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Symplectic Groupoids and Generalized Almost Contact Manifolds

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ABSTRACT

We obtain equivalent assertions among the integrability conditions of generalized almost contact manifolds, the condition of compatibility of source and target maps of symplectic groupoids with symplectic form and generalized contact maps.

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1 Introduction

A groupoid G consists of two sets G_1 and G_0, called arrows and the objects, respectively, with maps $s,t:G_1 \to G_0$ called source and target. It is equipped with a composition $m:G_2 \to G_1$ defined on the subset $G_2 = \{(g,h) \in G_1 \times G_1 \mid s(g) = t(h)\}$; an inclusion map of objects $e:G_0 \to G_1$ and an inversion map $i:G_1 \to G_1$. For a groupoid, the following properties are satisfied: $s(gh) = s(h)$, $t(gh) = t(g)$, $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $g(hf) = (gh)f$ whenever both sides are defined, $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$. Here we have used $gh, 1_x$ and g^{-1} instead of $m(g,h)$, $e(x)$ and $i(g)$. Generally, a groupoid G is denoted by the set of arrows G_1. From above definition it follows that a groupoid is a small category in which all morphisms are invertible.

A topological groupoid is a groupoid G_1 whose set of arrows and set of objects are both topological spaces whose structure maps s,t,e,i,m are all continuous and s,t are open maps.

A Lie groupoid is a groupoid G whose set of arrows and set of objects are both manifolds whose structure maps s,t,e,i,m are all smooth maps and s,t are submersions. The latter condition ensures that s and t-fibres are manifolds. One can see from above definition the space G_2 of composable arrows is a submanifold of $G_1 \times G_1$. We note that Lie groupoid introduced by Ehresmann [4].

On the other hand, Lie algebroids were first introduced by Pradines [10] as infinitesimal objects associated with the Lie groupoids. More precisely, a Lie algebroid structure on a real vector bundle A on a manifold M is defined by a vector bundle map $\rho_A : A \to TM$, the anchor of A, and an $\mathbb{R}$-Lie algebra bracket on $\Gamma(A),[,]_A$ satisfying the Leibniz rule

$$[\alpha, f\beta]_A = f[\alpha,\beta]_A + L_{\rho_A(\alpha)}(f)\beta$$

for all $\alpha,\beta \in \Gamma(A), f \in C^\infty(M)$, where $L_{\rho_A(\alpha)}$ is the Lie derivative with respect to the vector field $\rho_A(\alpha)$. And $\Gamma(A)$ denotes the set of sections in A.

On the other hand, Hitchin [7] introduced the notion of generalized complex manifolds by unifying and extending the usual notions of complex and symplectic manifolds. Later the notion of generalized Kähler manifold was introduced by Gualtieri [5] and submanifolds of such manifolds have been studied in many papers.

As an analogue of generalized complex structures on even dimensional manifolds, the concept of generalized almost contact manifolds were introduced in [8] and such manifolds have been also studied in, [14] and [12], [11].

Recently, Crainic [3] showed that there is a close relationship between the equations of a generalized complex manifold and a Lie groupoid. More precisely, he obtained that the complicated equations of such manifolds turn into simple structures for Lie groupoids.

In this paper, we investigate relationships between the normality conditions of generalized contact structures and symplectic groupoids. We showed that the equations of such manifolds are useful to obtain equivalent results on a symplectic groupoid.

2 Preliminaries

In this section we recall basic facts of Poisson geometry, Lie groupoids and Lie algebroids. More details can be found in [9] and [13]. A central idea in generalized geometry is that $TM \oplus T^*M$ should be thought of as a generalized tangent bundle to manifold M. If X and ξ denote a vector field and a dual vector field on M respectively, then we write (X,ξ) (or $X+\xi$) as a typical element of $TM \oplus T^*M$. The Courant bracket of two sections $(X,\xi),(Y,\eta)$ of $TM \oplus T^*M$ = TM is defined by

$$[(X,\xi),(Y,\eta)] = [X,Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi),$$

where d, L_X and i_X denote exterior derivative, Lie derivative and interior derivative with respect to X, respectively. The Courant bracket is antisymmetric but, it does not satisfy the Jacobi identity. We adapt the notions $\beta(\pi^{\#}\alpha) = \pi(\alpha,\beta)$ and $\omega_{\#}(X)(Y) = \omega(X,Y)$ which are defined as $\pi^{\#}: T^*M \to TM$, $\omega_{\#}: TM \to T^*M$ for any 1-forms α and β, 2-form ω and bivector field π, and vector fields X and Y. Also we denote by $[,]_\pi$, the bracket on the space of 1-forms on M defined by

$$[\alpha,\beta]_\pi = L_{\pi^\#\alpha}\beta - L_{\pi^\#\beta}\alpha - d\pi(\alpha,\beta).$$

On the other hand, a symplectic manifold is a smooth (even dimensional) manifold M with a non-degenerate closed 2-form $\omega \in \Omega^2(M)$. ω is called the symplectic form of M. Let G be a Lie groupoid on M and ω a form on Lie groupoid G, then ω is called multiplicative if

$$m^*\omega = pr_1^*\omega + pr_2^*\omega,$$

where $pr_i : G\times G \to G$, $i=1,2$, are the canonical projections. If a Lie groupoid G is endowed with a symplectic form which is multiplicative, then G is called symplectic groupoid.

We now recall the notion of Poisson manifolds. A Poisson manifold is a smooth manifold M whose function space $C^\infty(M, I\!R)$, is a Lie algebra with bracket $\{,\}$, such that the following properties are satisfied;

1. $\{f,g\} = -\{g,f\}$
2. $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0$
3. $\{fg,h\} = f\{g,h\}+g\{f,h\}$.

If M is a Poisson manifold, then there is a unique bivector π, called the Poisson bivector, and a unique homomorfizm $\pi^\# : T^*M \to TM$ of vector bundles with $\pi^\#(T^*M)\subset TM$ such that $\pi(df,dg)=\pi^\#(df)g=\{f,g\}$. It is also possible to define a Poisson manifold by using the bivector π. Indeed, a smooth manifold is a Poisson manifold if $[\pi,\pi]=0$, where $[,]$ denotes the Schouten bracket on the space of multivector fields.

We now give a relation between Lie algebroid and Lie groupoid. Given a Lie groupoid G on M, the associated Lie algebroid $A = Lie(G)$ has fibres $A_x = Ker(ds)_x = T_x(G(-,x))$, for any $x\in M$. Any $\alpha\in\Gamma(A)$ extends to a unique right-invariant vector field on G, which will be denoted by same letter α. The usual Lie bracket on vector fields induces the bracket on $\Gamma(A)$, and the anchor is given by $\rho = dt : A \to TM$.

Given a Lie algebroid A, an integration of A is a Lie groupoid G together with an isomorphism $A \cong Lie(G)$. If such a G exists, then it is called that A is integrable. In contrast with the case of Lie algebras, not every Lie algebroid admits an integration. However if a Lie algebroid is integrable, then there exists a canonical source simply connected integration G, and any other source simply connected integration is smoothly isomorphic to G. *From now on we assume that all Lie groupoids are source-simply-connected.*

In this section, finally, we recall the notion of IM form (infinitesimal multiplicative form) on a Lie algebroid [2]. More precisely, an IM form on a Lie algebroid A is a bundle map

$$u : A \to T^*M$$

satisfying the following properties

1. $\langle u(\alpha),\rho(\beta)\rangle = -\langle u(\beta),\rho(\alpha)\rangle$
2. $u([\alpha,\beta]) = L_\alpha(u(\beta)) - L_\beta(u(\alpha))$

$$+d\langle u(\alpha),\rho(\beta)\rangle$$

for $\alpha,\beta\in\Gamma(A)$, where $\langle,\rangle$ denotes the usual pairing between a vector space and its dual.

If A is a Lie algebroid of a Lie groupoid G, then a closed multiplicative 2-form ω on G induces an IM form u_ω of A by

$$\langle u_\omega(\alpha), X\rangle = \omega(\alpha, X).$$

For the relationship between IM form and closed 2-form we have the following.

Theorem 1 [2] *If A is an integrable Lie algebroid and if G is its integration, then $\omega \mapsto u_\omega$ is an one to one correspondence between closed multiplicative 2-forms on G and IM forms of A.*

Similar to 2-forms, given a Lie groupoid G, a (1,1)-tensor $J : TG \to TG$ is called multiplicative [3] if for any $(g,h) \in G \times G$ and any $v_g \in T_g G$, $w_h \in T_h G$ such that (v_g, w_h) is tangent to $G \times G$ at (g,h), so is (Jv_g, Jw_h), and

$$(dm)_{g,h}(Jv_g, Jw_h) = J((dm)_{g,h}(v_g, w_h)).$$

3 Lie Groupoids and Generalized Contact Structures

Let J be a complex structure on $M^{2n+1} \times \mathrm{IR}$ such that (i) J is invariant by translation along IR and (ii) $J(T\mathrm{IR}) \subseteq TM$. Then J is said to be M-adapted. If we denote by t the coordinate on IR, J is an M-adapted structure iff

$$J = F + dt \otimes Z - \xi \otimes \frac{\partial}{\partial t},$$

where $F \in End(TM), Z \in \chi(M), \xi \in \Omega^1(M)$. Accordingly, condition $J^2 = -Id$ becomes

$$F^2 = -Id + \xi \otimes Z, \xi \circ F = 0, FZ = 0, \xi(Z) = 1,$$

and the triple (F, Z, ξ) is called an almost contact structure on M. If the adapted structure J is integrable, the almost contact structure (F, Z, ξ) is normal and the normality condition is

$$N_F + Z \otimes d\xi = 0, \tag{1}$$

where N_F is the Nijenhuis tensor of F [1]. We note that a contact manifold is a smooth (odd dimensional) manifold M with 1-form $\xi \in \Omega^1(M)$ such that $\xi \wedge (d\xi)^n \neq 0$. ξ is called the contact form of M.

In this section we begin by giving a characterization for generalized contact manifolds, then we obtain certain relationships between generalized contact manifolds and symplectic groupoids. But we first recall that a generalized almost complex structure J is an endomorphism on TM such that $\mathrm{J}^2 = -Id$ and J is anti-symmetric with respect to the canonical symmetric bilinear operation given by

$$\langle \alpha + X, \beta + Y\rangle = \frac{1}{2}(\alpha(Y) + \beta(X)) \text{ forallsections } \alpha + X, \beta + Y \in \mathrm{TM}.$$

A generalized almost complex structure can be represented by classical tensor fields as follows:

$$\mathrm{J} = \begin{bmatrix} a & \pi^{\#} \\ \sigma_{\#} & -a^* \\ & \end{bmatrix} \tag{2}$$

where π is a bivector on M, σ is a 2-form on M, $a : TM \to TM$ is a bundle map, and $a^* : T^*M \to T^*M$ is dual of a.

We now give brief information on the geometry of generalized almost contact manifolds and its normality conditions taken from [11].

Definition 1 *A generalized, almost complex structure J on $M \times IR$ is said to be M-adapted if it has the following three properties (a) J is invariant by translation along IR, (b) $\mathrm{J}(TIR \oplus 0) \subseteq 0 \oplus T^*M$, (c) $\mathrm{J}(0 \oplus T^*IR) \subseteq TM \oplus 0$.*

The invariance of J by translations means that the Lie derivatives $L_{\partial\backslash\partial t}$ of the classical tensor fields of J (defined by (2)) vanish. If conditions (b), (c) are also imposed, it follows that the classical tensor fields of an *M*-adapted, generalized, almost complex structure are of the form

$$a = F, \pi = P + Z \wedge \frac{\partial}{\partial t}, \sigma = \theta + \xi \wedge dt,$$

where $P \in \Gamma \wedge^2 TM, \theta \in \Omega^2(M), Z \in \chi(M), \xi \in \Omega^1(M)$.

Furthermore, condition $J^2 = -Id$ is equivalent to

$$\begin{aligned} &F \circ P^{\#} = P^{\#} \circ F^{*}, \theta_{\#} \circ F = F^{*} \circ \theta_{\#}, i_Z \theta = 0, i_{\xi} P = 0, \\ &F^2 = -Id - P^{\#} \circ \theta_{\#} + \xi \otimes Z, F(Z) = 0, \xi \circ F = 0, \xi(Z) = 1. \end{aligned} \quad (3)$$

The triple (F, P, θ) defines and endomorphism F of TM of matrix form

$$\mathrm{F}\begin{bmatrix} X \\ \alpha \end{bmatrix} = \begin{bmatrix} F & P^{\#} \\ \theta_{\#} & -F^{*} \end{bmatrix}\begin{bmatrix} X \\ \alpha \end{bmatrix}. \quad (4)$$

The pair (Z, ξ) defines the endomorphism Z of TM of matrix form

$$\mathrm{Z}\begin{bmatrix} X \\ \alpha \end{bmatrix} = \begin{bmatrix} Z \otimes \xi & 0 \\ 0 & (Z \otimes \xi)^{*} \end{bmatrix}\begin{bmatrix} X \\ \alpha \end{bmatrix},$$

where $Z \otimes \xi : TM \to TM$ is the evaluation of ξ and $(Z \otimes \xi)^{*} : T^{*}M \to T^{*}M$ is the evaluation of Z. The conditions (3) are equivalent to

$$\flat_g \circ \mathrm{F} + \mathrm{F}^{*} \circ \flat_g = 0, \mathrm{F}^2 = -Id + \mathrm{Z}, \mathrm{F} \circ \mathrm{Z} = 0, \|Z \oplus \xi\|_g = 1. \quad (5)$$

(The first condition (5) ensures that P and θ are skew symmetric, and g is the neutral metric of TM .).

Definition 2 [11] *A generalized almost contact structure on* M *is a couple* $(\mathrm{F} \in End(\mathrm{TM}), (Z, \xi) \in \Gamma\mathrm{TM})$ *that satisfies (5). Equivalently the structure is a system of classical tensor fields* (P, θ, F, Z, ξ) *that satisfies (3).*

It is easy to see that a classical almost contact structure is a generalized almost contact structure with $P = 0, \theta = 0$

A generalized, almost contact structure will be called normal if the corresponding M-adapted, generalized, almost complex structure on $M \times IR$ is integrable. Thus, the normality conditions are [11]

$$\begin{aligned} &(1)[P, P] = 0, \\ &(2) FP^{\#} = P^{\#}F^{*}, \\ &F^{*}[\zeta, \eta]_P = \mathrm{L}_{P^{\#}\zeta} F^{*}\eta - \mathrm{L}_{P^{\#}\eta} F^{*}\zeta - dP(F^{*}\zeta, \eta), \\ &(3) F^2 + P^{\#}\theta_{\#} = -Id + \xi \otimes Z, \\ &N_F(X, Y) = P^{\#}(i_{(X \wedge Y)} d\theta) - (d\xi(X, Y))Z, \\ &(4) F^{*}\theta_{\#} = \theta_{\#}F, \\ &d\theta_F(X_1, X_2, X_3) = \Sigma_{Cycl(1,2,3)} d\theta(FX_1, X_2, X_3), \end{aligned} \quad (6)$$

$$(5) L_Z\xi = 0, L_Z F = 0, (L_{FX}\xi)(Y) - (L_{FY}\xi)(X) = 0,$$
$$L_Z P = 0, L_Z\theta = 0, L_{P^{\#}\alpha}\xi = 0.$$

We now give the definition of odd dimensional symplectic manifolds which has been given recently.

Definition 3 [6] *Suppose that M is a manifold of dimension $2n+1$ with a volume form Ω and a closed form ω of maximal rank. Then the triple (M,Ω,ω) is called an odd dimensional symplectic manifold.*

Example 1 [6] *Any contact manifold with a contact form ξ can be viewed as an odd dimensional symplectic manifold in the obvious way:*

$$\omega = d\xi$$

is the symplectic 2-form, ξ is the connection form, and hence the volume form is

$$\Omega = \xi \wedge \frac{(d\xi)^n}{n!}.$$

We now start to investigate relationships between normality conditions and Lie groupoids.

Lemma 1 *Let M be a generalized almost contact manifold. If P is a non-degenerate bivector field on $TM^* - Span\{\xi\}$, $d\xi$ is the inverse 2-form (defined by $(d\xi)_{\#} = (P^{\#})^{-1}$) and P satisfies (6) then $\theta = -d\xi - F^* d\xi + \xi \otimes (i_Z d\xi)$.*

Proof. For $X \in \chi(M)$, we apply $(d\xi)_{\#}$ to (6) and using the dual structure F^*, we have

$$(d\xi)_{\#} F^2(X) = -(d\xi)_{\#}(X) - (d\xi)_{\#}(P^{\#}\theta_{\#}(X)) + (d\xi)_{\#}(\xi(X)Z)$$
$$d\xi(F^2 X, Y) = -d\xi(X,Y) - \theta_{\#}(X)(Y) + d\xi(\xi(X)Z, Y).$$

Since $d\xi$ and F commute, we obtain

$$d\xi(FX, FY) = -d\xi(X,Y) - \theta(X,Y) + \xi(X) d\xi(Z,Y)$$
$$F^* d\xi(X,Y) = -d\xi(X,Y) - \theta(X,Y) + \xi(X) d\xi(Z,Y). \tag{7}$$

Since the equation (7) is hold for all X and Y, we get

$$\theta = -d\xi - F^* d\xi + \xi \otimes (i_Z d\xi). \tag{8}$$

From now on, when we mention a non-degenerate bivector field P, we mean it is non-degenerate on $TM^* - Span\{\xi\}$. We note that if $d\xi$ is the inverse 2-form of P, non-degenerate P on $TM^* - Span\{\xi\}$ implies that $d\xi$ is also non-degenerate on $TM - Span\{Z\}$.

(8) is called the twist of Hitchin pair $(d\xi, F)$. On the other hand, a symplectic+contact structure on M is a couple $(d\xi, I)$ consisting of a symplectic form $d\xi$ and a contact structure I on M, which commute.

Lemma 2 *Let M be a symplectic manifold, $d\xi$ is the symplectic form. Then $(d\xi, F)$ is a symplectic+contact structure if and only if $F^* d\xi = \xi \otimes (i_Z d\xi) - d\xi$.*

Proof. We will only prove the sufficient condition. It is trivial that $d(d\xi) = 0$ and $d(d\xi)_F = 0$, where $(d\xi)_F(X,Y) = d\xi(FX,Y)$. Since $F^* d\xi = -d\xi + \xi \otimes (i_Z d\xi)$, by using the following equation (see [3]),

$$i_{N_F(X,Y)}(d\xi) = i_{FX\wedge Y+X\wedge FY}(d(d\xi)_F) - i_{FX\wedge FY}(d(d\xi))$$

$$-i_{X\wedge Y}(d(F^* d\xi)), \quad (9)$$

we get $i_{N_F(X,Y)}(d\xi) = -i_{X\wedge Y}(d(F^* d\xi)) = -i_{X\wedge Y} d(\xi \otimes (i_Z d\xi)$. Then we have

$$i_{X\wedge Y} d(\xi \otimes (i_Z d\xi)) = d(\xi \otimes (i_Z d\xi))(X,Y,K)$$

$$= (d\xi \otimes (i_Z d\xi))(X,Y,K)$$

$$= d\xi(d\xi(X,Y)Z,K)$$

Thus we derive,

$$d\xi(N_F(X,Y),K) = -d\xi(d\xi(X,Y)Z,K).$$

Hence we get $N_F(X,Y) = -d\xi(X,Y)Z$, for $X,Y \in TM - span\{Z\}$ due to $d\xi$ is non-degenerate. Thus (1) (see also;[8]) implies that F is a contact structure. Then we have

$$F^* d\xi(X,Y) + d\xi(X,Y) - \xi \otimes (i_Z d\xi)(X,Y) = 0.$$

Since F^* is the dual contact structure, we get

$$d\xi(FX,FY) + d\xi(X,Y) - \xi(X)(i_Z d\xi)(Y) = 0.$$

Substituting FX by X, and using contact structure property.

$$d\xi(-X + \xi(X)Z, FY) + d\xi(FX,Y) = 0.$$

Hence we obtain

$$-d\xi(X,FY) + d\xi(FX,Y) = 0$$

which shows that $d\xi$ and F commute. The converse is clear.

Next we relate (1) and the 2-form $d\xi$.

Lemma 3 *Let P be a non-degenerate bivector on a generalized almost contact manifold M, and $d\xi$ the inverse 2-form (defined by $(d\xi)_\# = (P^\#)^{-1}$). Then P satisfies (1).*

Proof. Since $d\xi$ is a closed form, it is obvious due to Lemma 2.7 of [3].

Thus, we have the following result which shows that there is close relationship between condition (1) and a symplectic groupoid. Since $P^\#$ and $[,]_P$ define a Lie algebroid structure on T^*M, one can obtain the following result.

Theorem 2 *Let M be a generalized contact manifold. Then, there is a symplectic groupoid $(\Sigma, d\xi)$ over M.*

Proof. It is a well known fact that there is a one to one correspondence between integrable Poisson structures on M and symplectic groupoids over M. In fact, the condition (1) tells that P is an integrable Poisson structure.

We recall the following result from [3].

Lemma 4 [3] *Given a symplectic form ω, the associated non-degenerate bivector field P, i.e., ($P^\# = \omega_\#^{-1}$) and a bundle map a, then P and a satisfy (2) if and only if P and a commute and ω_a is closed.*

For our situation, $(d\xi)_F$ is closed we get the following.

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Remark 1 *Let* M *be a generalized almost contact manifold and* $d\xi$ *the symplectic form. Given a non-degenerate bivector* P *on* $TM - span\{\xi\}$ *(i.e.* $P^{\#} = ((d\xi)_{\#})^{-1}$*) and a map* $F : TM \to TM$*, then* P *and* F *satisfy (2) if and only if* $d\xi$ *and* F *commute.*

We now give a correspondence between generalized contact structures with non-degenerate P, and Hitchin pairs $(d\xi, F)$.

Proposition 1 *There is a one to one correspondence between generalized contact structures given by (2) with* P *non-degenerate, and Hitchin pairs* $(d\xi, F)$*. In this correspondence,* P *is the inverse of* $d\xi$*, and* θ *is the twist of the Hitchin pair* $(d\xi, F)$*.*

Proof. Since $(d\xi, F)$ is Hitchin pair, then $d\xi$ and $(d\xi)_F$ are closed. Using (9), we get

$$i_{N_F(X,Y)}(d\xi) = -i_{X\wedge Y}(d(F^* d\xi)). \qquad (10)$$

Since $\theta = -d\xi - F^* d\xi + \xi \otimes (i_Z d\xi)$, we derive

$$(d\xi)_{\#}(N_F(X,Y)) = -i_{X\wedge Y}(d(-\theta + \xi \otimes (i_Z d\xi))) \qquad (11)$$

Applying $P^{\#}$ to (11), then we get

$$\begin{aligned} N_F(X,Y) &= -P^{\#}(i_{X\wedge Y}(d(-\theta + \xi \otimes (i_Z d\xi)))) \\ N_F(X,Y) &= P^{\#}(i_{X\wedge Y}(d\theta) - P^{\#}(i_{X\wedge Y} d\xi \otimes (i_Z d\xi))) \\ N_F(X,Y) &= P^{\#}(i_{(X\wedge Y)} d\theta) - d\xi(X,Y)Z. \end{aligned} \qquad (12)$$

(12) is the second equation of (3). Now we show that $F^*\theta_{\#} = \theta_{\#} F$. From (7), we obtain

$$F^*\theta_{\#} = F^*(-(d\xi)_{\#} - (F^*(d\xi))_{\#} + (\xi \otimes (i_Z(d\xi)))_{\#}).$$

Hence, we have

$$F^*\theta_{\#} = -(d\xi)_{\#} F - (F^*(d\xi))_{\#} F + (\xi \otimes (i_Z(d\xi)))_{\#} F.$$

From definition of twist, we get

$$F^*\theta_{\#} = \theta_{\#} F.$$

This equation is the first equation of (4). Now, we will obtain

$$d\theta_F(X_1, X_2, X_3) = d\theta(FX_1, X_2, X_3) + d\theta(X_1, FX_2, X_3) + d\theta(X_1, X_2, FX_3)$$

which is second equation of (4). Writing the equation as

$$i_{X_1\wedge X_2}(d\theta_F) = i_{FX_1\wedge X_2 + X_1\wedge FX_2}(d\theta) + F^*(i_{X_1\wedge X_2}(d\theta))$$

and since $\theta = -d\xi - F^* d\xi + \xi \otimes (i_Z d\xi)$, then we should find

$$i_{X_1\wedge X_2}(d(-(F^*(d\xi))_F + (\xi \otimes (i_Z(d\xi)))_F) = i_{FX_1\wedge X_2 + X_1\wedge FX_2}(d(-F^* d\xi + \xi \otimes (i_Z d\xi)))$$
$$+ F^*(i_{X_1\wedge X_2}(d(-F^* d\xi + \xi \otimes (i_Z d\xi)))).$$

A straightforward computation shows that

$$i_{X_1\wedge X_2}(d((F^*(d\xi))_F) = i_{FX_1\wedge X_2 + X_1\wedge FX_2}(d(F^* d\xi)) + F^*(i_{X_1\wedge X_2}(d(F^* d\xi))).$$

Using (9), then we get

$$i_{X_1\wedge X_2}(d((F^*(d\xi))_F) = i_{FX_1\wedge X_2+X_1\wedge FX_2}(d(F^*d\xi)) - i_{N_F(X_1,X_2)}(d\xi)_F. \qquad (13)$$

Since $i_{N(X_1,X_2)}((d\xi)_F) = F^* i_{N(X_1,X_2)}(d\xi)$, applying $i_{X_1\wedge X_2}d(F^*d\xi) = -i_{N_F(X_1,X_2)}(d\xi)$ to (13), we have

$$i_{X_1\wedge X_2}(d((F^*(d\xi))_F) = i_{FX_1\wedge X_2+X_1\wedge FX_2}(d(F^*d\xi)) + F^*(i_{X_1\wedge X_2}(d(F^*d\xi))).$$

The converse is clear from Lemma 3 and Remark 1.

Note that it is well known that there is one to one correspondence between (1,1)-tensors F commuting with $d\xi$ and 2-forms on M. On the other hand, it is easy to see that (2) is equivalent to the fact that F^* is an IM form on the Lie algebroid T^*M associated Poisson structure P. Thus from the above discussion, Lemma 1 and Theorem 1, one can conclude with the following theorem.

Theorem 3 *Let M be a generalized contact manifold. Let P be an integrable Poisson structure on M, and $(\Sigma, d\xi)$ a symplectic groupoid over M. Then there exist multiplicative $(1,1)$-tensors I on Σ with the property that $(I, d\xi)$ is a Hitchin pair.*

Proof. From Theorem 3.3 of [3], we know that there is a one to one correspondence between (1,1)-tensors F on M satisfying (2) and multiplicative (1,1)-tensors I on Σ with the property that $(I, d\xi)$ is a Hitchin pair.

We recall the notion of generalized contact map between generalized contact manifolds. This notion is similar to the generalized holomorphic map given in [3].

Let $(M_i, \mathcal{F}_i)$, $i=1,2$, be two generalized contact manifolds, and let F_i, P_i, θ_i be the components of $\mathcal{F}_i$ in the matrix representation (4). A map $f: M_1 \to M_2$ is called generalized contact iff f maps F_1 into F_2, F_1 into F_2, P_1 into P_2, $f^*\theta_2 = \theta_1$ and $(df)\circ F_1 = F_2 \circ (df)$.

We now state and prove the following theorem which gives equivalent assertions between the condition (3), twist θ of $(d\xi, I)$ and contact maps for a symplectic groupoid over M.

Theorem 4 *Assume that (P,F) satisfy (1), (2) with integrable P, and let $(\Sigma, d\xi, I)$ be the induced symplectic groupoid over M and I the induced multiplicative (1, 1)-tensor. Then, for a 2-form θ on M, the following assertions are equivalent:*

1. (3) is satisfied,
2. $d\xi + I^*d\xi - \xi\otimes i_Z d\xi = s^*\theta - t^*\theta$,
3. $(t,s): \Sigma \to M\times \overline{M}$ is generalized contact map; (condition of generalized contact map on M is $(dt)\circ F_1 = F_2\circ(dt)$, this condition on $\overline{M}$ is $(ds)\circ F_1 = -F_2\circ(ds)$).

Proof. (i)$\Leftrightarrow$ (ii): Define $\phi = \tilde{\theta} - t^*\theta + s^*\theta$, such that $\tilde{\theta} = -d\xi - I^*d\xi + \xi\otimes(i_Z d\xi)$ and $A = ker(ds)|_M$. We know from Theorem 1 that closed multiplicative 2-form ψ on Σ vanishes if and only if IM form $u_\psi = 0$, i.e. $\psi(X,\alpha) = 0$, such that $X \in TM$, $\alpha \in A$. This case can be applied for forms with higher degree, i.e. 3-form ψ vanishes if and only if $\psi(X,Y,\alpha) = 0$.

Since $d\xi$ and $(d\xi)_I$ are closed, from (9) we get $i_{X\wedge Y}(d(I^*d\xi)) = -i_{N_I(X,Y)}d\xi$. Putting $\tilde{\theta} = -d\xi - I^*d\xi + \xi\otimes(i_Z d\xi)$, we obtain

$$i_{X\wedge Y}d(-\tilde{\theta} + \xi\otimes(i_Z d\xi)) = -i_{N_I(X,Y)}d\xi. \qquad (14)$$

Since $d\phi = 0 \Leftrightarrow d\phi(X,Y,\alpha) = 0$, we have

$$d\phi(X,Y,\alpha) = 0 \Leftrightarrow d\tilde{\theta}(X,Y,\alpha) - d(t^*\theta)(X,Y,\alpha) + d(s^*\theta)(X,Y,\alpha) = 0.$$

On the other hand, we obtain

$$d(t^*\theta)(X,Y,\alpha) = d\theta(dt(X), dt(Y), dt(\alpha)). \tag{15}$$

If we take $dt = \rho$ in (15) for A, we get

$$d(t^*\theta)(X,Y,\alpha) = d\theta(dt(X), dt(Y), \rho(\alpha)). \tag{16}$$

On the other hand, from [2] we know that

$$Id_\Sigma = m \circ (t, Id_\Sigma). \tag{17}$$

Differentiating (17), we obtain

$$X = dt(X). \tag{18}$$

Using (18) in (16), we get

$$d(t^*\theta)(X,Y,\alpha) = d\theta(X,Y,\rho(\alpha)).$$

In a similar way, we see that

$$d(s^*\theta)(X,Y,\alpha) = d\theta(ds(X), ds(Y), ds(\alpha)).$$

Since $\alpha \in kerds$, then $ds(\alpha) = 0$. Hence $d(s^*\theta) = 0$. Thus we obtain

$$d\tilde{\theta}(X,Y,\alpha) = d\theta(X,Y,\rho(\alpha)). \tag{19}$$

Using (14) in (19), we derive

$$d(\xi \otimes (i_Z d\xi))(X,Y,\alpha) + d\xi(N_I(X,Y),\alpha) = d\theta(X,Y,\rho(\alpha)). \tag{20}$$

On the other hand, it is clear that $\phi = 0 \Leftrightarrow \tilde{\theta} - t^*\theta + s^*\theta = 0$. Thus we obtain

$$\tilde{\theta}(X,\alpha) = \theta(X,\rho(\alpha)).$$

Since $\tilde{\theta} = -d\xi - I^* d\xi + \xi \otimes (i_Z d\xi)$, we get

$$-d\xi(X,\alpha) - d\xi(IX, I\alpha) + (\xi \otimes (i_Z d\xi))(X,\alpha) = \theta(X,\rho(\alpha)). \tag{21}$$

Since Poisson structure P is integrable, it defines a Lie algebroid whose anchor map is $P^\#$. Let us use $P^\#$ instead of ρ in (20) and (21), then we get

$$d(\xi \otimes (i_Z d\xi))(X,Y,\alpha) + d\xi(N_I(X,Y),\alpha) = +d\theta(X,Y,P^\#(\alpha)), \tag{22}$$

$$-d\xi(X,\alpha) - d\xi(IX, I\alpha) + (\xi \otimes (i_Z d\xi))(X,\alpha) = \theta(X,P^\#(\alpha)). \tag{23}$$

Since $d\xi(\alpha, X) = \alpha(X)$, $(d\xi)_I(\alpha, X) = \alpha(IX)$, from (22) we have

$$-\alpha(d\xi(X,Y)Z) - \alpha(N_F(X,Y)) = d\theta(X,Y,P^\#(\alpha))$$

$$= i_{X\wedge Y} d\theta(P^\#(\alpha))$$

$$= i_{X\wedge Y} d\theta(P^\#(\alpha))$$

$$= P(\alpha, i_{X\wedge Y} d\theta)$$

$$= -\alpha(P^\#(i_{X\wedge Y} d\theta)),$$

i.e. $\alpha(d\xi(X,Y)Z + N_F(X,Y)) = \alpha(P^\#(i_{X\wedge Y} d\theta))$. Since above equation holds for all non-degenerate α, we get

$$d\xi(X,Y)Z + N_F(X,Y) = P^{\#}(i_{X\wedge Y}d\theta). \quad (24)$$

On the other hand, from (23) we obtain

$$\alpha(X) + \alpha(F^2X) - \xi(X)\alpha(Z) = P(\alpha, i_X\theta)$$

$$= -\alpha(P^{\#}\theta_{\#}X).$$

Thus we get

$$F^2 + P^{\#}\theta_{\#} = -Id + \xi \otimes Z. \quad (25)$$

Then (i) $\Leftrightarrow$ (ii) follows from (24) and (25).

(ii) $\Leftrightarrow$ (iii): $d\xi + I^* d\xi - \xi \otimes i_Z d\xi = s^*\theta - t^*\theta$ says that (t,s) is compatible with 2-forms. Also it is clear that (t,s) and bivectors are compatible due to Σ is a symplectic groupoid. We will check the compatibility of (t,s) and $(1,1)$ -tensors. From compatibility condition of t and s, we will get $dt \circ I = F \circ dt$ and $ds \circ I = -F \circ ds$.

For all $\alpha \in A$ and $V \in \chi(\Sigma)$, we have

$$d\xi(\alpha, V) = d\xi(\alpha, dtV)$$

which is equivalent to

$$\alpha(V) = \langle u_{d\xi}(\alpha), dtV\rangle.$$

Since $u_{d\xi} = Id$ and $u_{(d\xi)_I} = F^*$, we get

$$\langle \alpha, F(dt(V))\rangle = \alpha(F(dt(V)))$$

$$= F^*\alpha(dt(V))$$

$$= \langle u_{(d\xi)_I}\alpha, V\rangle$$

$$= d\xi(\alpha, IV)$$

$$= d\xi(\alpha, dt(IV))$$

$$= \langle \alpha, dt(IV)\rangle.$$

Since this equation holds for all $\alpha \in A$, $F(dt) = dt(I)$. Using $s = t \circ i$,

$$F(ds(V)) = Fd(t \circ i)V$$

$$= F(dt(di(V)))$$

$$= -F(dt(V))$$

$$= -ds(IV),$$

which shows that $F(ds) = -ds(I)$. Thus proof is completed.

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