ISSN 2347-1921

The product NystrÖm method and Volterra-Hammerstien Integral Equation with A Generalized Singular Kernel

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Abstract

In this work, the existence of a unique solution of Volterra-Hammerstein integral equation of the second kind (**V-HIESK**) is discussed. The Volterra integral term (**VIT**) is considered in time with a continuous kernel, while the Fredholm integral term (**FIT**) is considered in position with a generalized singular kernel. Using a numerical technique, **V-HIESK** is reduced to a nonlinear system of Fredholm integral equations (**SFIEs**). Using product Nystrom method we have a nonlinear algebraic system of equations. Finally, some numerical examples when the kernel takes the logarithmic, and Carleman forms, are considered.

Keywords: product NystrÖm method; singular integral equation; nonlinear Volterra –Fredholm integral equation; logarithmic kernel; Carleman kernel.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics

Vol 6, No. 2
editor@cirworld.com
www.cirworld.com, member.cirworld.com

1. Introduction

Many authors have interested in solving the linear and nonlinear integral equation. Baker, in [1], obtained a numerical solution for the nonlinear Volterra integral equation of the second kind, when the kernel takes Abel's function form. Golberg, in[2], obtained, numerically, the solution of two-dimensional nonlinear Volterra integral equation by collocation and iterated collocation methods. In[3], Dzhuraev, analyzed the existence of asymptotic error expansion of the Nystrom solution for two–dimensional nonlinear Fredholm integral equation of the second kind. Many different cases for the linear and nonlinear integral equation with different kernels are discussed and solved by Abdou in [4,5,6]. In [7], Ezzati and Najafalizadeh, used Chebyshev polynomials to solve linear and nonlinear Volterra-Fredholm integral equations. Shazad, in [8], solved Volterra-Fredholm integral equation by using least squares technique.

In this work, we consider the **V–HIESK**

$$\mu\phi(x,t)=f(x,t)+\lambda\int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma(y,\tau,\phi(y,\tau))dyd\tau \tag{1}$$

The integral equation (1) is considered in time, for **VIT** and position for **FIT**. The functions $k(|g(x)-g(y)|)$, $F(t,\tau)$ and $f(x,t)$ are given and called the kernel of **FIT, VIT** and the free term, respectively. The constant μ defines the kind of the integral equation and λ is a real parameter (may be complex and has physical meaning). Also, Ω is the domain of integration with respect to position, and the time $t\in[0,T]$, $T<\infty$. While $\phi(x,t)$ is the unknown function to be determined in the space $L_2(\Omega)\times C[0,T]$.

2. The existence and uniqueness solution of V-HIE with a generalized singular kernel:

In this part, successive approximations method and Banach fixed point theorem will be used as sources to prove the existence and uniqueness solution of the integral equation (1) in the space $L_p[\Omega]\times C[0,T]$, where f,k,F and γ are known functions. $k(|g(x)-g(y)|)$ is called the generalized kernel of Hammerstein and $F(t,\tau)$ is called the kernel of Volterra with respect to time.

Also, the modified Schauder fixed point theorem will be considered to prove the existence of at least one solution of Eq. (1), when the Lipschitz condition is not satisfied.

2.1 The existence and uniqueness solution using Picard's method :

To discuss the existence and uniqueness solution of Eq. (1), we write it in the integral operator form

$$\overline{W}\phi(x,t)=\frac{1}{\mu}f(x,t)+\frac{\lambda}{\mu}W\phi(x,t),\qquad(\mu\neq0) \tag{2}$$

where

$$W\phi(x,t)=\int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma\big(\tau,y,\phi(y,\tau)\big)dy\,d\tau \tag{3}$$

Also, we assume the following conditions

a- The kernel of position $k(|g(x)-g(y)|)$, satisfies the discontinuity condition in $L_p[a,b]$

$$\left\{\int_\Omega\{\int_\Omega|k(|g(x)-g(y)|)|^p dx\}^{\frac{q}{p}}dy\right\}^{\frac{1}{q}}=c^* \qquad (p>1,\ c^*\text{ is a constant})$$

b-The kernel of time $F(t,\tau)\in C[0,T]$ satisfies $|F(t,\tau)|\le M$, M is a constant, $\forall t,\tau\in[0,T], 0\le\tau\le t\le T<\infty$

c- The given function $f(x,t)$ with its partial derivatives with respect to position x and time t are continuous in the space $L_p(\Omega)\times C[0,T]$, and its norm is defined as

$$\|f(x,t)\|_{L_p(\Omega)\times C[0,T]}=\max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega|f(x,\tau)|^p dx\right\}^{\frac{1}{p}}d\tau\right|=G \qquad ,(G\text{ is a constant})$$

d- The known continuous function $\gamma(t,x,\phi(x,t))$, for the constants $Q>P_1$ and $Q>Q_1$, satisfies the following conditions

(1) $\max\limits_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega|\gamma(\tau,x,\phi(x,\tau))|^p\,dx\right\}^{\frac{1}{p}}d\tau\right|\le Q_1\|\phi(x,t)\|_{L_p(\Omega)\times C[0,T]}$

(2) $|\gamma(t,x,\phi_1(x,t))-\gamma(t,x,\phi_2(x,t))|\le N(t,x)|\phi_1(x,t)-\phi_2(x,t)|$

where

$$\|N(t,x)\|_{L_p(\Omega)\times C[0,T]}=\max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega|N(\tau,x)|^p\,dx\right\}^{\frac{1}{p}}d\tau\right|=P_1<\infty$$

Using the method of successive approximations(Picard's method), we set

$$\mu\phi_n(x,t)=f(x,t)+\lambda\int_0^t\int_\Omega F(t,\tau)k(|g(x)-g(y)|)\gamma(\tau,y,\phi_{n-1}(y,\tau))dy\,d\tau\quad,(n>1)\tag{4}$$

with $\phi_0(x,t)=f(x,t)$

For ease of manipulation, it is convenient to introduce

$$\psi_n(x,t)=\phi_n(x,t)-\phi_{n-1}(x,t)\tag{5}$$

Hence, we get

$$\phi_n(x,t)=\sum_{i=1}^n\psi_i(x,t)\quad,\ \psi_0(x,t)=f(x,t)\tag{6}$$

From Eq. (1), we obtain

$$|\mu|\|\phi_n(x,t)-\phi_{n-1}(x,t)\|\le|\lambda|\left\|\int_0^t\int_\Omega|F(t,\tau)||k(|g(x)-g(y)|)||\gamma(\tau,y,\phi_{n-1}(y,\tau))-\gamma(\tau,y,\phi_{n-2}(y,\tau))|dy\,d\tau\right\|$$

with the aid of conditions (b) and (d-2) we have

$$|\mu|\|\phi_n(x,t)-\phi_{n-1}(x,t)\|\le|\lambda|M\left\|\int_0^t\int_\Omega|k(|g(x)-g(y)|)||N(\tau,y)|\,|(\phi_{n-1}(y,\tau)-\phi_{n-2}(y,\tau))|dy\,d\tau\right\|$$

Applying Hölder inequality to Hammerstein integral term, and taking in account (5), the above inequality becomes

$$|\mu|\|\psi_n(x,t)\|\le|\lambda|M\left\|\left\{\int_\Omega|k(|g(x)-g(y)|)|^p\,dy\right\}^{\frac{1}{p}}.\max_{0\le t\le T}\left|\{\int_0^t\left\{\int_\Omega|N(\tau,y)|^q\,|\psi_{n-1}(y,\tau)|^q\,dy\right\}^{\frac{p}{q}}d\tau\}\right|^{\frac{1}{p}}\right\|$$

Thus, we have

$$\|\psi_n(x,t)\|\le\frac{|\lambda|}{|\mu|}M\ \|N(t,x)\|\ \|\psi_{n-1}(x,t)\|.\max_{0\le t\le T}\left|\int_0^t\left\{\int_\Omega\{\int_\Omega|k(|g(x)-g(y)|)|^p\,dx\}^{\frac{q}{p}}dy\right\}^{\frac{1}{q}}d\tau\right|$$

In the light of the conditions (a) and (d-2), the last inequality reduces to

$$\|\psi_n(x,t)\|\le\sigma\|\psi_{n-1}(x,t)\|\ ,(\sigma=\frac{|\lambda|}{|\mu|}Mc^*QT),(n\ge1)\tag{7}$$

When $n=1$ the inequality (7) takes the form

$$\|\psi_1(x,t)\| \le \sigma G \tag{8}$$

By induction, we have

$$\|\psi_n(x,t)\| \le \sigma^n G \quad , n = 0,1,... \tag{9}$$

Since (9) is obviously true for $n = 0,1,...$; then it holds for all n. This bound makes the sequence $\{\phi_n(x,t)\}$ in (6) converges, so we can write

$$\phi(x,t) = \sum_{i=0}^{\infty} \psi_i(x,t) \tag{10}$$

The series (10) is uniformly convergent since the terms $\psi_i(x,t)$ are dominated by σ^i and $\sigma^i < 1$ for $i \to \infty$.

To prove that $\phi(x,t)$ defined by (10) satisfies Eq. (1), set

$$\phi(x,t) = \phi_n(x,t) + \Delta_n(x,t) \, , \qquad \left|\Delta_n(x,t) \underset{n\to\infty}{\to} 0\right| \tag{11}$$

In view of Eq. (4), we get

$$\left\|\phi(x,t) - \frac{1}{\mu} f(x,t) - \frac{\lambda}{\mu} \int_0^t \int_\Omega F(t,\tau) k(|g(x)-g(y)|) \gamma(\tau, y, \phi(y,\tau)) dy\, d\tau \right\|$$

$$\le \|\Delta_n(x,t)\| + \frac{|\lambda|}{|\mu|} \left\| \int_0^t \int_\Omega |F(t,\tau)| \, |k(|g(x)-g(y)|)| \, |\gamma(\tau,y,\phi(y,\tau)-\Delta_{n-1}(y,\tau)) - \gamma(\tau,y,\phi(y,\tau))| dy\, d\tau \right\| .$$

Using the conditions (b) and (d-2), and applying Hölder inequality to Hammerstein integral term, then with the aid of condition (a), we obtain

$$\left\|\phi(x,t) - \frac{1}{\mu} f(x,t) - \frac{\lambda}{\mu} \int_0^t \int_\Omega F(t,\tau) k(|g(x)-g(y)|) \gamma(\tau,y,\phi(y,\tau)) dy\, d\tau \right\| \le \|\Delta_n(x,t)\| + \sigma \|\Delta_{n-1}(x,t)\| . \tag{12}$$

So that, by taking n large enough, the right-hand side of (12) can be as small as desired. Thus, the function $\phi(x,t)$ satisfies

$$\mu\phi(x,t) = f(x,t) + \lambda \int_0^t \int_\Omega F(t,\tau) k(|g(x)-g(y)|) \gamma(\tau,y,\phi(y,\tau)) dy\, d\tau$$

and is therefore a solution of Eq. (1) .

To show that $\phi(x,t)$ is the only solution of Eq. (1), we assume the existence of another solution $\tilde{\phi}(x,t)$, then

$$\|\phi(x,t) - \tilde{\phi}(x,t)\| \le \frac{|\lambda|}{|\mu|} \left\| \int_0^t \int_\Omega |F(t,\tau)| \, |k(|g(x)-g(y)|)| \, |\gamma(\tau,y,\phi(y,\tau)) - \gamma(\tau,y,\tilde{\phi}(y,\tau))| dy\, d\tau \right\|,$$

Using the conditions (b) and (d-2) and applying Hölder inequality to Hammerstein integral term, then in view of the condition (a), the above inequality can be adapted in the form

$$\|\phi(x,t) - \tilde{\phi}(x,t)\| \le \sigma \|\phi(x,t) - \tilde{\phi}(x,t)\| \tag{13}$$

Since $\sigma < 1$, then the inequality (13) is true only if $\phi(x,t) = \tilde{\phi}(x,t)$; that is, the solution of Eq. (1) is unique.

ISSN 2347-1921

3. System of nonlinear integral equations with a generalized singular kernel in position

Consider the Volterra-Hammerstein integral equation with a generalized kernel (1) of the second kind.

We use a numerical technique to reduce the **V-FIE** (1) to a nonlinear system of Fredholm integral equations of the second kind.

Divided the interval $[0,T], 0 \le t \le T < \infty$, as $0 = t_0 < t_1 < ... < t_n = T$, where $t = t_i$, $i = 0,1,2,...,n$, then we get:

$$\mu\phi(x,t_i) = f(x,t_i) + \lambda \int_0^{t_i} \int_\Omega F(t_i,\tau) k(|g(x) - g(y)|) \gamma(y,\tau,\phi(y,\tau)) dy d\tau \tag{14}$$

Using the quadrate formula, the Volterra integral term in (14) becomes

$$\int_0^{t_i} F(t_i,\tau)\gamma(y,\tau,\phi(y,\tau)) d\tau = \sum_{j=0}^{i} w_j F(t_i,\tau_j)\gamma(y,t,\phi(y,t_j)) + R(x,t_i) \tag{15}$$

The values of *i* and the order of the truncation error R_i are depending on the number of derivatives of $F(t,\tau)$ for all $\tau \in [0,T]$, with respect to *t* and the w_j are the weights

Using (14) in (13), we have

$$\mu\phi(x,t_i) = f(x,t_i) + \lambda \sum_{j=0}^{i} w_j F(t_i,\tau_j) \int_\Omega k(|g(x) - g(y)|) \gamma(y,\tau,\phi(y,\tau_j)) dy + R_i(x) \tag{16}$$

Formula (16) can be adapted in the general form:

$$\mu\phi_i(x) - \lambda \sum_{j=0}^{i} w_j F_{ij} \int_\Omega k(|g(x) - g(y)|) \gamma(y,\phi_j(y)) dy = f_i(x) + R_i(x), \tag{17}$$

where,

$$\phi(x,t_i) = \phi_i(x),\ F(t_i,\tau_j) = F_{ij},\ x_i = y_i\ .$$

Formula (17) can be written in the form:

$$\mu\phi_i(x) - \mu_i^* \int_\Omega k(|g(x) - g(y)|) \gamma(y,\phi_i(y)) dy = \psi_i(x), \tag{18}$$

where,

$$\psi_i(x) = f_i(x) + R_i(x) + \lambda \sum_{j=0}^{i-1} w_j F_{i,j} \int_\Omega k(|g(x) - g(y)|) \gamma(y,\phi_j(y)) dy,\quad \mu_i^* = \lambda w_i F_{i,i}\,. \tag{19}$$

Formula (18) leads us to say that, we have *n* unknown functions $\phi_i(x)$ corresponding to the interval $[0,T]$, when $\mu = const \ne 0$, this is a system of Fredholm integral equation of the second kind, if $\mu = 0$ it is of the first kind.

4. The product NystrÖm method

In this section, we present the product Nystrom method [9] and [10] to obtain the numerical solution of the **SFIFs** of the second kind. Consider the integral equation,

$$\mu\phi(x) = \lambda \int_{-a}^{a} p(g(x),g(y)) \bar{k}(|g(x) - g(y)|) \gamma(y,\phi(y)) dy + \psi(x) \tag{20}$$

when the kernel $k(|g(x) - g(y)|)$ is singular within the range of integration. We can often factor out the singularity in $k(|g(x) - g(y)|)$ by writing

$$k(|g(x)-g(y)|)=p(g(x),g(y))\bar{k}(|g(x)-g(y)|), \tag{21}$$

where *p* and $\bar{k}$ are respectively badly behaved and well behaved functions of their arguments, respectively, $\phi(x)$ is the unknown function.

Divide the interval [-*a*,*a*] into *N* equal subinterval, where $x_i = y_i = a + ih,\ i = 0,1,...,N,\quad h = \frac{2a}{N}$ and *N* is even. We approximate the integral term by a product integration form of Simpsons rule, when $x = x_i$, we write

$$\sum_{j=0}^{N} w_{ij}\bar{k}(|g(x_i)-g(y_j)|)\gamma(y,\phi(y_j)) = \sum_{j=0}^{(N-2)/2}\int_{y_{2j}}^{y_{2j-2}} p(g(x_i),g(y))\bar{k}(|g(x_i)-g(y)|)\gamma(y,\phi(y))dy \tag{22}$$

where w_{ij} are the weights. Following the same way of Delves and Mohamed [10], we have

$$w_{i,0} = \beta_1(y_i) \quad , \quad w_{i,2j+1} = 2\gamma_{j+1}(y_i)$$
$$w_{i,2j} = \alpha_j(y_i) + \beta_{j+1}(y_i) \quad , \quad w_{i,N} = \alpha_{N/2}(y_i) \tag{23}$$

where

$$\alpha_j(y_i) = \frac{(g(h))^2}{g(2h)}\int_0^2 \zeta(\zeta-1)p(g(y_{2j-2})+\zeta g(h),g(y_i))d\zeta,$$

$$\beta_j(y_i) = \frac{(g(h))^2}{g(2h)}\int_0^2 (\zeta-1)(\zeta-2)p(g(y_{2j-2})+\zeta g(h),g(y_i))d\zeta,$$

$$\gamma_j(y_i) = \frac{(g(h))^2}{g(2h)}\int_0^2 \zeta(2-\zeta)p(g(y_{2j-2})+\zeta g(h),g(y_i))d\zeta. \tag{24}$$

Here, in (24), we introduce the variable $g(y) = g(y_{2j-2}) + \zeta g(h),\ 0 \le \zeta \le 2$. Therefore, the system (22) has a solution:

$$\Phi = [\eta I - \lambda W]^{-1}\Psi \tag{25}$$

where *I* is the identity matrix, and $|\eta I - \lambda W| \ne 0$.

5. The existence and uniqueness of the solution of the nonlinear algebraic system

In this part, the existence of a unique solution of the nonlinear algebraic system (23), will be proved. So, we prove the following lemma and theorem.

Lemma 1 (without proof)

In order to guarantee the normality and continuity of $\sum_{j=0}^{i}|w_{i,j}|$ of Eq. (23), i.e.

$$\sup_j \sum_{j=0}^{i}|w_{i,j}| < z,\quad (z \text{ is a constant}),\ \text{and}\ \lim_{i'\to i}\sup_{i,j}\sum_{j=0}^{i}|w_{i',j}-w_{i,j}| = 0.$$

We assume that the badly behaved kernel $p(g(x),g(y))$ of Eq. (20) satisfies the conditions

$$\int_{-a}^{a}|p(g(x),g(y)|dy \le L, \tag{26}$$

$$\lim_{x_i'\to x_i}\int_{-a}^{a}\left|p(g(x_i'),g(y))-p(g(x_i),g(y))\right|dx = 0;\quad x_i',x_i \in [-a,a] \tag{27}$$

ISSN 2347-1921

Theorem 1:

The nonlinear algebraic system of (23), when $i \to \infty$ is bounded and has a unique solution in Banach space ℓ^{∞} (ℓ^{∞} is the set of all continuous functions), where $\|\Phi\| = \sup_j |\phi_j|$, under the following conditions

1) $\sup_i |\psi_i| \le H < \infty$, (H is a constant).

2) $\sup_i \sum_{j=0}^{i} |w_{i,j}| |\bar{k}(|g(x_i) - g(y)|)| \le H^*$, (H^* is a constant).

3) The known set elements $\gamma(jh, \phi(jh))$, for constants $S > S_1$, $S > S_2$ satisfies the conditions

(a) $\sup_j |\gamma(jh, \phi(jh))| \le S_1 \|\Phi\|_{\ell^{\infty}}$.

(b) $\sup_j |\gamma(jh, \phi(jh)) - \gamma(jh, \theta(jh))| \le S_2 \|\Phi - \Theta\|_{\ell^{\infty}}$, $\|\Phi\| = \sup |\phi_j|$, $\forall j$.

Now, to prove this theorem, we must consider the following lemmas :

Lemma 2 (without proof):

Beside the conditions (a-d), the infinite series $\sum_{x=0}^{\infty} \theta(x_i)$, is uniformly convergent to a continuous set $\phi(x_i)$.

Lemma 3 (without proof):

The set of elements $\{\phi(x_i)\}_{i=1}^{N}$ represent a unique solution of nonlinear algebraic system (22).

Definition 1 :

The estimate local error $R_N^{(N)}(x_i)$ of the product Nystrom method, is given by

$$R_N^{(N)}(x_i) = \left| \int_{-a}^{a} k(|g(x_i) - g(y)|)\gamma(y, \phi(y))dy - \sum_{j=0}^{i} w_{ij} \bar{k}(x_i, x_j)\gamma(x_i, \phi_N(x_j)) \right|, \tag{28}$$

Also, it can be determined using the following formula

$$\phi_i - (\phi_i)_N = \sum_{j=0}^{i} w_{ij} \bar{k}(|g(x_i) - g(x_j)|)(\gamma(x_j, \phi(x_j)) - \gamma(x_j, \phi_N(x_j)) + R_N^{(N)}(x_i). \tag{29}$$

Definition 1 :

The product NystrÖm method is said to be convergent of order r , in $[-a, a]$, if for n sufficiently large, there exist a constant $D > 0$ independent of n such that

$$\|\phi(x) - \phi_n(x)\| \le Dn^{-r}. \tag{30}$$

6. Numerical examples

In this section, we apply the product NystrÖm method, to obtain the numerical solution of the **V-HIESK** with a generalized singular kernel using **Maple10** program. This leads to the required approximate solution of the **V-FIESK** (1) when the kernel $k(|g(x) - g(y)|)$ takes the forms of Carleman function, and logarithmic form.

6.1 Application for a Generalized Carleman Kernel

Example 1: Consider the integral equation:

$$\phi(x,t)=f(x,t)+\lambda\int_0^t\int_{-1}^1\left|x^4-y^4\right|^{-\upsilon}\tau^2\phi^k(y,\tau)dyd\tau,\qquad (0\le t\le T;|x|\le 1)$$

The product NystrÖm method is used to get the numerical solution for values of $\mu = 1$, at the times $t\in[0,0.03]$, $t\in[0,0.6]$, with $\lambda = 0.2500$, and 0.31579 , and we divided the position interval by $N = 21$ units, and $0 < v < 1/2$, v is called Poisson ratio.

the exact solution $\phi(x,t)=x^5t^6$.

Case1 : $\lambda = 0.2500$, $\nu = 0.1$:

Table (1)

T	*x*	*Exact sol.*	*Linear (k=1)*		*Nonlinear (k=2)*	
			Appr. sol .N.	*Err. N.*	*Appr. sol .N.*	*Err. N.*
0.03	-1.00	-7.29000E-10	-7.28588E-10	4.11642E-13	-7.288588E-10	4.11642E-13
	-0.60	-5.66870E-11	-5.60841E-11	6.02930E-13	-5.60841E-11	6.02930E-13
	-0.20	-2.33280E-13	3.42784E-13	5.76064E-13	3.42784E-13	5.76064E-13
	0.20	2.33280E-13	7.49329E-13	5.16049E-13	7.49329E-13	5.16049E-13
	0.60	5.66870E-11	5.72357E-11	5.50473E-13	5.72375E-11	5.50473E-13
	1.00	7.29000E-10	7.29516E-10	5.16725E-13	7.29516E-10	5.16725E-13
0.6	-1.00	-4.66560E-02	-4.72673E-02	6.11322E-04	-4.72690E-02	6.13059E-04
	-0.60	-3.62797E-03	-3.63796E-03	9.99015E-06	-3.63949E-03	1.15284E-05
	-0.20	-1.49299E-05	2.27504E-05	3.76803E-05	2.12492E-05	3.61791E-05
	0.20	1.49299E-05	4.90719E-05	3.41420E-05	4.75709E-05	3.26410E-05
	0.60	3.62797E-03	3.71381E-03	8.58430E-05	3.71227E-03	8.43054E-05
	1.00	4.66560E-02	4.73283E-02	6.72376E-04	4.73266E-02	6.70641E-04

Case2 : $\lambda = 0.31579$, $\nu = 0.12$:

Table (2)

T	*x*	*Exact sol.*	*Linear (k=1)*		*Nonlinear (k=2)*	
			Appr. sol .N.	*Err. N.*	*Appr. sol .N.*	*Err. N.*
0.03	-1.00	-7.29000E-10	-7.29015E-10	5.35178E-13	-7.28464E-10	5.35178E-13
	-0.60	-5.66870E-11	-5.66967E-11	8.24484E-13	-5.58625E-11	8.24484E-13
	-0.20	-2.33280E-13	5.56432E-13	7.89712E-13	5.56432E-13	7.89712E-13
	0.20	2.33280E-13	9.38483E-13	7.05203E-13	9.38483E-13	7.05203E-13
	0.60	5.66870E-11	5.74473E-11	7.60356E-13	5.74473E-11	7.60356E-13
	1.00	7.29000E-10	7.29711E-10	7.11948E-13	7.29711E-10	7.11948E-13
0.6	-1.00	-4.66560E-02	-4.74298E-02	7.73857E-04	-4.74321E-02	7.76145E-04
	-0.60	-3.62797E-03	-3.63628E-03	8.31667E-06	-3.63826E-03	1.02929E-05
	-0.20	-1.49299E-05	3.71229E-05	5.20528E-05	3.52041E-05	5.01340E-05
	0.20	1.49299E-05	6.19028E-05	4.69728E-05	5.99844E-05	4.50545E-05
	0.60	3.62797E-03	3.74126E-03	1.13298E-04	3.73929E-03	1.11323E-04
	1.00	4.66560E-02	4.75124E-02	8.56468E-04	4.75101E-02	8.54181E-04

Example 2: Consider the integral equation:

$$\phi(x,t)=f(x,t)+\lambda\int_0^t\int_{-1}^1\left|\sin(x)-\sin(y)\right|^{-\upsilon}\tau^2\phi^k(y,\tau)dyd\tau,$$

The values of $\mu = 1$, at the times $t\in[0 , 0.006]$, $t\in[0 , 0.03]$, with $\lambda= 0.111111, 0.13636$, and we divided the position interval by $N = 21$ units. Exact solution $\phi(x,t) = t\sin(x)$.

Case1 : λ = 0.111111 , ν = 0.05:

Table (3)

T	*x*	*Exact sol.*	*Linear (k=1)*		*Nonlinear (k=3)*	
			Appr. sol .N.	*Err. N.*	*Appr. sol .N.*	*Err. N.*
0.006	-1.00	-5.04882E-03	-5.04132E-03	7.49836E-06	-5.04132E-03	7.49836E-06
	-0.60	-3.38785E-03	-3.39321E-03	5.35731E-06	-3.39321E-03	5.35731E-06
	-0.20	-1.19201E-03	-1.20232E-03	1.03059E-05	-1.20232E-03	1.03059E-05
	0.20	1.19201E-03	1.17786E-03	1.41497E-05	1.17786E-03	1.41497E-05
	0.60	3.38785E-03	3.35534E-03	3.25130E-05	3.35534E-03	3.25130E-05
	1.00	5.04882E-03	5.03215E-03	1.66675E-05	5.03215E-03	1.66675E-05
0.03	-1.00	-2.52441E-02	-2.52066E-02	3.74978E-05	-2.52066E-02	3.74966E-05
	-0.60	-1.69392E-02	-1.69661E-02	2.68232E-05	-1.69661E-02	2.68341E-05
	-0.20	-5.96007E-03	-6.01167E-03	5.15915E-05	-6.01167E-03	5.15920E-05
	0.20	5.96007E-03	5.88926E-03	7.08178E-05	5.88926E-03	7.08173E-05
	0.60	1.69392E-02	1.67765E-02	1.62710E-04	1.67765E-02	1.62709E-04
	1.00	2.52441E-02	2.51607E-02	8.34051E-05	2.51607E-02	8.34039E-05

Case2 : λ = 0.13636 , ν = 0.06:

Table (4)

T	*x*	*Exact sol.*	*Linear (k=1)*		*Nonlinear (k=3)*	
			Appr. sol .N.	*Err. N.*	*Appr. sol .N.*	*Err. N.*
0.006	-1.00	-5.04882E-03	-5.03856E-03	1.02656E-05	-5.03856E-03	1.02656E-05
	-0.60	-3.38785E-03	-3.39717E-03	9.31579E-06	-3.39717E-03	9.31580E-06
	-0.20	-1.19201E-03	-1.20880E-03	1.67900E-05	-1.20880E-03	1.67900E-05
	0.20	1.19201E-03	1.16999E-03	2.20252E-05	1.16999E-03	2.20252E-05
	0.60	3.38785E-03	3.33849E-03	4.93593E-05	3.33849E-03	4.93593E-05
	1.00	5.04882E-03	5.02308E-03	2.57381E-05	5.02308E-03	2.57381E-05
0.03	-1.00	-2.52441E-02	-2.51982E-02	5.13069E-05	-2.51928E-02	5.13051E-05
	-0.60	-1.69392E-02	-1.69858E-02	4.65927E-05	-1.69858E-02	4.65945E-05
	-0.20	-5.96007E-03	-6.04403E-03	8.39553E-05	-6.04403E-03	8.39560E-05
	0.20	5.96007E-03	5.84995E-03	1.10121E-04	5.84995E-03	1.10121E-04
	0.60	1.69392E-02	1.66924E-02	2.46783E-04	1.66924E-02	2.46781E-04
	1.00	2.52441E-02	2.51154E-02	1.28669E-04	2.51154E-02	1.28668E-04

6.2 Application for a Generalized logarithmic kernel.

Example 1: Consider the integral equation:

$$\phi(x,t) = f(x,t) + \lambda\int_0^t\int_{-1}^1 \ln\left|x^4 - y^4\right| \tau^2\phi^k(y,\tau)\,dy\,d\tau,$$

The product NystrÖm method is used to get approximate solution for values of $\mu = 1$, $\lambda = 0.25, 0.6666666667$, $t\in[0,0.03]$, $t\in[0 , 0.6]$ and $N = 21$. Exact solution $\phi(x,t) = x^5 t^6$.

Case1 : λ = 0.25:

Table (5)

T	x	Exact sol.	Linear (k=1)		Nonlinear (k=2)	
			Appr. sol .N.	Err. N.	Appr. sol .N.	Err. N.
0.03	-1.00	-7.29000E-10	-7.28954E-10	4.56279E-14	-7.28954E-10	4.56279E-13
	-0.60	-5.66870E-11	-5.66465E-11	4.04445E-14	-5.66465E-11	4.04445E-14
	-0.20	-2.33280E-13	-2.09324E-13	2.39555E-14	-2.09324E-13	2.39555E-14
	0.20	2.33280E-13	2.44331E-13	1.10515E-14	2.44331E-14	1.10515E-14
	0.60	5.668704E-11	5.66855E-11	1.45837E-15	5.66855E-11	1.45837E-15
	1.00	7.290000E-10	7.28992E-10	7.21890E-15	7.28992E-10	7.21890E-15
0.6	-1.00	-4.66560E-02	-4.72913E-02	6.35393E-04	-4.72888E-02	6.32852E-04
	-0.60	-3.62797E-03	-3.67495E-03	4.69817E-05	-3.67408E-03	4.61133E-05
	-0.20	-1.49299E-05	-1.35588E-05	1.37110E-06	-1.30299E-05	1.89993E-06
	0.20	1.49299E-05	1.58609E-05	9.31073E-07	1.63898E-05	1.45992E-06
	0.60	3.62797E-03	3.67751E-03	4.95456E-05	3.67838E-03	5.04141E-05
	1.00	4.66560E-02	4.72939E-02	6.37919E-04	4.72964E-02	6.40461E-04

Case2 : λ = 0.6666666667:

Table (6)

T	x	Exact sol.	Linear (k=1)		Nonlinear (k=2)	
			Appr. sol .N.	Err. N.	Appr. sol .N.	Err. N.
0.03	-1.00	-7.29000E-10	-7.28878E-10	1.21666E-13	-7.28878E-10	1.21666E-13
	-0.60	-5.66870E-20	-5.65792E-11	1.07834E-13	-5.65792E-11	1.07834E-13
	-0.20	-2.33280E-13	-1.69417E-13	6.38621E-14	-1.69417E-13	6.38621E-14
	0.20	2.33280E-13	2.62734E-13	2.94547E-14	2.62734E-13	2.94547E-14
	0.60	5.66870E-11	5.66831E-11	3.89978E-15	5.66831E-11	3.89978E-15
	1.00	7.29000E-10	7.28980E-10	1.92547E-14	7.28980E-10	1.92547E-14
0.6	-1.00	-4.66560E-02	-4.83897E-02	1.73374E-03	-4.83828E-02	1.72680E-03
	-0.60	-3.62797E-03	-3.75601E-03	1.28041E-04	-3.75364E-03	1.25671E-04
	-0.20	-1.49299E-05	-1.10893E-05	3.84054E-06	-9.64301E-06	5.28361E-06
	0.20	1.49299E-05	1.75158E-05	2.58592E-06	1.89590E-05	4.02915E-06
	0.60	3.62797E-03	3.76316E-03	1.35198E-04	3.76553E-03	1.37569E-04
	1.00	4.66560E-02	4.83967E-02	1.74079E-03	4.84037E-02	1.74773E-03

Example 2: Consider the integral equation:

$$\phi(x,t) = f(x,t) + \lambda \int_0^t \int_{-1}^1 \ln\left|e^{x^2} - e^{y^2}\right| \tau^2 \phi^k(y,\tau)\,dy\,d\tau,$$

The values of μ = 1, λ = 0.001, 0.01, 0, $t\in$[0,0.03], $t\in$[0,0.6], and N = 21. Exact solution $\phi(x,t) = e^{x^5} t^3$.

Case1 : λ = 0.001:

Table (7)

T	x	Exact sol.	Linear (k=1)		Nonlinear (k=2)	
			Appr. sol .N.	Err. N.	Appr. sol .N.	Err. N.
0.03	-1.00	9.93274E-06	1.10992E-05	1.16653E-06	1.10992E-05	1.16653E-06
	-0.60	2.49800E-05	2.55617E-05	5.81741E-07	2.55617E-05	5.81741E-07
	-0.20	2.69913E-05	2.72749E-05	2.83542E-07	2.72749E-05	2.83541E-07
	0.20	2.70002E-05	2.84545E-05	1.45428E-06	2.84545E-05	1.45428E-06
	0.60	2.78570E-05	2.85914E-05	7.3434E-07	2.85914E-05	7.34343E-07
	1.00	4.87315E-05	4.91498E-05	4.18244E-07	4.91498E-05	4.18244E-07
0.6	-1.00	7.94619E-02	8.87986E-02	9.33669E-04	8.87994E-02	9.33752E-03
	-0.60	1.99840E-01	2.04514E-01	4.67468E-03	2.04505E-01	4.66542E-03
	-0.20	2.15930E-01	2.18224E-01	2.29406E-03	2.18211E-01	2.28055E-03
	0.20	2.16002E-01	2.27663E-01	1.16614E-02	2.27649E-01	1.16474E-02
	0.60	2.22856E-01	2.28755E-01	5.89917E-03	2.28744E-01	5.88770E-03
	1.00	3.89852E-01	3.93225E-01	3.37348E-03	3.93220E-01	3.36786E-03

Case2 : λ = 0.01:

Table (8)

T	x	Exact sol.	Linear (k=1)		Nonlinear (k=2)	
			Appr. sol .N.	Err. N.	Appr. sol .N.	Err. N.
0.03	-1.00	9.93274E-06	2.32346E-05	1.33019E-05	2.32346E-05	1.33019E-05
	-0.60	2.49800E-05	3.15763E-05	6.59627E-06	3.15763E-05	6.59627E-06
	-0.20	2.69913E-05	3.04458E-05	3.45447E-06	3.04458E-05	3.45447E-06
	0.20	2.70002E-05	4.36878E-05	1.66875E-05	4.36878E-05	1.66875E-05
	0.60	2.78570E-05	3.62679E-05	8.41092E-06	3.62679E-05	8.41091E-06
	1.00	4.87315E-05	5.39592E-05	5.22773E-06	5.39529E-05	5.22773E-06
0.6	-1.00	7.94619E-02	1.86093E-01	1.06632E-01	1.86045E-01	1.06583E-01
	-0.60	1.99840E-01	2.52903E-01	5.30634E-02	2.52782E-01	5.29423E-02
	-0.20	2.15930E-01	2.43869E-01	2.79381E-02	2.43720E-01	2.77893E-02
	0.20	2.16002E-01	3.49950E-01	1.33948E-01	3.49780E-01	1.33777E-01
	0.60	2.22856E-01	2.90476E-01	6.76201E-02	2.90346E-01	6.74901E-02
	1.00	3.89852E-01	4.32009E-01	4.21574E-02	4.31941E-01	4.20890E-02

7. Conclusion

(1) Due to the kernel of Carleman and logarithmic kernels, when the values of λ and υ are increasing and the values of the time T kept fixed, the error is increasing, where the atomic bond between the particles of the material is increasing.

(2) When the values of time T are increasing and the values of λ, υ and N kept fixed, the error is increasing.

(3) For linear and nonlinear case, the result error increase as well as the values of T increases for different of λ, υ.

(4) The error for linear case is smaller than nonlinear case, for different values λ, υ.

ISSN 2347-1921

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