bCl_\theta(F^-(B))) \bigcup bCl_\theta(bInt_\theta(F^-(B)))$ for every subset B of Y.

**Proof:** This Proof is Similar to that of theorem (3.2), and is thus omitted.

**Definition 3.4:** Let x be a point of a space X and V a subset of X. The set V is called a β-θ-neighborhood of x in X if there exists a β-θ-open set A of X such that $x \in A \subset V$.

**Theorem 3.5:** For a multifunction $F: X \rightarrow Y$, the following statements are equivalent:

a) F is Upper β-θ-continuous;

b) $F^+(V) \in \beta\theta\Sigma(X, T)$ for every open set V of Y;

c) $F^-(V)$ is β-θ-closed in X for every closed set V of Y;

d) $\beta Cl_\theta(F^-(B)) \subset F^-(Cl(B))$ for every $B \subset Y$;

e) For each point $x \in X$ and each neighborhood V of F(x), $F^+(V)$ is an β-θ-Neighborhood of x;

f) For each point $x \in X$ and each neighborhood V of F(x), there exists an β-θ-Neighborhood U of x such that, $F(U) \subset V$;

g) $\beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^-(B))) \subset F^-(Cl(B))$ for every subset B of Y;

h) $F^+(Int(B)) \subset \beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^+(B)))$ for every subset B of Y.

**Proof:** (a) ⇒ (b): Let V be any open set of a space Y and $x \in F^+(V)$, there exists $U \in \beta\theta\Sigma(X, x)$ such that $F(U) \subset V$. Therefore, $x \in U \subset \beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(U))) \subset \beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^+(V)))$ Then we have, $F^+(V) \subset \beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^+(V)))$ and hence, $F^+(V) \in \beta\theta\Sigma(X, T)$.

(b) ⇒ (c): This Proof follows immediately from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(c) ⇒ (d): for any subset B of Y, Cl(B) is closed in Y and $F^-(Cl(B))$ is β-θ-closed in X. Therefore, we obtain, $\beta Cl_\theta(F^-(B)) \subset F^-(Cl(B))$.

(d) ⇒ (c): Let V be any closed set of Y. Then we have, $\beta Cl_\theta(F^-(V)) \subset F^-(Cl(V)) = F^-(V)$. This shows that, $F^-(V)$ is β-θ-closed in X.

(b) ⇒ (e): Let $x \in X$ and V be a neighborhood of F(x). Then there exists an open set A of Y such that $F(x) \subset A \subset V$. Therefore, we obtain $x \in F^+(A) \subset F^+(V)$. Since $F^+(A) \in \beta\theta\Sigma(X, T)$, Then $F^+(V)$ is an β-θ-neighborhood of x.

(e) ⇒ (f): Let $x \in X$ and V be a neighborhood of F(x). Put $U = F^+(V)$, Then U is an β-θ-neighborhood of x and $F(U) \subset V$.

(f) ⇒ (a): Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. Then V is a neighborhood of F(x). There exists a β-θ-neighborhood U of x such that $F(U) \subset V$. Therefore there exists, $A \in \beta\theta\Sigma(X, T)$ such that $x \in A \subset U$, hence $F(A) \subset V$.

(c) ⇒ (g): For any subset B of Y, Cl(B) is closed in Y and by part (c), we have, $F^-(Cl(B))$ is β-θ-close in X. This means $\beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^-(B))) \subset \beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^-(Cl(B)))) \subset F^-(Cl(B))$.

(g) ⇒ (h): By replacing Y–B instead of B in part (g), we have: $\beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^+(Y-B))) \subset F^+(Cl(Y-B))$, and therefore,

$F^+(Int(B)) \subset \beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^+(B)))$.

(h) ⇒ (b): Let V be any open set of Y. Then, by using (h) we have $F^+(V) \in \beta\theta\Sigma(X, T)$ and this completes the proof.

**Theorem 3.6:** For a multifunction $F: (X, T) \rightarrow (Y, T^*)$, the following statements are equivalent:

a) F is lower β-θ-continuous;

b) $F^-(V) \in \beta\theta\Sigma(X, T)$ for every open set V of Y;

c) $F^+(V)$ is β-θ-closed in X for every closed set V of Y;

d) $\beta Cl_\theta(F^+(B)) \subset F^+(Cl(B))$ for every $B \subset Y$;

e) $F(\beta Cl_\theta(A) \subset Cl(F(A))$ for every $A \subset X$;

f) $\beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^+(B))) \subset F^+(Cl(B))$ for every subset B of Y;

g) $F^-(Int(B)) \subset \beta Cl_\theta(\beta Int_\theta(\beta Cl_\theta(F^-(B)))$ for every subset B of Y.

**Proof:** Similar to the proof of Theorem (3.5), thus is omitted.

**Theorem 3.7:** Let F: (X, T) → (Y, $T^*$) and $F^*$: (Y, $T^*$) → (Z, $T^{**}$) be a multifunctions, If F: X → Y is upper (lower) b-θ-continuous (resp. upper (lower) β-θ-continuous) multifunction and $F^*$: Y→ Z is upper (lower) simecontinuous multifunction, then $F^*$o F: X→Z is an upper (lower) b-θ-continuous (resp. upper (lower) β-θ-continuous) multifunction.

**Proof:** Let V be any open subset of Z. using the definition of $F^*$o F, we obtain:

$(F^* o F)^+(V) = F^+(F^{*+}(V))$ (resp . $(F^* o F)^-(V) = F^-(F^{*-}(V))$). Since $F^*$ is upper (lower) simecontinuous multifunction, it follows that, $F^{*+}(V)$ (resp . $F^{*-}(V)$) is an open set. Since F is upper (lower) b-θ-continuous (resp. upper (lower) β-θ-continuous) multifunction, it follows that $F^+(F^{*+}(V))$ (resp . $F^-(F^{*-}(V))$) is an b-θ-open (resp. β-θ-open) sets. It show that $F^*$o F is an upper (lower) b-θ-continuous (resp. upper (lower) β-θ-continuous) multifunction.

**Remark 3.8:** For a multifunction F: (X, T) → (Y, $T^*$), the following implications are hold:

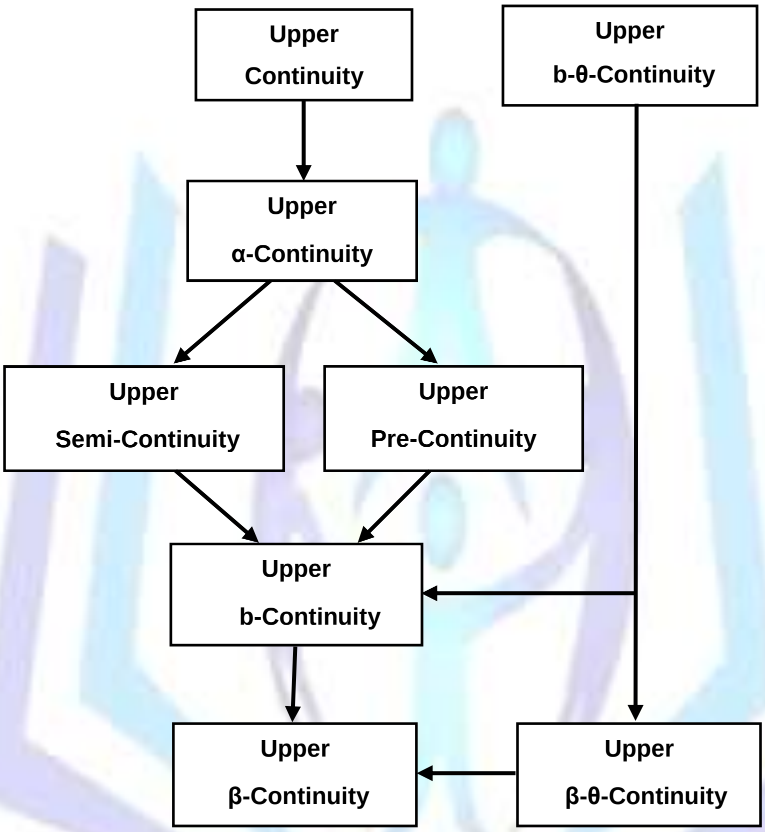


**Fig 2: The relationships between upper b-θ(resp. β-θ)-Continuous Multifunctions and some other well-known types of continuous Multifunctions**

However the converses are not true in general by Examples (3.1), (3.2), (3.3), (3.4), (3.5), of [1], and the following examples.

**Example 3.9:** Let X = Y = {1, 2, 3}, Define a topology T= {Ø, X, {1}, {2}, {1, 2}, {1, 3}} on X and a topology $T^*$= {Ø, Y, {1}, {2}, {1, 2}} on Y and let F: (X, T) → (Y, $T^*$) be a multifunction defined as follows: F(x) = {x} for each x∈X. Then F is upper b-continuous and upper β-continuous but not upper b-θ-continuous because {1, 2}∈$T^*$ and $F^+$({1, 2}) = {1, 2} is not b-θ-open in (X, T).

**Example 3.10:** Let (X, T) and (Y, $T^*$) be define as in example (3.9). And let F: (X, T) → (Y, $T^*$) be a multifunction defined as follows: F(x) = {x} for all x∈X. Then F is upper β-continuous but not upper β-θ-continuous because {1, 2}∈$T^*$ and $F^+$({1, 2}) = {1, 2} is not β-θ-open in (X, T).

**Example 3.11:** Let X = Y = {1, 2, 3, 4}, Define a topology T= {Ø, X, {1}, {2, 3}, {1,2, 3}} on X and a topology $T^*$= {Ø, Y, {1}, {3, 4}, {1, 3, 4}} on Y and let F: (X, T) → (Y, $T^*$) be a multifunction defined as follows: F(x) = {x} for each x∈X. Then F is upper β-continuous but not upper b-continuous because {3, 4}∈$T^*$ and $F^+$({3, 4}) = {3, 4} is not b-open in (X, T).

**Example 3.12:** Let X = Y = {1, 2, 3, 4}, Define a topology T= {Ø, X, {1}, {2, 3}, {1,2, 3}} on X and a topology $T^*$= {Ø, Y, {1}, {2,4}, {1, 2,4}} on Y and let F: (X, T) → (Y, $T^*$) be a multifunction defined as follows: F(x) = {x} for each x∈X. Then F is upper β-θ-continuous but not upper b-θ-continuous, because {2, 4}∈$T^*$ and $F^+$({2, 4}) = {2, 4} is not b-θ-open in (X, T).

For a multifunction F: (X, T) → (Y, $T^*$), the graph multifunction $G_F$: X → X×Y is defined as follows: $G_F(x) = \{x\}\times F(x)$ for every $x \in X$.

**Lemma 3.13:** For a multifunction [11] F: (X, T) → (Y, $T^*$), the following hold:

a) $G_F^+(A\times B) = A \cap F^+(B)$,

b) $G_F^-(A\times B) = A \cap F^-(B)$, for any subsets $A \subset X$ and $B \subset Y$.

**Theorem 3.14:** Let F: X → Y be a multifunction such that F(x) is compact for each $x \in X$. Then F is upper b-θ-continuous (resp. upper β-θ-continuous) if and only if $G_F$: X → X × Y is upper b-θ-continuous (resp. upper β-θ-continuous).

**Proof:** **(Necessity),** suppose that a multifunction F: (X, T) → (Y, $T^*$), is upper b-θ-continuous (resp. upper β-θ-continuous). Let $x \in X$ and H be any open set of X×Y containing $G_F(x)$. For each, $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y)\times V(y) \subset H$.

The family of $\{V(y): y \in F(x)\}$ is an open cover of F(x) and F(x) is compact. Therefore, there exist a finite number of points, say, $y_1, y_2, \ldots, y_n$ in F(x) such that $F(x) \subset \bigcup \{ V(y_i): 1\le i \le n \}$. Set $U = \cap \{U(y_i): 1\le i \le n\}$ and $V = \bigcup \{V(y_i): 1\le i \le n\}$. Then U and V are open in a space X and Y, respectively, and $\{x\}\times F(x) \subset U\times V \subset H$. Since F is upper b-θ-continuous (resp. upper β-θ-continuous). There exists, $U_o \in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) such that $F(U_o) \subset V$. By Lemma (3.13), we have, $U \cap U_o \subset U \cap F^+(V) = G_F^+(U\times V) \subset G_F^+(H)$. Therefore, we get, $U \cap U_o \in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) and $G_F(U \cap U_o) \subset H$. This shows that $G_F$ is upper b-θ-continuous (resp. upper β-θ-continuous).

**(Sufficiency),** suppose that $G_F$: X → X×Y is upper b-θ-continuous (resp. upper β-θ-continuous). Let $x \in X$ and V be any open set of Y containing F(x). Since X×V is open in X×Y and $G_F(x) \subset X\times V$, there exists $U \in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) such that, $G_F(U) \subset X\times V$. Therefore, by Lemma (3.13) we have, $U \subset G_F^+(X\times V) = F^+(V)$ and hence, $F(U) \subset V$. This shows that F is upper b-θ-continuous (resp. upper β-θ-continuous).

**Theorem 3.15:** A multifunction F: (X, T) → (Y, $T^*$), is lower b-θ-continuous (resp. lower β-θ-continuous) if and only if $G_F$: X → Y, is lower b-θ-continuous (resp. lower β-θ-continuous).

**Proof:** **(Necessity),** suppose that a multifunction F: (X, T) → (Y, $T^*$) is lower b-θ-continuous (resp. lower β-θ-continuous). Let $x \in X$ and H be any open set of X×Y such that, $x \in G_F^-(H)$. Since $H \cap (\{x\}\times F(x)) \neq \varnothing$, There exists $y \in F(x)$ such that, $(x, y) \in H$ and hence $(x, y) \in U\times V \subset H$ for some open sets $U \subset X$ and $V \subset Y$. Since $F(x) \cap V \neq \varnothing$, there exists $U_o \in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) such that, $U_o \subset F^-(V)$. By Lemma (3.13) we have:

$U \cap U_o \subset U \cap F^-(V) = G_F^-(U \times V) \subset G_F^-(H)$. Moreover, $x \in U \cap U_o \in B\theta\Sigma(X, T)$ (resp. $\beta\theta\Sigma(X, T)$) and hence $G_F$ is lower b-θ-continuous (resp. lower β-θ-continuous).

**(Sufficiency),** Suppose that $G_F$ is lower b-θ-continuous (resp. lower β-θ-continuous). Let $x \in X$ and V be an open set in a space Y such that $x \in F^-(V)$. Then X×V is open in X×Y and $G_F(x) \cap (X\times V) = (\{x\}\times F(x)) \cap (X\times V) = \{x\}\times(F(x) \cap V) \neq \varnothing$. Since $G_F$ is lower b-θ-continuous (resp. lower β-θ-continuous). There exists $U \in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) such that $U \subset G_F^-(X\times V)$. By Lemma (3.13) we obtain $U \subset F^-(V)$. This shows that a multifunction F is lower b-θ-continuous (resp. lower β-θ-continuous).

**Definition 3.16:** A subset A of a topological space (X, T) is said to be:

a) α-paracompact [25] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X;

b) α-regular [26] if for each $a \in A$ and each open set U of X containing *a*, there exists an open set G of X such that $a \in G \subset Cl(G) \subset U$.

**Lemma 3.17:** If A is an α-regular and α-paracompact [26] set of a space X and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset Cl(G) \subset U$.

For a multifunction F: (X, T) → (Y, $T^*$), by Cl(F): (X, T) → (Y, $T^*$) [27] we denote a multifunction defined as follows: $(ClF)(x) = Cl(F(x))$ for all point $x \in X$. Similarly, we can define $bCl_\theta F$: X → Y, $\beta Cl_\theta F$: X → Y, αCl*F*: X → Y, sCl*F*: X → Y, pCl*F*: X → Y, bCl*F*: X → Y, βCl*F*: X → Y.

**Lemma 3.18:** If F: (X, T) → (Y, $T^*$) is a multifunction such that F(x) is α-regular and α-paracompact for each $x \in X$, then: $G^+(V) = F^+(V)$ for each open set V of Y ; where G denotes Cl(*F*), $bCl_\theta(F)$, $\beta Cl_\theta(F)$, αCl(*F*), sCl(*F*), pCl(*F*), bCl(*F*) or βCl(*F*).

**Proof:** The proof is similar to that of Lemma (3.3) of [1].

**Theorem 3.19:** Let F: (X, T) → (Y, $T^*$) be a multifunction such that, F(x) is α-regular and α-paracompact for every $x \in X$. Then the following properties are equivalent:

a) F is upper b-θ-continuous (resp. upper β-θ-continuous);

b) Cl(*F*) upper b-θ-continuous (resp. upper β-θ-continuous);

c) $bCl_\theta(F)$ is upper b-θ-continuous (resp. upper β-θ-continuous);

d) $\beta Cl_\theta(F)$ is upper b-θ-continuous (resp. upper β-θ-continuous);

e) $\alpha Cl(F)$ is upper b-θ-continuous (resp. upper β-θ-continuous);

f) $sCl(F)$ is upper b-θ-continuous (resp. upper β-θ-continuous);

g) $pCl(F)$ is upper b-θ-continuous (resp. upper β-θ-continuous);

h) $bCl(F)$ is upper b-θ-continuous (resp. upper β-θ-continuous);

i) $\beta Cl(F)$ is upper b-θ-continuous (resp. upper β-θ-continuous).

**Proof:** We set $G=Cl(F)$, $bCl_\theta(F)$, $\beta Cl_\theta(F)$, $\alpha Cl(F)$, $sCl(F)$, $pCl(F)$, $bCl(F)$ or $\beta Cl(F)$. Suppose that F is upper b-θ-continuous (resp. upper β-θ-continuous). Let $x\in X$ and V be any open set of Y containing G(x). By lemma (3.18) we have, $x\in G^+(V) = F^+(V)$ and hence there exists $U\in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) such that $F(U)\subset V$ . Since F(u) is α-paracompact and α-regular for each $u\in U$, By lemma (3.17), There exists an open set H such that, $F(u)\subset H\subset Cl(H)\subset V$ ; hence $G(u)\subset Cl(H)\subset V$ for every $u\in U$. Therefore, we obtain $G(U)\subset V$. This shows that G upper b-θ-continuous (resp. upper β-θ-continuous).

**(Conversely),** suppose that G is upper b-θ-continuous (resp. upper β-θ-continuous). Let $x\in X$ and V be any open set of Y containing F(x). By Lemma (3.18) we have $x\in F^+(V) = G^+(V)$ and hence $G(x)\subset V$ . There exists $U\in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) such that, $G(U)\subset V$. Therefore, we obtain $U\subset G^+(V) = F^+(V)$ and hence $F(U)\subset V$ . This shows that a multifunction F is upper b-θ-continuous (resp. upper β-θ-continuous).

**Lemma 3.20:** If F: (X, T) → (Y, T*) is a multifunction, then, for each open set V of Y $G^-(V) = F^-(V)$, where G denotes $Cl(F)$, $bCl_\theta(F)$, $\beta Cl_\theta(F)$, $\alpha Cl(F)$, $sCl(F)$, $pCl(F)$, $bCl(F)$ or $\beta Cl(F)$.

**Proof:** The proof is similar to that of Lemma (3.4) of [1].

**Theorem 3.21:** For a multifunction F: (X, T) → (Y, T*), the following properties are equivalent:

a) F is lower b-θ-continuous (resp. lower β-θ-continuous);

b) $Cl(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous);

c) $bCl_\theta(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous);

d) $\beta Cl_\theta(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous);

e) $\alpha Cl(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous);

f) $sCl(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous);

g) $pCl(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous);

h) $bCl(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous);

i) $\beta Cl(F)$ is lower b-θ-continuous (resp. lower β-θ-continuous).

**Proof:** By using Lemma (3.20) this is shown similarly as in Theorem (3.19).

Recall that, for two multifunctions $F: X_1 \to Y_1$ and $F^*: X_2 \to Y_2$, the product multifunction $F\times F^*: X_1\times X_2 \to Y_1\times Y_2$ is defined as follows: $(F\times F^*)(x_1, x_2) = F(x_1) \times F^*(x_2)$, for each $x_1\in X_1$ and $x_2\in X_2$,

**Lemma 3.22:** For two multifunctions [28] $F: X_1 \to Y_1$ and $F^*: X_2 \to Y_2$, the following hold:

a) $(F\times F^*)^+(A\times B) = F^+(A) \times F^{*+}(B)$.

b) $(F\times F^*)^-(A\times B) = F^-(A) \times F^{*-}(B)$. for any $A\subset X_1$ and $B\subset X_2$.

**Theorem 3.23:** If $F: X_1 \to Y_1$ and $F^*: X_2 \to Y_2$ are upper b-θ-continuous (resp. upper β-θ-continuous) multifunctions, then $F\times F^*: X_1\times X_2 \to Y_1\times Y_2$ is upper b-θ-continuous (resp. upper β-θ-continuous).

**Proof:** Let $(x_1,x_2)\in X_1\times X_2$ and H be any open set of $Y_1\times Y_2$ containing $F(x_1)\times F^*(x_2)$. There exist open sets $V_1$ and $V_2$ of $Y_1$ and $Y_2$, respectively, such that $F(x_1)\times F^*(x_2)\subset V_1\times V_2\subset H$. since F and $F^*$ are upper b-θ-continuous (resp. upper β-θ-continuous) there exist $U_1\in B\theta\Sigma(X_1, x_1)$ (resp. $U_1\in\beta\theta\Sigma(X_1, x_1)$) and $U_2\in B\theta\Sigma(X_2, x_2)$ (resp. $U_2\in\beta\theta\Sigma(X_2, x_2)$) such that, $F(U_1)\subset V_1$ and $F^*(U_2)\subset V_2$ . By lemma (3.22) we obtain: $U_1\times U_2\subset F^+(V_1)\times F^{*+}(V_2) = (F\times F^*)^+(V_1\times V_2)\subset (F\times F^*)^+(H)$. Therefore, we have: $U_1\times U_2\in B\theta\Sigma(X_1\times X_2, (x_1,x_2))$ (resp. $U_1\times U_2\in\beta\theta\Sigma(X_1\times X_2, (x_1,x_2))$ and $F\times F^*(U_1\times U_2)\subset H$. this shows that $F\times F^*$ is upper b-θ-continuous (resp. upper β-θ-continuous).

**Theorem 3.24:** If $F: X_1 \to Y_1$ and $F^*: X_2 \to Y_2$ are lower b-θ-continuous (resp. lower β-θ-continuous) multifunctions, then $F\times F^*: X_1\times X_2 \to Y_1\times Y_2$ is lower b-θ-continuous (resp. lower β-θ-continuous).

**Proof:** This Proof is similar to that of theorem (3.23) thus omitted.

## 4. SOME PROPERTIES OF UPPER AND LOWER b-θ(β-θ)-CONTINUOUS MULTIFUNCTIONS

**Definition 4.1:** Let A be a subset of a topological space (X, T). The b-θ-frontier (resp. β-θ-frontier) of A, denoted by $bFr_\theta(A)$ (resp. $\beta Fr_\theta(A)$) is defined by:

$bFr_\theta(A)$ (resp. $\beta Fr_\theta(A)$) = $bCl_\theta(A) \cap bCl_\theta(X - A) = bCl_\theta(A) - bInt_\theta(A)$ (resp. $\beta Cl_\theta(A) \cap \beta Cl_\theta(X-A) = \beta Cl_\theta(A) - \beta Int_\theta(A)$).

**Theorem 4.2:** Let F: (X, T) → (Y, $T^*$) be a multifunction. The set of all points x of X is not upper b-θ-continuous (resp. upper β-θ-continuous) is identical with the union of the b-θ-frontier (resp. β-θ-frontier) of the upper inverse images of open sets containing F(x).

**Proof:** Let $x \in X$ at which F is not upper b-θ-continuous (resp. upper β-θ-continuous). Then, there exists an open set V of Y containing F(x) such that $U \cap (X - F^+(V)) \neq \emptyset$ for every $U \in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$). Therefore, $x \in bCl_\theta(X - F^+(V)) = X - bInt_\theta(F^+(V))$ (resp. $\beta Cl_\theta(X - F^+(V)) = X - \beta Int_\theta(F^+(V))$ and $x \in F^+(V)$.Thus, we obtain $x \in bFr_\theta(F^+(V))$ (resp. $\beta Fr_\theta(F^+(V))$).

**(Conversely)**, if F is upper b-θ-continuous (resp. upper β-θ-continuous) at x, suppose that V is an open set of Y containing F(x) such that $x \in bFr_\theta(F^+(V))$ (resp. $\beta Fr_\theta(F^+(V))$). Then there exists $U \in B\theta\Sigma(X, x)$ (resp. $\beta\theta\Sigma(X, x)$) such that $U \subset F^+(V)$;hence $x \in bInt_\theta(F^+(V))$ (resp. $\beta Int_\theta(F^+(V))$ This contradicts and hence F not upper b-θ-continuous (resp. upper β-θ-continuous) at x.

**Theorem 4.3:** Let F: (X, T) → (Y, $T^*$) be a multifunction. The set of all points x of X is not lower b-θ-continuous (resp. lower β-θ-continuous) is identical with the union of the b-θ-frontier (resp. β-θ-frontier) of the lower inverse images of open sets meeting F(x).

**Proof:** The proof is shown similarly as in Theorem (4.2).

In the following (D, >) is directed set, $(F_\lambda)$ is a net of a multifunction $F_\lambda$: X → Y for every $\lambda \in D$ and F is a multifunction from X into Y.

**Definition 4.4:** Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from a space X to Y. A multifunction $F^*$: (X, T) → (Y, $T^*$) is defined as follows: for each $x \in X$, $F^*(x) = \{y \in Y$: for each open neighborhood V of y and each $\eta \in D$, there exists $\lambda \in D$ such that, $\lambda > \eta$ and $V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit [29] of the net $(F_\lambda)_{\lambda \in D}$.

**Definition 4.5:** A net $(F_\lambda)_{\lambda \in D}$ is said to be equally upper b-θ(resp. equally upper β-θ) continuous at $x_0 \in X$ if for every open set $V_\lambda$ containing $F_\lambda(x_0)$, there exists a $U \in B\theta\Sigma(X, x_0)$ (resp. $U \in \beta\theta\Sigma(X, x_0)$) such that, $F_\lambda(U) \subset V_\lambda$ for all $\lambda \in D$.

**Theorem 4.6:** Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from a space X into a compact space Y, If the following are satisfied:

a) $\bigcup\{F_\eta(x): \eta > \lambda\}$ is closed in Y for each $\lambda \in D$ and each $x \in X$,

b) $(F_\lambda)_{\lambda \in D}$ is equally upper b-θ(resp. equally upper β-θ) continuous on X,

Then, $F^*$ upper b-θ (resp. upper β-θ) continuous on X,

**Proof:** From definition (4.4) and a part (a) we have, $F^*(x) = \bigcap\{(\bigcup\{F_\eta(x): \eta > \lambda\}): \lambda \in D\}$. Since the net $(\bigcup\{F_\eta(x): \eta > \lambda\})_{\lambda \in D}$ is a family of closed sets having the finite intersection property and Y is compact, it is follow that $F^*(x) \neq \emptyset$ for each $x \in X$. Now, let $x_0 \in X$ and let $V \in T^*$ such that $V \neq Y$ and $F^*(x_0) \subset V$. Since $F^*(x_0) \bigcap (Y-V) = \emptyset$, $F^*(x_0) \neq \emptyset$, and $(Y-V) \neq \emptyset$, $\bigcap\{(\bigcup\{F_\eta(x_0): \eta > \lambda\}): \lambda \in D\} \bigcap (Y-V) = \emptyset$ and hence, $\bigcap\{(\bigcup\{F_\eta(x_0) \bigcap (Y-V): \eta > \lambda\}): \lambda \in D\} = \emptyset$. Since Y is compact and the family $\{(\bigcup\{F_\eta(x_0) \bigcap (Y-V): \eta > \lambda\}): \lambda \in D\}$ is a family of closed sets with the empty intersection, there exists $\lambda \in D$ such that for each $\eta \in D$ with $\eta > \lambda$ we have, $F_\eta(x_0) \bigcap (Y-V) = \emptyset$; therefore, $F_\eta(x_0) \subset V$. Since the net $(F_\lambda)_{\lambda \in D}$ is equally upper b-θ(resp. equally upper β-θ) continuous on X, there exists, $U \in B\theta\Sigma(X, x_0)$ (resp. $U \in \beta\theta\Sigma(X, x_0)$) such that, $F_\eta(U) \subset V$ for each, $\eta > \lambda$, therefore, $F_\eta(x) \bigcap (Y-V) = \emptyset$ for each $x \in U$. Then we have, $\bigcup\{F_\eta(x) \bigcap (Y-V): \eta > \lambda\})\} = \emptyset$, and hence $\bigcap\{(\bigcup\{F_\eta(x): \eta > \lambda\}): \lambda \in D\} \bigcap (Y-V) = \emptyset$. This implies that $F^*(U) \subset V$. if V=Y, then it is clear that for each $U \in B\theta\Sigma(X, x_0)$ (resp. $U \in \beta\theta\Sigma(X, x_0)$) we have $F^*(U) \subset V$. Hence $F^*$ is upper b-θ(resp. upper β-θ) continuous at $x_0$. Since $x_0$ is arbitrary, the proof completes.

**Definition 4.7:** A space X is called b-compact or γ-compact [2] (resp. β-compact [30]) if every cover of X by b-open sets has a finite subcover (resp. if every cover of X by β-open sets has a finite subcover).

**Definition 4.8:** A space X is called b-closed [19] (resp. β-closed [30]) if every cover of X by b-open (resp. β-open) sets has a finite subfamily whose b-closures (resp. β- closures) cover X.

**Remark 4.9:** Every b-compact [31] (resp. β-compact [32]) space is b-closed (resp. β- closed).

**Theorem 4.10:** If X is a b-closed space [33] (resp. β-closed space [32]) then every cover of X by b-θ-open (resp. β-θ-open) set has a finite subcover.

**Theorem 4.11:** Let F: (X, T) → (Y, $T^*$) be an upper b-θ-continuous (resp. upper β-θ-continuous) surjective multifunctions such that F(x) is compact for each x ∈ X. If X is b-closed (resp. β-closed) space then Y is compact.

**Proof:** Let {$V_\lambda$: λ∈Δ} be an open cover of Y. For each x∈X, F(x) is compact and there exists a finite subset Δ(x) of Δ such that F(x)⊂⋃{$V_\lambda$: λ∈Δ(x)}. Set V(x)=⋃{$V_\lambda$: λ∈Δ(x)}. Since F is upper b-θ-continuous (resp. upper β-θ-continuous), there exists U(x)⊂BθΣ(X, T) (resp. βθΣ(X, T)) containing x such that F(U(x))⊂V(x). The family {U(x): x∈X} is an b-θ-open (resp. β-θ-open) cover of X and there exist a finite number of points, say, $x_1$, $x_2$ , . . . , $x_n$ in X such that X =⋃{U($x_i$): 1≤ i ≤ n}.Therefore, we have :

Y = F(X) = $F(\bigcup_{i=1}^{n} U(xi)) = \bigcup_{i=1}^{n} F(U(xi)) \subset \bigcup_{i=1}^{n} V(xi) = \bigcup_{i=1}^{n} \bigcup_{\lambda\in\Delta(xi)} V\lambda$ .This shows that Y is compact.

**Corollary 4.12:** Let F: X → Y be an upper b-θ-continuous (resp. upper β-θ-continuous) surjective multifunctions such that F(x) is compact for each x∈X. If a space X is b-compact (resp. β-compact) space then Y is compact space.

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