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Enlargement of A Local Group To A Group

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ABSTRACT

Local lie groups are introduced by Cartan [1]. Local groups are local lie groups without topologically property. The aim of this paper is to find conditions that a local group X is contained in a group G. It is showed in the example 3.2 that every local groups is not gllobaly associative . Then it may not be extended to a group.

Indexing terms/Keywords

Local group; Gllobaly associative; Enlargeable groups; Monodrome.

Academic Discipline And Sub-Disciplines

Pure Mathematics, Algebra, Group, Topological groups, Lie groups.

SUBJECT CLASSIFICATION

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TYPE (METHOD/APPROACH)

Algebra, Group, Lie group.

1. INTRODUCTION

The studies for lie groups; a special case of local lie groups, goes back to 1936's [1]. Elie Cartan showed that every local lie group contains a neighborhood of identity which is homeomorphic to a neighborhood of the identity of a lie group [1], [6, Theorem 84] and Pontryagin spotted that a local lie group is basis for a lie group [6]. Then the question arose as to whether every local lie group is contained in a lie group. Olver showed that if a local lie group has the associative law then it embeds into a lie group [5].

Local groups are local lie groups without topologically property, which means that the group multiplication and inversion operations only been defined for elements sufficiently near the identity.

In this paper we show that under what conditions a local group X in center of group G can be extended to a group G. The idea is motivated by [2].

At the rest of the paper, in section 2, we give definitions which will be needed in the next section. In the section3, the monodrome of a local group is defined. In the example 3.2 is showed the local X isn't enlargeable to a group. We need the property gllobaly associative local groups. Then we extend a local group to group and this group is not necessarily unique. But If the group G is the X-monodrome then it is unique.

2. Primary Definitions

In this section we give definitions which will be needed another section.

Definition 2.1 If X is a set, $D^{(n)} \subset X^n$ is subset of the cartesian product X^n of n copies of X and

$$f^{(n)} : D^{(n)} \to X \ , \ f^{(n)}(x_1,...,x_n) = x_1 \ ... \ x_n \ ,$$

then $f^{(n)}$ will be called an n **-array** local operation on X . Denote $f^{(2)}(x,y)$ by xy .

Definition 2.2 A triple $(X, f^{(2)}, D^{(2)})$ is a **local group** if X is a set and a subset $D^{(2)} \subset X \times X$ and $f^{(2)} : D^{(2)} \to X$ a binary local operation such that [7]:

1. If xy and yz exist then either both $(xy)z$ and $x(yz)$ exist and $(xy)z = x(yz)$ or both $(xy)z$ and $x(yz)$ do not exist;
2. There exists an element $e \in D$ such that ex and xe exist for every $x \in D^{(1)}$ and $xe = ex$;

3. for every $x \in D^{(1)}$ there exist an unique $x^{-1} \in D^{(1)}$ such that xx^{-1} and $x^{-1}x$ exist and $xx^{-1} = x^{-1}x$;

4. If xy exists then $y^{-1}x^{-1}$ exists and $(xy)^{-1} = y^{-1}x^{-1}$.

Definition 2.3 Let X be a local group, we call X an $n-$ **assocative** if

1. local operation $f^{(n)}$ is defined for every $k < n$.
2. there exists $f^{(k)}(x_1,...,x_k)f^{(l)}(x_{k+1},...,x_n)$ for every $k,l < n$ such that $k+l = n$ and

$$f^{(k)}(x_1,...,x_k)f^{(l)}(x_{k+1},...,x_n) = f^{(n)}(x_1,...,x_n).$$

Definition 2.4 We call X the **global associative** if:

1. conditions in Definition 1.3 for every n hold;
2. for all local operation $f_1^{(n)}$, $f_2^{(n)}$ and for every n-tuple $(x_1, x_2,..., x_n) \in D^{(n)} \subset X^n$,

$$f_1^{(n)}(x_1,...,x_n) = f_2^{(n)}(x_1,...,x_n).$$

Definition 2.5 A $n-$ array local opration in a local group X is called a **word** if it is an $n-$ assocative.

A map of local groups $\phi : X \to X'$, will be called a **homomorphism** of local group, if $x, y \in X$ and $xy \in X$ then $\phi(x)\phi(y)$ exists in X' and $\phi(xy) = \phi(x)\phi(y)$.

With these morphisms the local groups form a category which contains the subcategory of groups.

Definition 2.6 A homomorphism of local groups $f : (X,.) \to (X',*)$ is called **strong** if for every $x, y \in X$, the existence of $f(x) * f(y)$ implies that $x.y \in X$.

A morphism is called a monomorphism (epimorphism) if it is injective (surjective).

Definition 2.7 A subset H of a local group will be called **sublocal group** (symmetric subset) if it contains the identity and also if $x \in H$ then $x^{-1} \in H$.

3. The Monodrome a Group Of a Local Group

In [8,9] enlargement of a local group X and monodrome were introduced.

Definition 3.1 We say that a local group X is **enlargeable** if there exists a group G and a morphism $\phi : X \to G$ such that $\phi : X \to \phi(X)$ is a isomorphism related to the equivalent class.

Example 3.2 Let

$X = \{1, a, b, c, d, ab, bc, cd, de, (ab)c, a(bc), (cd)e, c(de), b(cd)e, h = ((a(bc))d)e, k = a(b(cd)e) \mid (ab)c = a(bc), (cd)e = c(de)\}$.

Now X is a local group, with the action

$$1 * x = x * 1 = x,\ \ x * x^{-1} = x^{-1} * x = 1 \text{ for every } x \in X,$$

but X cannot be a local subgroup of a group, since $h \neq k$. If a group G is an enlargement of X, then X is isomorphic to $\phi(X)$ (ϕ as in Definition 2.1). Since $h, k \in X$, $h \neq k$.then $\phi(h) \neq \phi(k)$ in G, that is G is not a group and this is a contradiction.

So a local group X may not be extended to a group, but there exists a sublocal group of X in which it is enlargeable to a group.

Example 3.3 Let $X' = \{1, a, b, ab\}$ with the action as in the Example 3.2, such that

$$a^{-1} = a, b^{-1} = b, ((ab)^{-1} = b^{-1}a^{-1}).$$

Then X' is a sublocal group of X, which embeds in S_3 (permutation group of order 3) with :

$$1 \mapsto \mathrm{I},\ a \mapsto (231),\ b \mapsto (321).$$

We see that X' is a non-abelian local group and is enlargeable to a group.

So, there exists a sublocal group X' of a local group X which is enlargeable but X is not enlargeable to a group.

Definition 3.4 Let X be a local group and G a group and $X \subset G$. Then G is called an X-**monodrome** if

1. X generates G (i.e: X is the smallest closed sublocal group in G which generate G)
2. For a group H and every homomorphism $\psi : X \to H$ there exists a homomorphism $\nu : G \to H$ such that the following diagram commutes.

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow \psi & \\ G & \underset{\nu}{\rightarrow} & H \end{array} \qquad (3.1)$$

Note 3.5 Let G be an enlargement of X. Note that ϕ: $X \longrightarrow \phi(X)$ should be a strong isomorphism of local groups.

We consider $X = \{-1,0,1\}$ with the following actions:

$$1 * \{-1\} = \{-1\} * 1 = 0 \quad \text{and} \quad x * 0 = 0 * x = x \quad \forall\, x \in X$$

Then X is a local group. It cannot be extended to $\mathbb{Z}_3$. For if, ϕ: $X \longrightarrow \mathbb{Z}_3$ is the identity map, we have $1 * 1 = 2$ in $\mathbb{Z}_3$.
On the other hand, we know that $1 * 1 \notin X$ and $2 = 1 * 1 = \phi(1) * \phi(1)$.

which is a contradictions. But, X can be extend to $\mathbb{Z}_5$ and $\mathbb{Z}$ such that $\mathbb{Z}$ is an X-monoderome.

Lemma 3.6 (Uniqueness) Let G be a group which is an X-monodrome with embedding ϕ. Suppose H is a group, $\psi : X \to H$ a homomorphism, H is generated by $\psi(X)$ and $\nu' : H \to G$ is a homomorphism. If the following diagram commutes

$$\begin{array}{ccc} & & X \\ & \psi \swarrow & \downarrow \\ H & \underset{\nu'}{\rightarrow} & G \end{array} \qquad (3.2)$$

then ν' is a isomorphism and H is an X-monodrome with embedding ψ.

Proof: Combining the diagrams (3.1) and (3.2), we obtain

$$\begin{array}{ccccc} & & X & & \\ & \psi \swarrow & \downarrow \searrow \psi & & \\ H & \underset{\nu'}{\rightarrow} & G & \underset{\nu}{\rightarrow} & H \end{array}$$

Since $\nu(\nu'(\psi(x))) = \psi(x)$ for every $x \in X$ and H is generated by $\psi(X)$, then $\nu \circ \nu' = Id_H$.

$$H \underset{\nu'}{\rightarrow} G \underset{\nu}{\rightarrow} H$$

In fact, $\nu' : H \to G$ is surjective, because $\nu' \circ \psi(X) = \phi(X)$ generates G. Hence, ν' is a isomorphism with the inverse ν. □

Definition 3.7 Let F be a free group on a local group X. Then, $u \in F$ is called an **e-element** if $u = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \ldots x_n^{\varepsilon_n}$,where $\varepsilon_i \in \{1,-1\}$ and $x_1, x_2, \ldots, x_n \in X$ such that $f^{(n)}(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_n^{\varepsilon_n})$ is an $n-$associative and is equaled to the identity e in X.

A local group X can be embedded in the factor group F over a normal subgroup N which contains the e-elements.

Theorem 3.8 (main theorem) Let X be a local group with the global associative property and F the free group on X. Let $N \subset F$ be the set of all e-elements of F. Then N is a normal subgroup of F and if $\phi : F \to \dfrac{F}{N}$ denotes the natural homomorphism, then the restriction of ϕ to X is injective and $\frac{F}{N}$ is a monodrome for X with embedding $\phi : X \to \frac{F}{N}$.

Proof: We can easily obtain that N is normal and that $\phi : X \to \frac{F}{N}$ is a homomrphism.

Let $\psi : X \to H$ be a homomorphism into a group H that is generated with X. We will prove that there exists a commutative diagram

$$\begin{array}{ccc} & & X \\ & \psi \swarrow & \downarrow \phi \\ H & \underset{\nu}{\leftarrow} & \frac{F}{N} \end{array}$$

Indeed, since F is freely generated by X , the mapping $\psi: X \longrightarrow H$ can be extended to homomorphism $\gamma: F \longrightarrow H$. This gives a commutative diagram $\gamma \circ \iota = \psi$.

$$\begin{array}{ccc} X & & \\ \iota \downarrow & \searrow \psi & \\ F & \underset{\gamma}{\rightarrow} & H \end{array}$$

Where ι is the inclusion map. Since $\psi: X \longrightarrow H$ is a homomorphism, we have $\gamma(u) = e$ for every e-element $u \in F$. Thus $\gamma(N) = e$. It follows that $\gamma: F \longrightarrow H$ factor through $\phi: F \longrightarrow \frac{F}{N}$.

$$\begin{array}{ccc} & & F \\ & \gamma \swarrow & \downarrow \phi \\ H & \underset{\nu}{\leftarrow} & \frac{F}{N} \end{array}$$

Hence $\frac{F}{N}$ is monodrome. We have $\psi: X \longrightarrow H$ is the embedding map then ϕ is an embedding. □

4. Conclusion

The most common version of Hilbert's fifth problem asks whether every locally Euclidean topological group is a Lie group.

Goldbring in [3] solved this problem for local Lie groups, by methods from nonstandard analysis.

Now we define local groups, if the products $x.y; y.z; x.(y.z)$ and $(x.y).z$ are all defined, then $x.(y.z) = (x.y).z$. This condition is called "local associativity" [5]. A much stronger condition for a local group is "global associativity" in which, given any finite sequence of elements from the local group and two different ways of introducing parentheses in the sequence, if both products thus formed exist, then these two products are in fact equal.

we show that under condition "global associative" a local group X in center of group G can be extended to a group G by methods "construction of free groups".

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