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Solutions and comparison theorems for anticipated backward stochastic differential equations with stochastic Lipschitz generators

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Abstract
This paper deals with a class of anticipated backward stochastic differential equations, we extend results of Peng and Yang (2009) to the case in which the generator satisfies stochastic Lipschitz conditions. The existence and uniqueness of solutions for anticipated backward stochastic differential equation as well as comparison theorem are obtained.

Keywords Anticipated backward stochastic differential equations; Backward stochastic differential equations; Existence and uniqueness; Comparison theorem

Mathematics Subject Classification (2000) 60H10

1 Introduction

Consider the following backward stochastic differential equations (BSDEs for short):

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s,$$

Where ξ is the terminal condition, T is the time horizon, f is the generator. This type of equation was firstly considered by Pardoux and Peng (1990), where the existence and uniqueness of adapted solutions were established under the uniform Lipschitz conditions on f . Since then, BSDEs have been studied extensively, and have gradually become an important mathematical tool in many fields such as financial mathematics, optimal control, stochastic games and partial differential euqations.

Recently, a new type of BSDE, called anticipated backward stochastic differential equations (anticipated BSDE for short), was introduced by Peng and Yang(2009). The anticipated BSDE has the following form:

$$\begin{cases} -\mathrm{d}Y_t = f(t, Y_t, Z_t, Y_{(t+\delta(t))}, Z_{(t+\zeta(t))})\mathrm{d}t - Z_t \mathrm{d}W_t, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K]; \\ Z_t = \eta_t, & t \in [T, T+K], \end{cases} \tag{1.1}$$

Where $\delta(\cdot):[0,T] \to \mathrm{R}^+ \setminus \{0\}$, $\zeta(\cdot):[0,T] \to \mathrm{R}^+ \setminus \{0\}$ are continuous functions and satisfying:

(i) there exists a constant $K \geq 0$ such that for each $t \in [0,T]$,

$$t + \delta(t) \leq T + K, \quad t + \zeta(t) \leq T + K;$$

(ii) there exists a constant $L \geq 0$ such that for each $t \in [0,T]$ and each nonnegative integrable function $g(\cdot)$,

$$\int_t^T g(s + \delta(s))\mathrm{d}s \leq L \int_t^{T+K} g(s)\mathrm{d}s, \qquad \int_t^T g(s + \zeta(s))\mathrm{d}s \leq L \int_t^{T+K} g(s)\mathrm{d}s.$$

We mention that, Peng and Yang proved in Peng and Yang(2009) that the equation (1.1) has a unique adapted solution under the Lipschitz conditions, furthermore, they established a comparison theorem, which required that generators of the anticipated backward stochastic differential equations cannot depend on the anticipated term of Z and one of them must be increasing in the anticipated term of Y .

Motivated by Peng and Yang(2009) and Hou(2013), the present paper deals with a class of anticipated backward stochastic differential equations under the stochastic Lipschitz conditions, the existence and uniqueness of solutions are given, furthermore, a more general comparison theorem in which the generators of the anticipated backward

stochastic differential equations break through the above restrictions are obtained.
The rest of the paper is organized as follows: in section2, we list some notations and some conditions, in section 3, we will give our main results, and in sections 4, we prove them.

2 Preliminaries

Let $\{W_t; t\geq 0\}$ be a d-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t; t\geq 0\}$ be its natural filtration. suppose $T\geq 0$ is given. For all $n\in\mathbb{N}$, denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. For any $s\in[0,T]$, we use the following notation:

$L^2(\mathcal{F}_T;\mathbb{R}^m)$ ={$\mathbb{R}^m$ -valued $\mathcal{F}_T$ measurable random variable ξ satisfying that $E[|\xi|^2]<+\infty$};

$L^2_{\mathcal{F}}(s,T;\mathbb{R}^m)$ ={$\mathbb{R}^m$ -valued and $\mathcal{F}_t$ -adapted process $\{\varphi_t\}_{t\in[s,T]}$ such that $E[\int_s^T|\varphi_t|_2\,\mathrm{d}t]<+\infty$};

$S^2_{\mathcal{F}}(s,T;\mathbb{R}^m)$ ={continuous process in $L^2_{\mathcal{F}}(s,T;\mathbb{R}^m)$ such that $E[sup_{t\in[s,T]}|\varphi_t|^2]<+\infty$ }.

If $m=1$, we denote them by $L^2(F_T)$, $L_F{}^2(0,T)$ and $S^2_{\mathcal{F}}(0,T)$. The above L^2 are all separable Hilbert spaces.

The setting of our problem is as follows: to find a pair of $\mathcal{F}_t$ -adapt processes $(Y_\cdot, Z_\cdot)\in S^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^{m\times d})$ satisfying anticipated BSDE (1.1).

In this paper, we assume that for all $s\in[0,T]$,

$$f(s,w,y,z,\xi,\eta):\Omega\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(\mathcal{F}_r;\mathbb{R}_m)\times L^2(\mathcal{F}_{r'};\mathbb{R}_{m\times d})\to L^2(\mathcal{F}_s;\mathbb{R}_m),$$

where $r,r'\in[s,T+K]$, and f satisfies the following conditions:

(H1) f satisfies the stochastic Lipschitz continuous in (y,z,ξ,η), non-uniformly with respect to (w,t), i.e., there exists four $(\mathcal{F}_t)$ -progressively measurable and positive stochastic process $\mu_1(w,t)$, $\mu_2(w,t)$, $\nu_1(w,t)$ and $\nu_2(w,t)$: $\Omega\times[0,T]\to\mathbb{R}^+$, $M\in\mathbb{R}^+$ satisfying

$$\int_0^T\mu_1(t)+\nu_1^2(t)+\mu_2(t)\mathrm{d}t+\nu_2^2(t)\mathrm{d}t\leq M,$$

Such $\forall s\in[0,T], y,y'\in\mathbb{R}, z,z'\in\mathbb{R}^d, \xi,\xi'\in L^2_{\mathcal{F}}(t,T+K), \eta,\eta'\in L^2_{\mathcal{F}}(t,T+K;\mathbb{R}^d), r,\bar{r}\in[t,T+K]$, we have

$$|f(w,t,y,z,\xi_r,\eta_{\bar{r}})-f(w,t,y',z',\xi_r{}',\eta_{\bar{r}}{}')|$$
$$\leq\mu_1(w,t)|y-y'|+\nu_1(w,t)|z-z'|+E^{\mathcal{F}_t}[\mu_2(w,t)|\xi_r-\xi_r{}'|+\nu_2(w,t)|\eta_{\bar{r}}-\eta_{\bar{r}}{}'|].$$

(H2) $E[\int_0^T|f(s,0,0,0,0)|^2\,\mathrm{d}s]<\infty$.

3 Existence and uniqueness result for anticipated BSDEs

In this section, inspired by Peng and Yang(2009) and Hou(2013), we will extend the result obtained in Peng and Yang(2009) (see Theorem 1 in the following). Before that, Let us present two results which will be useful in the proof of the theorem.

Lemma 1 For a fixed $\xi\in L^2(\mathcal{F}_T)$ and $g_0(\cdot)$ which is an $\mathcal{F}_t$ -adapted process satisfying $E[(\int_0^T|g_0(t)|\,\mathrm{d}t)^2]<+\infty$, there exists a unique pair of processes $(y_\cdot,z_\cdot)\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{1+d})$ satisfying the following BSDE:

$$y_t=\xi+\int_t^T g_0(s)\mathrm{d}s-\int_t^T z_s\mathrm{d}W_s,\qquad t\in[0,T].$$

If $g_0(\cdot)\in L^2_{\mathcal{F}}(0,T)$, then $(y_\cdot,z_\cdot)\in S^2_{\mathcal{F}}(0,T)\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$.

Lemma 2 If the assumption(ii) holds, then for all $t\in[0,T],\bar{u}(t)\geq 0$ and nonnegative integrable function $g(\cdot)$, we have

$$\int_t^T e^{\int_0^s \bar{u}(r)\mathrm{d}r} g(s+\delta(s))\mathrm{d}s \leq L\int_t^{T+K} e^{\int_0^s \bar{u}(r)\mathrm{d}r} g(s)\mathrm{d}s$$

Proof. In fact,

$$\begin{aligned}\int_t^T e^{\int_0^s \bar{u}(r)\mathrm{d}r} g(s+\delta(s))\mathrm{d}s &= \int_t^T e^{\int_0^{s+\delta(s)} \bar{u}(r)\mathrm{d}r} e^{-\int_s^{s+\delta(s)} \bar{u}(r)\mathrm{d}r} g(s+\delta(s))\mathrm{d}s\\ &\leq \int_t^T e^{\int_0^{s+\delta(s)} \bar{u}(r)\mathrm{d}r} g(s+\delta(s))\mathrm{d}s\\ &\leq L\int_t^{T+K} e^{\int_0^s \bar{u}(r)\mathrm{d}r} g(s)\mathrm{d}s.\end{aligned}$$

where (ii) was used in the last equation.

The following Theorem 3.3 is the first result of this paper, which extend the result obtained in Peng and Yang(2009) by weakening the Lipschitz condition to stochastic Lipschitz condition on coefficients.

Theorem 1 Suppose that f satisfies (H1) and (H2), and δ,ζ satisfy (i) and (ii). Then for any given terminal conditions $\xi_\cdot\in S^2_{\mathcal{F}}(T,T+K;\mathbb{R}^m)$ and $\eta_\cdot\in L^2_{\mathcal{F}}(T,T+K;\mathbb{R}^{m\times d})$, the anticipated BSDE (1.1) has a unique solution, that is, there exists a unique pair of $\mathcal{F}_t$-adapted processes $(Y_\cdot,Z_\cdot)\in S^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m)\times L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^{m\times d})$ satisfying (1.1).

Proof. We fix $\overline{\mu}(t)=4\mu_1^2(t)+4\nu_1^2(t)+4L\mu_2^2(t)+4L\nu_2^2(t)+1$, where $\mu_1(w,t)$, $\mu_2(w,t)$, $\nu_1(w,t)$ and $\nu_2(w,t)$ is the stochastic Lipschitz coefficients of f given in (H2), and introduce a norm in Banach space $L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m)$:

$$\|\nu(\cdot)\|_{\overline{\mu}(r)}=(E[\int_0^{T+K} e^{\int_0^t \overline{\mu}(r)\mathrm{d}r}|\nu_t|^2\,\mathrm{d}t])^{\frac{1}{2}}.$$

Clearly, it is equivalent to the original norm of $L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m)$. But it is more convenient to use this norm to construct a contraction mapping that allows us to apply the Fixed Point Theorem.

For $\forall(y_t,z_t)_{t\in[0,T+K]}\in S^2_{\mathcal{F}}(0,T+K)\times L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^d)$, by the suppose (H2), we have

$$\begin{aligned}&|f(s,y_s,z_s,y_{s+\delta(s)},z_{s+\zeta(s)})|\\ &\leq |f(s,0,0,0,0)|+\mu_1(s)|y_s|+\nu_1(s)|z_s|+E^{\mathcal{F}_s}[\mu_2(s)|y_{s+\delta(s)}|+\nu_2(s)|z_{s+\zeta(s)}|].\end{aligned}$$

Then

$$\begin{aligned}&E[(\int_0^T|f(t,y_t,z_t,y_{t+\delta(t)},z_{t+\zeta(t)})|^2\,\mathrm{d}t)]\\ &\leq 5E[(\int_0^T|f(t,0,0,0,0)|^2\,\mathrm{d}t)]+5E[(\int_0^T\mu_1(t)|y_t|\mathrm{d}t)^2]+5E[(\int_0^T\nu_1(t)|z_t|\mathrm{d}t)^2]\end{aligned}$$

$$
\begin{aligned}
&+5E[(\int_0^T \mu_2(t)E^{\mathcal{F}_t}|y_{t+\delta(t)}|\mathrm{d}t)^2]+5E[(\int_0^T \nu_2(t)E^{\mathcal{F}_t}|z_{t+\zeta(t)}|\mathrm{d}t)^2]\\
&\le 5E[(\int_0^T |f(t,0,0,0,0)|^2\,\mathrm{d}t)]+5E[\sup_{t\in[0,T]}|y_t|^2(\int_0^T \mu_1(t)\mathrm{d}t)^2]\\
&+5E[\int_0^T |\nu_1(t)|^2\,\mathrm{d}t\int_0^T |z_t|^2\,\mathrm{d}t]+5E[\sup_{t\in[0,T]}(E^{\mathcal{F}_t}|y_{t+\delta(t)}|)^2(\int_0^T \mu_1(t)\mathrm{d}t)^2]\\
&+5E[\int_0^T |\nu_2(t)|^2\,\mathrm{d}t\int_0^T (E^{\mathcal{F}_t}|z_{t+\zeta(t)}|)^2\mathrm{d}t].
\end{aligned}
$$

Applying Doob's martingale inequality,

$$
\begin{aligned}
E[\sup_{t\in[0,T]}(E^{\mathcal{F}_t}|y_{t+\delta(t)}|)^2] &\le E[\sup_{t\in[0,T]}(E^{\mathcal{F}_t}[\sup_{r\in[0,T+K]}|y_r|])^2]\\
&\le 4E[\sup_{r\in[0,T+K]}|y_r|^2].
\end{aligned} \tag{3.1}
$$

Thus we have

$$
\begin{aligned}
&E[(\int_0^T |f(t,y_t,z_t,y_{t+\delta(t)},z_{t+\zeta(t)})|^2\,\mathrm{d}t)]\\
&\le 5E[(\int_0^T |f(t,0,0,0,0)|^2\,\mathrm{d}t)]+5M^2E[\sup_{t\in[0,T]}|y_t|^2]+5ME[\int_0^T |z_t|^2\,\mathrm{d}t]\\
&+5M^2E[\sup_{r\in[0,T+K]}|y_r|^2]+5MLE[\int_0^{T+K}|z_t|^2\,\mathrm{d}t]<+\infty.
\end{aligned}
$$

So

$$
E[|\xi_T+\int_0^T f(s,y_s,z_s,y_{s+\delta(s)},z_{s+\zeta(s)}\mathrm{d}s|^2]<+\infty.
$$

which means $\{E^{\mathcal{F}_t}[\xi_T+\int_0^T f(s,y_s,z_s,y_{s+\delta(s)},z_{s+\zeta(s)}\mathrm{d}s]\}_{t\in[0,T+K]}$ is a square integrable martingale. According to the martingale representation theorem, there exists a unique $(Z_t)_{t\in[0,T+K]}\in L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^{k\times d})$ such that

$$
\begin{aligned}
E^{\mathcal{F}_t}[\xi_T+\int_0^T f(s,y_s,z_s,y_{s+\delta(s)},z_{s+\zeta(s)})\mathrm{d}s] &= E[\xi_T+\int_0^T f(s,y_s,z_s,y_{s+\delta(s)},z_{s+\zeta(s)})\mathrm{d}s]\\
&+\int_0^t Z_s\mathrm{d}W_s, \qquad 0\le t\le T.
\end{aligned} \tag{3.2}
$$

Let

$$
Y_t := E^{\mathcal{F}_t}[\xi_T+\int_t^T f(s,y_s,z_s,y_{s+\delta(s)},z_{s+\zeta(s)})\mathrm{d}s], \qquad 0\le t\le T. \tag{3.3}
$$

Obviously, $(Y,Z)\in L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})$. Equation (3.2) and (3.3) have constructed a mapping from $L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})$ to $L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^m\times\mathbb{R}^{m\times d})$, and we denote it by ϕ,

Set

$$
\begin{cases}
Y_t=\xi_T+\int_t^T f(s,y_s,z_s,y_{(s+\delta(s))},z_{(s+\zeta(s))})\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s, & t\in[0,T];\\
Y_t=\xi_t, & t\in[T,T+K];\\
Z_t=\eta_t, & t\in[T,T+K].
\end{cases} \tag{3.4}
$$

That is

$$\phi[(y_., z_.)] = (Y_., Z_.).$$

If ϕ is a contractive mapping with respect to the norm $\| v(\cdot) \|_{\overline{\mu}(r)}$, by the fixed point theorem, there exists a unique $(Y_., Z_.) \in L^2_{\mathcal{F}}(0, T+K; \mathrm{R}^m \times \mathrm{R}^{m\times d})$ satisfying (3.2) and (3.3), i.e. (3.4).

Now we prove that ϕ is a contraction mapping under the norm $\| \cdot \|_{\overline{u}(r)}$. For two arbitrary elements $(y_., z_.)$ and $(y_.', z_.')$ in $L^2_{\mathcal{F}}(0, T+K; \mathrm{R}^m \times \mathrm{R}^{m\times d})$, set $(Y_., Z_.) = \phi[(y_., z_.)]$ and $(Y_.', Z_.') = \phi[(y_.', z_.')]$. Denote their differences by

$$(\hat{y}_., \hat{z}_.) = ((y - y')_., (z - z')_.), \qquad (\hat{Y}_., \hat{Z}_.) = ((Y - Y')_., (Z - Z')_.).$$

We apply It$\hat{o}$'s formula to $e^{\int_0^T \overline{\mu}(r)\mathrm{d}r} |\hat{Y}_t|^2$, consider the suppose (H1), we have

$$\begin{aligned}
&|\hat{Y}_0|^2 + \int_0^T e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} (\overline{\mu}(r)|\hat{Y}_t|^2 + |\hat{Z}_t|^2)\mathrm{d}t \\
&= 2\int_0^T e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} \langle \hat{Y}_t, f(t, y_t, z_t, Y_{(t+\delta(t))}, Z_{(t+\zeta(t))} - f(t, y_t', z_t', Y_{(t+\delta(t))}', Z_{(t+\zeta(t))}' \rangle \mathrm{d}t \\
&\quad -2\int_0^T e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} \langle \hat{Y}_t, \hat{Z}_t \mathrm{d}W_t \rangle \\
&\le 2\int_0^T e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} |\hat{Y}_t| (\mu_1(t)|\hat{y}_t| + \nu_1(t)|\hat{y}_t| + \mu_2(t)E^{\mathcal{F}_t}|\hat{y}_{t+\delta(t)}| + \nu_2(t)E^{\mathcal{F}_t}|\hat{z}_{t+\zeta(t)}|)\mathrm{d}t \\
&\quad -2\int_0^T e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} \langle \hat{Y}_t, \hat{Z}_t \mathrm{d}W_t \rangle.
\end{aligned}$$

Then take expectations on both sides, noticing that the last term in the right-hand side is a martingale and use the inequality $2ab \le \varepsilon a^2 + b^2/\varepsilon,$ by Fubini theorem, we have

$$\begin{aligned}
&E[\int_0^{T+K} e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} (\overline{\mu}(r)|\hat{Y}_t|^2 + |\hat{Z}_t|^2)\mathrm{d}t] = E[\int_0^T e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} (\overline{\mu}(r)|\hat{Y}_t|^2 + |\hat{Z}_t|^2)\mathrm{d}t] \\
&\le E[\int_0^T e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} (4\mu_1^2(t)|\hat{Y}_t|^2 + \frac{1}{4}|\hat{y}_t|^2 + 4\nu_1^2(t)|\hat{Y}_t|^2 + \frac{1}{4}|\hat{z}_t|^2 \\
&\quad + 4L\nu_2^2(t)|\hat{Y}_t|^2 + \frac{1}{4L}(E^{\mathcal{F}_t}|\hat{y}_{t+\delta(t)}|)^2 + 4L\nu_1^2(t)|\hat{Y}_t|^2 + \frac{1}{4L}(E^{\mathcal{F}_t}|\hat{z}_{t+\delta(t)}|)^2)\mathrm{d}t] \\
&\le E[\int_0^{T+K} e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} (4\mu_1^2(t) + 4\nu_1^2(t) + 4L\mu_2^2(t) + 4L\nu_2^2(t))|\hat{Y}_t|^2 + \frac{1}{2}|\hat{y}_t|^2 + \frac{1}{2}|\hat{z}_t|^2 \,\mathrm{d}t].
\end{aligned}$$

Because $\overline{\mu}(t) = 4\mu_1^2(t) + 4\nu_1^2(t) + 4L\mu_2^2(t) + 4L\nu_2^2(t) + 1$, then

$$E[\int_0^{T+K} e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} (|\hat{Y}_t|^2 + |\hat{Z}_t|^2)\mathrm{d}t] \le \frac{1}{2} E[\int_0^{T+K} e^{\int_0^t \overline{\mu}(r)\mathrm{d}r} (|\hat{y}_t|^2 + |\hat{z}_t|^2)\mathrm{d}t].$$

or

$$\|(\hat{Y}_\cdot,\hat{Z}_\cdot)\|_{\bar{\mu}_r} \le \frac{1}{\sqrt{2}}\|(\hat{y}_\cdot,\hat{z}_\cdot)\|_{\bar{\mu}_r}.$$

Consequently, ϕ is a strict contraction mapping of $L^2_{\mathrm{F}}(0,T+K;\mathrm{R}^m\times\mathrm{R}^{m\times d})$. It follows by the Fix Point Theorem that (1.1) has a unique solution $(Y_\cdot,Z_\cdot)\in L^2_{\mathrm{F}}(0,T+K;\mathrm{R}^m\times\mathrm{R}^{m\times d})$. Since f satisfies (H1) and (H2) and since δ,ζ satisfy (i) and (ii) , we have $f(\cdot,Y_\cdot,Z_\cdot,Y_{\cdot+\delta(\cdot)},Z_{\cdot+\zeta(\cdot)})\in L^2_{\mathrm{F}}(0,T;\mathrm{R}^m)$. Thus, by Lemma 3.1, we obtain $Y_\cdot\in S^2_{\mathrm{F}}(0,T+K;\mathrm{R}^m)$.

4 General comparison theorems

In this section, we will give a more general comparison theorem in which the generators of the anticipated BSDEs are allowed to contain the anticipated term of Z. The main approach we adapt is to consider an anticipated BSDE as a series of BSDEs then apply the well-known comparison for 1-dimensional BSDEs.

We assume that $f(w,t,y,z):\Omega\times[0,T]\times\mathrm{R}^m\times\mathrm{R}^{m\times d}$ satisfies the following conditions:

(A1) f satisfies the stochastic Lipschitz continuous in (y,z), non-uniformly with respect to (w,t), i.e., there exists two (F_t)-progressively measurable and positive stochastic process $\mu(t),\nu(t):\Omega\times[0,T]\to\mathrm{R}^+$ satisfies

$$\left\|\int_0^T \mu(w,t)+\nu^2(w,t)\mathrm{d}t\right\|_\infty<\infty.$$

(A2) $E[(\int_0^T|f(t,0,0)|\mathrm{d}t)^2]<\infty.$

Lemma 3 *Let $T\le+\infty,\xi,\xi'\in L^2(\Omega,\mathrm{F}_t,P;\mathrm{R})$. g and g' be two generators of BSDEs, $(y_\cdot,z_\cdot)$ and $(y'_\cdot,z'_\cdot)$ be respectively a solution to $BSDE(\xi,T,g)$ and $BSDE(\xi',T,g')$. If $\xi\le\xi',\mathrm{d}P-a.s.$, and one of the following two statements holds true:*

(i) g satisfies (A1) and (A2) and

$$g(t,y_t',z_t')\le g'(t,y_t',z_t')\qquad \mathrm{d}P\times\mathrm{d}t-a.s.,$$

(ii) g' satisfies (A1) and (A2) and

$$g(t,y_t,z_t)\le g'(t,y_t,z_t)\qquad \mathrm{d}P\times\mathrm{d}t-a.s.,$$

then for each $t\in[0,T]$, we have

$$y_t\le y_t',\qquad \mathrm{d}P-a.s..$$

From lemma 4.1, the following corollary is immediate.

Corollary 1 *Assume that $T\le+\infty$, and one of g and g' satisfies (A1) and (A2), Let $\xi,\xi'\in L^2(\Omega,\mathrm{F}_t,P;\mathrm{R})$, $(y_\cdot,z_\cdot)$ and $(y'_\cdot,z'_\cdot)$ be respectively a solution to $BSDE(\xi,T,g)$ and $BSDE(\xi',T,g')$. If $\xi\le\xi',\mathrm{d}P-a.s.,\forall y,z,\quad g(t,y,z)\le g'(t,y,z)\quad \mathrm{d}P\times\mathrm{d}t-a.s.$, then for each $t\in[0,T]$,*

$$y_t\le y_t',\qquad \mathrm{d}P-a.s..$$

Lemma 4 Putting $t_0=T$, we define by iteration

$$t_i:=min\{t\in[0,T]:min\{s+\delta(s),s+\zeta(s)\}\ge t_{i-1},s\in[t,T]\},\quad i\ge1, \tag{4.1}$$

Set $N := max\{i : t_{i-1} > 0\}$. Then N is finite, $t_N = 0$ and

$$[0,T] = [0, t_{N-1}] \cup [t_{N-1}, t_{N-2}] \cup \cdots \cup [t_2, t_1] \cup [t_1, T].$$

Proof. Let us first prove that N is finite. For this purpose, we apply the method of reduction to absurdity. Suppose N is infinite. From the definition of $\{t_i\}_{i=1}^{+\infty}$, we know

$$min\{t_i + \delta(t_i), t_i + \zeta(t_i)\} = t_{i-1}, \quad i = 1,2,\cdots. \text{(4.2)}$$

Since $\delta(\cdot)$ and $\zeta(\cdot)$ are continuous and positive, thus obviously we have $t_i \le t_i (i = 1,2,\cdots).$ Therefore $\{t_i\}_{i=1}^{+\infty}$ converges as a strictly monotone and bounded series. Denote its limit by $\bar{t}$. Letting $i \to \infty$ on both sides of (4.1), we get

$$min\{\bar{t} + \delta(\bar{t}), \bar{t} + \zeta(\bar{t})\} = \bar{t}.$$

Hence $\delta(\bar{t}) = 0$ or $\zeta(\bar{t}) = 0$, which is just a contradiction since both δ and ζ are positive.

Next we will show that $t_N = 0.$ In fact, the following holds obviously:

$$min\{t_N + \delta(t_N), t_N + \zeta(t_N)\} \ge t_N,$$

which implies $t_N = 0,$ or else we can find a $\bar{t} \in [0, t_N)$ due to the continuity of $\delta(\cdot)$ and $\zeta()\cdot$ such that

$$min\{s + \delta(s), s + \zeta(s)\} \ge t_N, for\ all\ s \in [\tilde{t}, T),$$

from which we know that $\tilde{t}$ is an element of the series sa well.

Lemma 5 Suppose $(Y^{(j)}, Z^{(j)})(j = 1,2)$ are the solutions of anticipated BSDEs(1) respectively. Then for fixed $i \in \{1,2,\cdots,N\}$, over time interval $[t_i, t_{i-1}]$, anticipated BSDE(1) are equivalent to the following anticipated BSDEs:

$$\begin{cases} -\bar{Y}_t^j = f_j(t, \bar{Y}_t^j, \bar{Z}_t^j, \bar{Y}_{t+\delta(t)}^j, \bar{Z}_{t+\zeta(t)}^j)\mathrm{d}t - \bar{Z}_t^j \mathrm{d}W_t, & t \in [t_i, t_{i-1}]; \\ \bar{Y}_t^j = Y_t^{(j)}, & t \in [t_{i-1}, T+K]; \\ \bar{Z}_t^j = Z_t^{(j)}, & t \in [t_{i-1}, T+K], \end{cases} \quad (4.3)$$

which are also equivalent to the following BSDEs with terminal conditions $Y_{t_{i-1}}^{(j)}$ respectively:

$$\tilde{Y}_t^j = \tilde{Y}_{t_{i-1}}^j + \int_t^{t_{i-1}} f_j(s, \tilde{Y}_s^j, \tilde{Z}_s^j, \tilde{Y}_{s+\delta(s)}^j, \tilde{Z}_{s+\delta(s)}^j)\mathrm{d}s - \int_t^{t_{i-1}} \tilde{Z}_s^j \mathrm{d}W_s. \quad (4.4)$$

That is to say,

$$Y_t^j = \bar{Y}_t^j = \tilde{Y}_t^j, \qquad Z_t^j = \bar{Z}_t^j = \tilde{Z}_t^j = \frac{\mathrm{d}\langle \tilde{Y}^j, W\rangle_t}{\mathrm{d}t}, \quad t \in [t_i, t_{i-1}], j = 1,2,$$

where $\langle \tilde{Y}^j, W\rangle$ is the variation process generated by $\tilde{Y}_t^j$ and the Brownian motion W.

Proof. We only need to prove the equivalence between anticipated BSDE(4.2) and BSDE(4.3). It is obvious that for each $s \in [t_i, t_{i-1}], s + \delta(s) \ge t_{i-1}, s + \zeta(s) \ge t_{i-1}$, thus $(\bar{Y}_{t+\delta(t)}^{(j)}, \bar{Z}_{t+\zeta(t)}^{(j)}) = (Y_{t+\delta(t)}^{(j)}, Z_{t+\zeta(t)}^{(j)})$ in the generator of anticipated BSDE(4.2). Clearly

$f_j(\cdot,\cdot,\cdot,Y^{(j)}_{t+\delta(t)},Z^{(j)}_{t+\zeta(t)})$ satisfies the Lipschitz condition as well as the integrable condition since f_j satisfies (H1) and (H2). Thus BSDE(4.3) has a unique adapted solution.

Moreover, it is obvious that $(Y^{(j)}_t, Z^{(j)}_t)_{t\in[t_i,t_{i-1}]}$ satisfies both anticipated BSDE(4.2) and BSDE(4.3). Then from the uniqueness of anticipated BSDE's solution and that of BSDE's, we can easily obtain the desired equalities.

Theorem 2 $Let(Y^{(j)},Z^{(j)})\in S^2_{\mathcal{F}}(0,T+K;\mathbb{R})\times L^2_{\mathcal{F}}(0,T+K;\mathbb{R}^d)$ (j=1,2) be the unique solutions to anticipated BSDEs (1) respectively. If

(i) $\xi^{(1)}_s \ge \xi^{(2)}_s, s\in[T,T+K], a.e., a.s.;$

(ii) for all $t\in[0,T],(y,z)\in\mathbb{R}\times\mathbb{R}^d,\theta^{(j)}\in S^2_{\mathcal{F}}(t,T+K;\mathbb{R})(j=1,2)$ such that $\theta^{(1)}\ge\theta^{(2)},\theta^{(j)}_{r}{}_{r\in[t,T]}$ is a continuous semimartingale and $(\theta^{(j)}_r)_{r\in[T,T+K]}=(\xi^{(j)}_r)_{r\in[T,T+K]}$,

$$f_1(t,y,z,\theta^{(1)}_{t+\delta(t)},\eta^{(1)}_{t+\zeta(t)})\ge f_2(t,y,z,\theta^{(2)}_{t+\delta(t)},\eta^{(2)}_{t+\zeta(t)}), \tag{4.5}$$

$$f_1(t,y,z,\theta^{(1)}_{t+\delta(t)},\frac{\mathrm{d}\langle\theta^{(1)},W\rangle_r}{\mathrm{d}r}|_{r=t+\zeta(t)})\ge f_2(t,y,z,\theta^{(2)}_{t+\delta(t)},\frac{\mathrm{d}\langle\theta^{(2)},W\rangle_r}{\mathrm{d}r}|_{r=t+\zeta(t)})\quad a.e.,a.s., \tag{4.6}$$

$$f_1(t,y,z,\xi^{(1)}_{t+\delta(t)},\frac{\mathrm{d}\langle\theta^{(1)},W\rangle_r}{\mathrm{d}r}|_{r=t+\zeta(t)})\ge f_2(t,y,z,\xi^{(2)}_{t+\delta(t)},\frac{\mathrm{d}\langle\theta^{(2)},W\rangle_r}{\mathrm{d}r}|_{r=t+\zeta(t)})\quad a.e.,a.s., \tag{4.7}$$

then $Y^{(1)}_t\ge Y^{(2)}_t, a.e., a.s.$

Proof. Consider the anticipated BSDE(1.1) one time interval by one time interval. For the first step, we consider the case when $t\in[t_1,T]$. According to Lemma 5, we can equivalently consider the following BSDE:

$$\widetilde{Y}^{(j)}_t=\xi^{(j)}_T+\int_t^T f_j(s,\widetilde{Y}^{(j)}_s,\widetilde{Z}^{(j)}_s,\xi^{(j)}_{s+\delta(s)},\eta^{(j)}_{s+\delta(s)})\mathrm{d}t-\int_t^T\widetilde{Z}^{(j)}_s\mathrm{d}W_s,$$

from which we have

$$\widetilde{Z}^{(j)}_t=\frac{\mathrm{d}\langle\widetilde{Y}^{(j)}_t,W\rangle_t}{\mathrm{d}t},\quad t\in[0,T]. \tag{4.8}$$

Noticing that $\xi^{(j)}\in S^2_{\mathcal{F}}(T,T+K;\mathbb{R})(j=1,2)$ and $\xi^{(1)}\ge\xi^{(2)}$, from (4.3) in (ii), we can get, for $s\in[t_1,T],y\in\mathbb{R},z\in\mathbb{R}^d$,

$$f_1(s,y,z,\xi^{(1)}_{s+\delta(s)},\eta^{(1)}_{s+\zeta(s)})\ge f_2(t,y,z,\xi^{(2)}_{s+\delta(s)},\eta^{(2)}_{s+\zeta(s)}).$$

According to the comparison theorem for 1-dimension BSDEs(Corollary 1), we can get

$$\widetilde{Y}^{(1)}_t\ge\widetilde{Y}^{(2)}_t,\quad a.e.,a.s.$$

Consequently,

$$Y^{(1)}_t\ge Y^{(2)}_t,\quad t\in[t_1,T],\quad a.e.,a.s. \tag{4.9}$$

For the second step, we consider the case when $t\in[t_2,t_1]$. Similarly, according to Lemma 5, we can consider the following BSDE equivalently:

$$\widetilde{Y}_t^{(j)} = Y_{t_1}^{(j)} + \int_t^{t_1} f_j(s, \widetilde{Y}_s^{(j)}, \widetilde{Z}_s^{(j)}, Y_{s+\delta(s)}^{(j)}, Z_{s+\zeta(s)}^{(j)})\mathrm{d}t - \int_t^{t_1} \widetilde{Z}_s^{(j)}\mathrm{d}W_s,$$

from which we have $\widetilde{Z}_t^{(j)} = \frac{\mathrm{d}\langle \widetilde{Y}_t^{(j)}, W\rangle_t}{\mathrm{d}t}$, for $t \in [t_2, t_1]$, Noticing (4.6) and (4.7), according to (ii), we have, for $s \in [t_2, t_1], y \in \mathrm{R}, z \in \mathrm{R}^d$,

$$f_1(s, y, z, Y_{s+\delta(s)}^{(1)}, Z_{s+\zeta(s)}^{(1)}) \geq f_2(t, y, z, Y_{s+\delta(s)}^{(2)}, Z_{s+\zeta(s)}^{(2)}).$$

Applying the comparison theorem for BSDEs again, we can finally get

$$Y_t^{(1)} \geq Y_t^{(2)}, t \in [t_2, t_1], \quad a.e., a.s.$$

Similarly to the above steps, we can give the proofs for the other cases when $t \in [t_3, t_2], [t_4, t_3], \cdots [t_N, t_{N-1}]$.

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