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On a nonlocal boundary value problem of a coupled system of Volterra functional integro-dierential equations.

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Abstract

In this paper, we study the existence of a unique solution for the nonlocal boundary-value problem of coupled system of Volterra functional integro-differential equations.

Keywords:.Functional integral equation, nonlocal boundary conditions, coupled system, Lipschitz condition, fxed point

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1 .INTRODUCTION

The coupled systemof Volterra functional integro-differential equations appear in many applications that arise in the fields of mathematical analysis, nonlinear functio -nalanalysis, mathematical physics, and engineering . An interesting feature ofcoup-led system of functional integro-differential equations is their role in the study of many problems . Here we are study the existence of nonlocal boundary value problem of coupled system of Volterra functional integro-differential equations.

$$\frac{dx}{dt} = f_1\left(t, \int_0^t g_1\left(s, \frac{dy}{ds}\right) ds\right), \qquad t \in (0,1) \qquad (1)$$
$$\frac{dy}{dt} = f_2\left(t, \int_0^t g_2\left(s, \frac{dx}{ds}\right) ds\right) \qquad t \in (0,1) \; .$$

With the nonlocal boundary conditions

$$y(\tau) = \beta\, y(\xi), \tau \in [0,1), \quad \xi \in (0,1], \quad \beta \neq 1 \qquad (3)$$

We are concerned here with the existence of solutions$x, y \in C[0,1]$ and $x, y \in AC[0,1]$ of **the problem (1)-(3).**

2 .Functional integral equations

Consider the boundary value problem of the coupled system of Volterra functional integro-differential equations (1) with the nonlocal boundary conditions (2) - (3).

Let $\frac{dx}{dt} = u$, and $\frac{dy}{dt} = v$, in (1) we obtain the coupled system of functional integral equations

$$u(t) = f_1\, (t, \int_0^t g_1(s, v(s)) ds) \, , \qquad t \in (0,1)$$
$$(4)$$

$$v(t) = f_2\, (t, \int_0^t g_2(s, u(s)) ds), \qquad t\; (0,1)$$

where

$$x(t) = x(0) + \int_0^t u(s) ds \; (5)$$

and

$$y(t) = y(0) + \int_0^t v(s) ds. \qquad (6)$$

Now using the boundary condition (2), we obtain

$$x(\tau) = x(0) + \int_0^\tau u(s) ds$$

$$x(\xi) = x(0) + \int_0^\xi u(s) ds,$$

then

$x(0) + \int_0^{\tau} u(s)ds = \alpha\, x(0) + \alpha \int_0^{\xi} u(s)ds$.

$$x(0) = \frac{\alpha}{1-\alpha}\int_0^{\xi} u(s)ds - \frac{1}{1-\alpha}\int_0^{\tau} u(s)ds.$$

Substituting in (5), we obtain

$$x(t) = \frac{\alpha}{1-\alpha}\int_0^{\xi} u(s)ds - \frac{1}{1-\alpha}\int_0^{\tau} u(s)ds + \int_0^{t} u(s)ds. \qquad (7)$$

Also, using the boundary condition (3), and substituting in (6), we obtain

$$y(t) = \frac{\beta}{1-\beta}\int_0^{\xi} v(s)ds - \frac{1}{1-\beta}\int_0^{\tau} v(s)ds + \int_0^{t} v(s)ds. \qquad (8)$$

3. Existence a unique continuous solution

Consider the coupled system of the functional integral equations (4) under the following assumptions.

(1) f_i: $[0,1] \times R \to R$ are continuous and satisfy the Lipschitz condition with constant Li

$$| f_i(t,u) - f_i(t,v)| \le L_i\, |u - v|, \qquad i = 1,2$$

(2) g_i : $[0,1] \times R \to R$are measurable in $s \in [0,1]$ and satisfy the Lipschitz condition

$$| g_i(s,u(s)) - g_i(s,v(s))| \le k_i(s)\, |u(s) - v(s)|,$$

where

$$\int_0^t k_i(s)ds \le M_i\,, \qquad i = 1,2$$

Let $X = \{ U = (u,v): u,v \in C[0,1] \}$ and its norm defined as

$$\| (u,v)\| = \| u \| + \| v \| = sup| u(t)| + sup\, |v(t)|. \qquad t \in [0,1]$$

Now for the existence of a unique continuous solution for the coupled system of the functional integral equations (4) we have the following theorem.

Theorem 1. Let the assumptions (1)-(2) be satisfied. If $Li\, Mi < 1,\ i = 1,2,$ then the coupled system of the functional integral equations (4) has a unique continuous solution in X.

Proof. Define the operator F with the coupled system of thefunctional integral equations (4) by ,

$$F(u,v) = (F_1\, v, F_2 u)$$

where

$$F_1\, v = f_1\,(t, \int_0^t g_1(s,v(s))ds)$$

$$F_2 u = f_2\,(t, \int_0^t g_2(s,u(s))ds)$$

Firstly, to prove $F : X \to X$.

Let $u,v \in C[0,1]$, and $t1, t2 \in [0,1]$, such that $t1 < t2$, and $| t2 - t_1| < \delta$,

then

$$|F_1 v(t_2) - F_1 v(t_1)\,| = |\, f_1(t_2, \int_0^{t_2} g_1(s,v(s))ds) - f_1(t_1, \int_0^{t_1} g_1(s,v(s))ds)\,|$$

$$= | f_1(t_2, \int_0^{t_2} g_1(s, v(s))ds) \; - f_1(t_1, \int_0^{t_1} g_1(s, v(s))ds)$$

$$+ f_1(t_2, \int_0^{t_1} g_1(s, v(s))ds) \; - f_1(t_2, \int_0^{t_1} g_1(s, v(s))ds)|$$

$$\leq | f_1(t_2, \int_0^{t_2} g_1(s, v(s))ds) \; - f_1(t_2, \int_0^{t_1} g_1(s, v(s))ds)|$$

$$+| f_1(t_2, \int_0^{t_1} g_1(s, v(s))ds) \; - f_1(t_1, \int_0^{t_1} g_1(s, v(s))ds)|$$

$$\leq L_1 \int_0^{t_2} g_1(s, v(s))ds - \int_0^{t_1} g_1(s, v(s))ds)|$$

$$+| f_1(t_2 , \int_0^{t_1} g_1(s, v(s))ds) \; - f_1(t_1, \int_0^{t_1} g_1(s, v(s))ds)|.$$

Hence$F1v\,(t) \in C[0\,,1]\,, \forall\, v\,(t) \in C[0\,,1]$.

Smillarly, $F2\,u(t) \in C[0\,,1]\,, \forall\, u(t) \in C[0\,,1]$.

Now since $F\,(u, v) = (F1v\; F2\,u)$

$$F(u, v)(t_2) - F(u, v)(t_1) = F(u(t_2), v(t_2)) - F(u(t_1), v(t_1))$$

$$= (F_1 v(t_2), F_2 u(t_2\,)) - (F_1 v(t1), F_2 u(t1))$$

$$= \big(F_1 v(t2) - F_1 v(t1), F_2\, u(t2) - F_2 u(t1)\big).$$

Then

$$\| F(u, v)(t_2) - F(u, v)(t_1)\| = \| F_1 v(t_2) - F_1 v(t_1)\| + \| F_2 u(t_2) - F_2 u(t_1)\|.$$

This prove that$F : X \rightarrow X$.

Secondly, to prove that $F$is a contraction, we have this following.

Let $z = (u, v) \in X\,, z1 = (u_1, v_1) \in X$.

$$F(u, v) = (F_1 v(\,t\,), F_2 u(\,t\,))$$

$$F(u_1, v_1) = (F_1 v_1(\,t\,)\,, F_2\, u_1(\,t\,))$$

$$|F_1\, v(\,t\,) \; - F_1 v_1(\,t)| = | f_1(t, \int_0^{t} g_1(s, v(s))ds) \; - f_1\,(t, \int_0^{t} g_1(s, v_1(s))ds)\,|$$

$$\leq L_1 | \int_0^{t} g_1(s, v(s))ds - \int_0^{t} g_1(s, v_1(s))ds|$$

$$\leq L_1 | \int_o^{t} [g_1(s, v(s)) - g_1(s, v_1(s))]ds|$$

$$\leq L_1 \int_o^{t} |g_1(s, v(s)) - g_1(s, v_1(s))|ds$$

$$\leq L_1 \int_0^{t} k_1(s)|v(s) - v_1(s)|ds$$

$$\leq L_1 \int_0^{t} k_1(s) sup|v(s) - v_1(s)|ds$$

$$\leq L_1\|v - v_1\| \int_0^t k_1(s)ds$$

$$\leq L_1M_1\|v - v_1\|$$

$$i.e \qquad \|F_1v - F_1v_1\| \leq L_1M_1\|v - v_1\|.$$

Since $L1M1 < 1$,then$F1$ is a contraction.

Similarly,

$$\|F_2u - F_2u_1\| \leq L_2M_2 \|u - u_1\| .$$

Since$L_2M_2 < 1$, then $F_2$is a contraction.

Hence

$\|F(u,v) - F(u_1,v_1)\| = \|(F_1v, F_2u) - (F_1v_1, F_2u_1)\|$

$$= \|F_1v - F_1v_1, F_2u - F_2u_1\|$$

$$= \|F_1v - F_1v_1\| + \|F_2u - F_2u_1\|$$

$$\leq max\{L_1M_1, L_2M_2\}\|(u,v) - (u_1,v_1)\|.$$

Let $LM = max\{ L_1 M_1, L_2 M_2\}$

then

$$\| F (u,v) - F (u_1,v_1) \| \leq LM \| (u,v) - (u_1,v_1) \| .$$

Since$LM < 1$, then$F$ is a contraction, and by using Fixed point Theorem[(6)] then there exists a unique solution in X of the coupled system of the functional integral equation (4) .

4 .The boundary value problem

Consider now the problem (1)-(3) .

Theorem 2. Let the assumptions of the Theorem 1 be satisfied, then there exists a unique $solution\ x, y \in C[0,1]$of the problem(1)-(3).

Proof. The solutions of the problem (1) - (3) is given by

$$x(t) = \frac{\alpha}{1-\alpha}\int_0^{\xi} u(s)ds - \frac{1}{1-\alpha}\int_0^{\tau} u(s)ds + \int_0^t u(s)ds \in C[0,1]$$

and

$$y(t) = \frac{\beta}{1-\beta}\int_0^{\xi} v(s)ds - \frac{1}{1-\beta}\int_0^{\tau} v(s)ds + \int_0^t v(s)ds \in C[0,1] .$$

Where

$$u(t) = f_1 (t, \int_0^t g_1(s, v(s))ds) \in C[0,1] ,$$

$$v(t) = f_2 (t, \int_0^t g_2(s, u(s))ds) \in C[0,1] .$$

Then from Theorem 1 we can deduce that there exists of a unique solution of the problem(1) - (3).

5 . Unique L^1 –solution

Consider the coupled system of the functional integral equations (4) under the following assumptions.

(i) $f: [0,1] \times R \rightarrow R$are measurable $in\ t \in [0,1]$ and satisfy the Lipschitz condition with constantL_i

$$| f_i(t,u) - f_i(t,v)| \leq L_i |u - v|,$$

and $f_i(t,0) \in L^1[0,1]$, $i = 1,2$

(ii) $g_i: [0,1] \times R \to R$ are measurable in $s \in [0,1]$ and satisfy the Lipschitz condition

$$| g_i(s,u(s)) - g_i(s,v(s))| \leq K_i \, |u(s) - v(s)|,$$

and $g_i(s,0) \in L^1[0,1]$, $i = 1,2$

(iii) $\int_0^1 f_i(t,0)\,dt \leq r_i$ and $\int_0^1 g_i(s,0)\,dt \leq b_i i = 1,2.$

Define $Y = \{ U = (u, v) : u, v \in L^1[0,1] \}$ and its norm defined as

$$\| (u,v) \| = \| u \| + \| v \|$$

$$= \int_a^b |u(t)|dt + \int_a^b |v(t)|dt.$$

Now for the existence of integrable solution for the coupled system of the functional integral equations (4) we have the following theorem

Theorem 3. Let the assumptions (i)-(iii) be satisfied. If $L_i K_i < 1$, $i = 1,2$, then the coupled system of the functional integral equation (4) has a unique solution in Y.

Proof. Define the operator G associated with the functional integral equations (4) by

$$G(u,v) = (G_1 v, G_2 u)$$

where

$$G_1 v = f_1\,(t, \int_0^t g_1(s,v(s))ds)$$

$$G_2 u = f_2(t, \int_0^t g_2(s,u(s))ds)$$

Firstly, to prove $G : Y \to Y$.

$Let\ (u,v) \in Y.$

Now, to prove $G_1 v : L^1[0,1] \to L^1[0,1]$.

$$| f_1(t,u)| - |f_1(t,0)| \leq | f_1(t,u) - f_1(t,0) | \leq L_1|u|$$

$$|f_1(t,u)| \leq L_1|u| + |f_1(t,0)|.$$

Also ,

$$|g_1(s,v(s))| \leq K_1|v(s)| + |g_1(s,0)|.$$

Hence

$$|f_1(t,u)| = |G_1 v(t)|$$

$$= |f_1(t, \int_0^t g_1(s,v(s))ds)| \leq L_1 |\int_0^t g_1(s,v(s))ds| + |f_1(t,0)|.$$

Integrating both sides with respect to t, we obtain

$$\int_0^1 |f_1(\mathrm{t}, \int_0^t g_1(s, v(s))ds)|\, dt \le \int_0^1 [L_1| \int_0^t g_1(s, v(s))ds| + |f_1(t,0)|]\, dt.$$

$$\le \int_0^1 [L_1 \int_0^t |g_1(s, v(s))|ds + |f_1(t,0)|]\, dt$$

$$\le \int_0^1 [\, L_1 \int_0^t K_1|v(s)|ds + |g_1(s,0)| + |f_1(t,0)|]dt$$

$$\le L_1 \int_0^1 \int_0^t K_1|v(s)|dsdt + \int_0^1 g_1(s,0)dt + \int_0^1 |f_1(t,0)|dt$$

$$\le L_1\, K_1 \int_0^1 |v(s)|dt + \int_0^1 |g_1(s,0)|dt + \int_0^1 |f_1(t,0)|dt$$

$$\le L_1 K_1 \|v\|_{L^1} + b_1 + r_1$$

i.e

$$\|G_1 v\|_{L^1} \le L_1 K_1 \|v\|_{L^1} + b_1 + r_1$$

This prove that $G_1 v$: $L^1[0\,,1] \to L^1[0{,}1]$.

Similarly,

$$\|G_2 u\|_{L^1} \le L_2 K_2 \|u\|_{L^1} + b_2 + r_2$$

Then $\quad G_2 u$: $L^1[0{,}1] \to L^1[0{,}1]$.

Hence

$$\| G(u,v) \| = \| (G_1 v\,, G_2 u) \| = \| G_1 v \| + \| G_2\, u \|$$

$$= L_1\, K_1 \|v\|_{L^1} + L_2\, K_2 \|u\|_{L^1} + b_1 + b_2 + r_1 + r_2.$$

Then $G : Y \to Y$.

Secondly, to prove G is a contraction, we have the following

Let $z = (u\,, v) \in Y$, $\quad z_1 = (u_1\,, v_1) \in Y$

and $G(u\,,v) = (G_1 v\; G_2\, u)$, $G(u_1\,, v_1) = (G_1 v_1\,, G_2\, u_1)$.

$|G_1\, v - G_1\, v_1| = |\, f_1\,(t, \int_0^t g_1(s, v(s))ds) - f_1\,(t, \int_0^t g_1(s, v_1(s))ds)|$

Integrating both sides with respect to t we obtain

$$\int_0^1 |\mathrm{G}_1\, v - \mathrm{G}_1 v_1|dt \le \int_0^1 |f_1\,(\mathrm{t}, \int_0^t g_1(s, v(s))ds) - f_1\,(\mathrm{t}, \int_0^t g_1(s, v_1(s))ds)|dt$$

$$\le \int_{\mathbf{0}}^{\mathbf{1}} \mathrm{L}_1 | \int_0^t g_1(s, v(s))ds - \int_0^t g_1(s, v_1(s))ds|dt$$

$$\le \int_{\mathbf{0}}^{\mathbf{1}} \mathrm{L}_1 \int_0^t |g_1(s, v(s)) - g_1(\, s, v_1(s))|\, ds\, dt$$

$$\le L_1 \int_0^1 \int_0^t K_1 |v(s) - v_1(s)|dsdt$$

$$\leq L_1K_1\int_0^1 |v(s)-v_1(s)|dt$$

$$\leq L_1K_1\|v-v_1\|_{L^1}$$

i.e

$$\|G_1v-G_1v_1\|_{L^1}\leq L_1K_1\|v-v_1\|_{L^1}$$

Since $L_1K_1<1$, then G_1 is a contraction.

As done before we obtain

$$\|G_2u-G_2u_1\|_{L^1}\leq L_2K_2\|u-u_1\|_{L^1}$$

Since $L_2K_2<1$, then G_2 is a contraction.

Hence

$$\| G(u,v)-G(u_1,v_1)\| = \|(G_1v,G_2u)-(G_1v_1,G_2u_1)\|$$
$$= \|G_1v-G_1v_1, G_2u-G_2u_1\|$$
$$= \|G_1v-G_1v_1\|+\|G_2u-G_2u_1\|$$
$$\leq max\{L_1K_1, L_2K_2\}\|(u,v)-(u_1,v_1)\|.$$

Let $LK = max\{L_1K_1, L_2K_2\}$

then

$$\|G(u,v)-G(u_1,v_1)\|_{L^1}\leq LK\|(u,v)-(u_1,v_1)\|_{L^1}$$

Since $LK<1$ then G is a contraction, and by using Fixed point Theorem[(6)] then there exists a unique solution in Y of the coupled system of the functional integral equations (4) .

4 .The boundary value problem

Consider now the problem (1)-(3) .

Theorem 2. Let the assumptions of the Theorem 3 be satisfied,then there exists a uniquesolution $x, y \in AC[0,1]$ of the problem(1)-(3).

Proof. The solutions of the problem (1) - (4) is given by

$$x(t) = \frac{\alpha}{1-\alpha}\int_0^{\xi} u(s)ds - \frac{1}{1-\alpha}\int_0^{\tau} u(s)ds + \int_0^t u(s)ds \in AC[0,1]$$

and

$$y(t) = \frac{\beta}{1-\beta}\int_0^{\xi} v(s)ds - \frac{1}{1-\beta}\int_0^{\tau} v(s)ds + \int_0^t v(s)ds \in AC[0,1].$$

Where

$$u(t) = f_1(t,\int_0^1 g_1(s,v(s))ds) \in L^1[0,1],$$

$$v(t) = f_2(t,\int_0^1 g_2(s,u(s))ds) \in L^1[0,1].$$

Then from Theorem 3 we can deduce that there exists of a unique solution of the problem (1) - (3).

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