On Two Absolute Index-Summability Methods

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ABSTRACT: In this paper we have established a relation between the Summability methods $X - \left| \overline{N}, p_n \right|_k$, $Y - \left| A \right|_k , k \geq 1$ and $Y - \left| A, f(\delta) \right|_k , k \geq 1, \delta \geq 0.$

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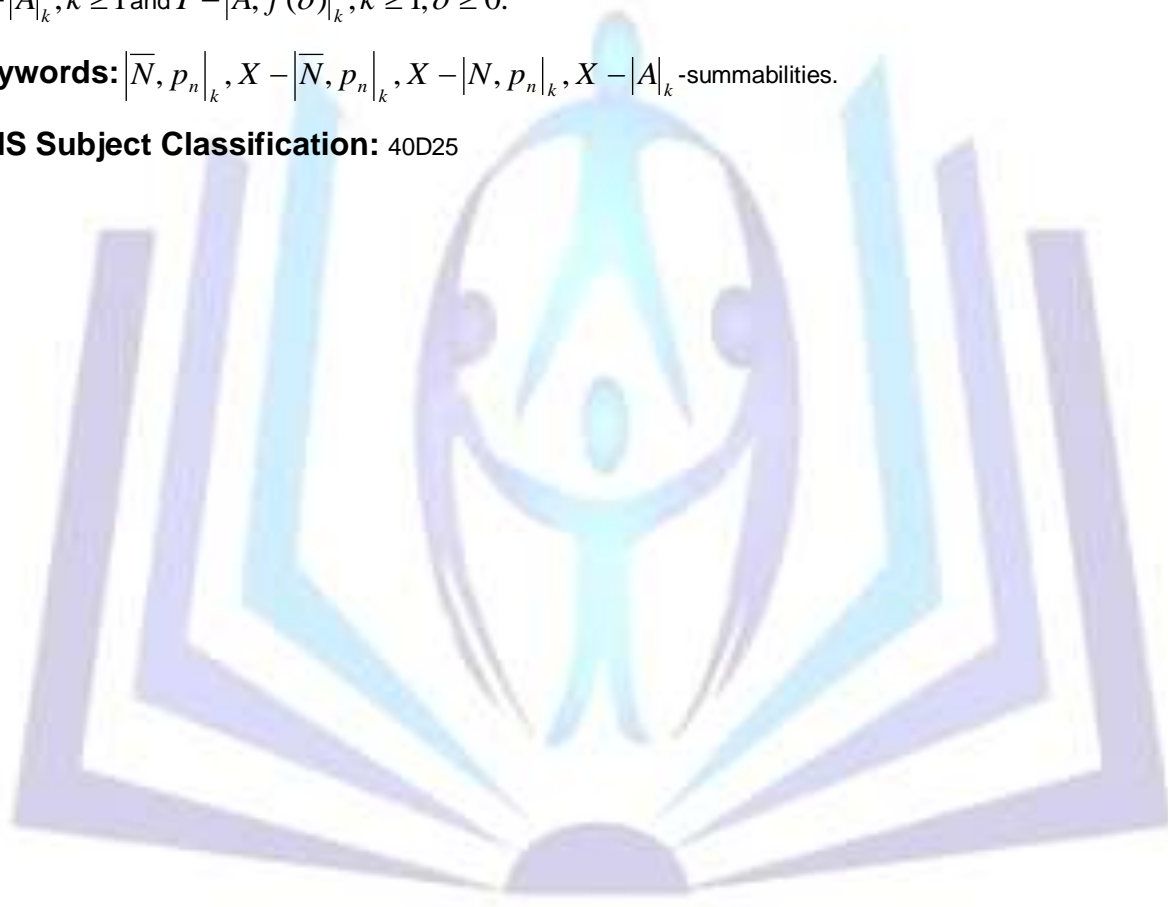
1. Introduction:

Let $\sum a_n$ be an infinite series and $\{s_n\}$ the sequence of partial sums. Let $\{p_n\}$ be a sequence of non-negative numbers with $P_n = \sum_{v=0}^{n} p_v$ for all $n \in N$. The sequence –to-sequence transformation

(1.1)
$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v , P_n \neq 0$$

defines $\left|\overline{N}, p_n\right|$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$. The series $\sum a_n$ is said to be summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$, **[4]** if

(1.2)
$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|t_n - t_{n-1}\right|^k < \infty ,$$

The sequence –to-sequence transformation

(1.3)
$$\tau_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v , P_n \neq 0 ,$$

defines $\left|N, p_n\right|$-mean of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be summable $\left|N, p_n\right|_k$, $k \geq 1$, if

(1.4)
$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\tau_n - \tau_{n-1}\right|^k < \infty .$$

The series $\sum a_n$ is said to be summable $X - \left|\overline{N}, p_n\right|_k$, $k \geq 1$, if

(1.5)
$$\sum_{n=1}^{\infty} X_n^{k-1} \left|t_n - t_{n-1}\right|^k < \infty$$

where $\{X_n\}$ is a sequence of positive real constants. Similarly, $\sum a_n$ is said to be summable $X - \left|N, p_n\right|_k$, $k \geq 1$, if

(1.6)
$$\sum_{n=1}^{\infty} X_n^{k-1} \left|\tau_n - \tau_{n-1}\right|^k < \infty .$$

Let $A = (a_{nk})$ be a $\infty \times \infty$ matrix. The series $\sum a_n$ is said to be summable $X - |A|_k$, $k \geq 1$, if

(1.7)
$$\sum_{n=1}^{\infty} X_n^{k-1} \left|T_n - T_{n-1}\right|^k < \infty ,$$

$\sum a_n$ is said to be summable $X - |A, \delta|_k$, $k \geq 1, \delta \geq 0$, if

(1.8)
$$\sum_{n=1}^{\infty} X_n^{\delta k + k - 1} \left|T_n - T_{n-1}\right|^k < \infty$$

and $\sum a_n$ is said to be summable $X - \left|A, f(\delta)\right|_k$, $k \geq 1, \delta \geq 0$, if

(1.9)
$$\sum_{n=1}^{\infty}\left(f\left(\delta\right)\right)^{k} X_{n}^{k-1}\left|T_{n}-T_{n-1}\right|^{k} < \infty,$$

where the sequence –to-sequence transformation $\{T_n\}$ is given by

(1.10)
$$T_n = \sum_{k=1}^{\infty} a_{nk} s_k .$$

Clearly $X-\left|A, f\left(\delta\right)\right|_{k}$, $k \geq 1, \delta \geq 0$ reduces to $X-\left|A, \delta\right|_{k}$, $k \geq 1, \delta \geq 0$ if $f\left(\delta\right)=X_{n}^{\delta}$.

2. Known Theorems:

Dealing with the index summability method Bor has established the following theorems:

Theorem-A[1]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$

  i) $np_n = O(P_n)$ ii) $P_n = O(np_n)$.

If $\sum a_n$ is summable $\left|C,1\right|_{k}$ then it is summable $\left|\overline{N}, p_n\right|_{k}$, $k \geq 1$.

Theorem-B[2]:

Let $\{p_n\}$ be a sequence of positive real constants such that as $n \to \infty$

  i) $np_n = O(P_n)$ ii) $P_n = O(np_n)$.

If $\sum a_n$ is summable $\left|\overline{N}, p_n\right|_{k}$ then it is summable $\left|C,1\right|_{k}$, $k \geq 1$.

Subsequently Bor and Thorpe established the following result.

Theorem-C[3]:

Let $\{p_n\}$ and $\{q_n\}$ be the sequence of positive real constants such that

  i) $p_n Q_n = O(P_n q_n)$ ii) $P_n q_n = O(p_n Q_n)$.

then the series $\sum a_n$ is summable $\left|\overline{N}, q_n\right|_{k}$ whenever it is summable $\left|\overline{N}, p_n\right|_{k}$, $k \geq 1$.

Further, Tripathi established

Theorem-D[6]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

  i) $q_n p_n = O(p_n q_n)$ ii) $Q_n = O(q_n X_n)$ iii) $Y_n p_n = O(p_n)$.

If $\sum a_n$ is summable $X-\left|\overline{N}, p_n\right|_{k}$, then it is summable $Y-\left|\overline{N}, q_n\right|_{k}$, $k \geq 1$.

Extending the above result , Misra, Misra and Routa established the following theorem replacing $Y-\left|\overline{N}, q_n\right|_{k}$, $k \geq 1$ by $Y-\left|N, q_n\right|_{k}$, $k \geq 1$, in Theorem-D.

Theorem-E[7]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

  i) $q_n p_n = O(p_n q_n)$ ii) $Q_n = O(q_n X_n)$ iii) $Y_n p_n = O(p_n)$

iv) $\displaystyle\sum_{n=v+1}^{n+1}\frac{1}{Q_n}=O\left(\frac{1}{Q_\gamma}\right)$ v) $\displaystyle\sum_{v=1}^{n-1}a_v^{\frac{k}{k-1}}=O(1), k\neq 1.$

If $\sum a_n$ is summable $X-\left|\overline{N},p_n\right|_k$, then it is summable $Y-\left|N,q_n\right|_k$, $k\geq 1$.

Further, generalizing the above theorem to matrix summability Padhy, panda, Misra and Misra established the following theorem:

Theorem-F[5]:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

i) $a_{nn}P_n=O(p_n),$ ii) $\dfrac{1}{a_{nn}}=O(X_n),$ iii) $Y_np_n=O(P_n),$

iv) $\displaystyle\sum_{n=r}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}A_{nr}=O(a_{rr})$, where $A_{nk}=\displaystyle\sum_{v=k}^{n}a_{nv}$

and, for the infinite triangular matrix $A=\left(a_{nk}\right)_{\infty\,x\,\infty}$,

v) $\displaystyle\sum_{r=1}^{n}A_{nr}=O(1).$

Then $\sum a_n$ is $Y-\left|A,\delta\right|_k$, $k\geq 1, \delta\geq 0$ summable whenever $\sum a_n$ is summable $X-\left|\overline{N},p_n\right|_k$, $k\geq 1$.

In what follows, in the present paper, further generalizing the above theorem we establish

3. Main Theorem:

Suppose $\{p_n\}$, $\{q_n\}$, $\{X_n\}$ and $\{Y_n\}$ are sequences of positive real constants such that

(3.1)   i) $a_{nn}P_n=O(p_n),$

(3.2)   ii) $\dfrac{1}{a_{nn}}=O(X_n),$

(3.3)   iii) $Y_np_n=O(P_n),$

(3.4)   iv) $\displaystyle\sum_{n=r}^{m+1}\left(f(\delta)\right)^k\left(\frac{P_n}{p_n}\right)^{k-1}A_{nr}=O(a_{rr})$, where $A_{nk}=\displaystyle\sum_{v=k}^{n}a_{nv}$

and, for the infinite triangular matrix $A=\left(a_{nk}\right)_{\infty\,x\,\infty}$,

(3.5)   v) $\displaystyle\sum_{r=1}^{n}A_{nr}=O(1).$

Then $\sum a_n$ is $Y-\left|A,f(\delta)\right|_k$, $k\geq 1, \delta\geq 0$ summable, whenever $\sum a_n$ is summable $X-\left|\overline{N},p_n\right|_k$, $k\geq 1$.

4. Proof of The Theorem:

If $\{t_n\}$ is the nth $\left|\overline{N},p_n\right|$ -mean of $\sum a_n$,then

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

$$= \frac{1}{P_n} \{ p_0 a_0 + p_1(a_0 + a_1) + \cdots + p_n(a_0 + a_1 + \cdots + a_n) \}$$

$$= \frac{1}{P_n} \{ P_n a_0 + (P_n - p_0) a_1 + \cdots + (P_n - p_{n-1}) a_n \}$$

$$= \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v$$

Then

$$\Delta t_n = t_n - t_{n-1}$$

$$= \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1}) a_v - \frac{1}{P_{n-1}} \sum_{v=0}^{n-1} (P_{n-1} - P_{v-1}) a_v$$

$$= \frac{1}{P_n} \sum_{v=1}^{n} (P_n - P_{v-1}) a_v - \frac{1}{P_{n-1}} \sum_{v=0}^{n-1} (P_{n-1} - P_{v-1}) a_v$$

$$= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \sum_{v=1}^{n} P_{v-1} a_v$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v$$

Hence,

$$\frac{P_n P_{n-1}}{p_n} \Delta t_n = \sum_{v=1}^{n} P_{v-1} a_v$$

and

$$\frac{P_{n-1} P_{n-1}}{p_{n-1}} \Delta t_{n-1} = \sum_{v=1}^{n-1} P_{v-1} a_v$$

Thus,

$$a_n = \frac{P_n}{p_n} \Delta t_n - \frac{P_{n-2}}{p_{n-1}} \Delta t_{n-1}$$

Further, if $\{T_n\}$ is the nth A-mean of $\sum a_n$, where $A = (a_{nk})_{\infty x \infty}$ triangular matrix,

$$T_n = \sum_{k=0}^{n} a_{nk} s_k$$

$$= \sum_{k=0}^{n} A_{nk} a_k \quad , A_{nk} = \sum_{v=k}^{n} a_{nv}$$

Then,

$$T_n - T_{n-1} = \sum_{k=0}^{n} A_{nk} a_k - \sum_{k=0}^{n-1} A_{n-1,k} a_k$$

$$= \sum_{k=1}^{n} (A_{nk} - A_{n-1,k}) a_k$$

$$= \sum_{k=1}^{n} (A_{nk} - A_{n-1,k}) \left(\frac{P_k}{p_k} \Delta t_k - \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1} \right)$$

$$= \sum_{k=1}^{n} A_{nk} \frac{P_k}{p_k} \Delta t_k - \sum_{k=1}^{n} A_{nk} \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1} - \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_k}{p_k} \Delta t_k + \sum_{k=1}^{n-1} A_{n-1,k} \frac{P_{k-2}}{p_{k-1}} \Delta t_{k-1}$$

$$= S_1 + S_2 + S_3 + S_4 \text{ (say)}.$$

Now,

$$\sum_{n=1}^{m+1} (f(\delta))^k Y_n^{k-1} |T_n - T_{n-1}|^k \leq \sum_{n=1}^{m+1} (f(\delta))^k Y_n^{k-1} |S_1 + S_2 + S_3 + S_4|^k = \sum_{i=1}^{4} \sum_{n=1}^{m+1} Y_n^{\delta k + k - 1} |S_i|^k$$

(By Minokowski's inequality)

Our Theorem will be established if we show that $\sum_{n=1}^{m+1} Y_n^{\delta k + k - 1 \, 1} |S_i|^k < \infty$, $\forall i = 1, 2, 3, 4$.

$$\sum_{n=1}^{m+1} (f(\delta))^k Y_n^{k-1} |S_1|^k = \sum_{n=1}^{m+1} (f(\delta))^k Y_n^{k-1} \left| \sum_{r=1}^{n} A_{nr} \frac{P_r}{p_r} \Delta t_r \right|^k$$

$$\leq \sum_{n=1}^{m+1} (f(\delta))^k Y_n^{k-1} \sum_{r=1}^{n} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k A_{nr} \left(\sum_{r=1}^{n} A_{nr} \right)^{k-1}$$

(Using Holder's inequality)

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} (f(\delta))^k Y_n^{k-1} A_{nr}, \text{ by (3.5)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r} \right)^k |\Delta t_r|^k \sum_{n=r}^{m+1} (f(\delta))^k \left(\frac{P_n}{p_n} \right)^{k-1} A_{nr}, \text{ by (3.3)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{1}{a_{rr}} \right)^k |\Delta t_r|^k a_{rr}, \text{ (using 3.4)}$$

$$= O(1) \sum_{r=1}^{m+1} X_r^{k-1} |\Delta t_r|^k, \text{ (using 3.2)}$$

$$= O(1).$$

Next

$$\sum_{n=1}^{m+1} (f(\delta))^k Y_n^{k-1} |S_2|^k = \sum_{n=1}^{m+1} (f(\delta))^k Y_n^{k-1} \left| \sum_{r=1}^{n} A_{nr} \frac{P_{r-2}}{p_{r-1}} \Delta t_{r-1} \right|^k$$

$$\leq \sum_{n=1}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} \sum_{r=1}^{n} \left(\frac{P_{r-1}}{p_{r-1}}\right)^k \left|\Delta t_{r-1}\right|^k A_{nr} \left(\sum_{r=1}^{n} A_{nr}\right)^{k-1}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_{r-1}}{p_{r-1}}\right)^k \left|\Delta t_{r-1}\right|^k \sum_{n=r}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} A_{nr} \text{, by (3.5)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_{r-1}}{p_{r-1}}\right)^k \left|\Delta t_{r-1}\right|^k \sum_{n=r}^{m+1} \left(f\left(\delta\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} A_{nr} \text{, by (3.3)}$$

$$= O(1) \sum_{r=2}^{m+1} \left(\frac{P_r}{p_r}\right)^k \left|\Delta t_r\right|^k a_{rr} \text{ , (using 3.4)}$$

$$= O(1) \sum_{r=2}^{m+1} \left(\frac{1}{a_{rr}}\right)^k \left|\Delta t_r\right|^k a_{rr}$$

$$= O(1) \sum_{r=2}^{m+1} X_r^{k-1} \left|\Delta t_r\right|^k \text{ , (using 3.2)}$$

$$= O(1) .$$

Also,

$$\sum_{n=1}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} \left|S_3\right|^k = \sum_{n=1}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} \left|\sum_{r=1}^{n=1} A_{n-1,r} \frac{P_r}{p_r} \Delta t_r\right|^k$$

$$\leq \sum_{n=1}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} \sum_{r=1}^{n-1} \left(\frac{P_r}{p_r}\right)^k \left|\Delta t_r\right|^k A_{n-1,r} \left(\sum_{r=1}^{n} A_{n-1,r}\right)^{k-1}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{P_r}{p_r}\right)^k \left|\Delta t_r\right|^k \sum_{n=r+1}^{m+1} \left(f\left(\delta\right)\right)^k \left(\frac{P_n}{p_n}\right)^{k-1} A_{n-1,r} \text{, using (3.5)}$$

$$= O(1) \sum_{r=1}^{m+1} \left(\frac{1}{a_{rr}}\right)^k \left|\Delta t_r\right|^k a_{rr} \text{, (using 3.4)}$$

$$= O(1) \sum_{r=1}^{m+1} X_r^{k-1} \left|\Delta t_r\right|^k \text{ , (using 3.2)}$$

$$= O(1) .$$

Finally,

$$\sum_{n=1}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} \left|S_4\right|^k = \sum_{n=1}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} \left|\sum_{r=1}^{n=1} A_{n-1,r} \frac{P_{r-2}}{p_{r-2}} \Delta t_{r-1}\right|^k$$

$$\leq \sum_{n=1}^{m+1} \left(f\left(\delta\right)\right)^k Y_n^{k-1} \sum_{r=1}^{n-1} \left(\frac{P_{r-2}}{p_{r-2}}\right)^k \left|\Delta t_{r-1}\right|^k A_{n-1,r} \left(\sum_{r=1}^{n} A_{n-1,r}\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{m+1}\big(f(\delta)\big)^k Y_n^{k-1}\sum_{r=1}^{n-1}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k \big|\Delta t_{r-1}\big|^k A_{n-1,r}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{P_{r-1}}{p_{r-1}}\right)^k \big|\Delta t_{r-1}\big|^k \sum_{n=r+1}^{m+1} A_{n-1,r}\big(f(\delta)\big)^k Y_n^{k-1}$$

$$= O(1)\sum_{r=1}^{m}\left(\frac{1}{a_{r-1,r-1}}\right)^k \big|\Delta t_{r-1}\big|^k a_{r-1,r-1}, \text{ (using 3.4)}$$

$$= O(1)\sum_{r=1}^{m} X_{r-1}^{k-1}\big|\Delta t_{r-1}\big|^k \text{ , (using 3.2)}$$

$$= O(1).$$

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