Necessary Condition for Cubic Planer three Connected Graph to be Non-Hamiltonian and proof of Barnette's Conjecture

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Abstract

A conjecture of Barnette's states that every three connected cubic bipartite planer graph is Hamiltonian. This problem has remained open since its formulation .This paper has a threefold purpose. The first is to provide survey of literature surrounding the conjecture. The second is to give the necessary condition for cubic planer three connected graph to be non Hamiltonian, and finally I shall prove the Barnett's conjecture. For the proof of different results using to prove the results I illustrate most of the results by using counter examples.

Keywords: cubic graph; Hamiltonian cycle; planer graph; bipartite graph; faces; sub graphs; degree of graph.

Introduction and previous work

It is not an easy task to prove the Barnette's conjecture by direct method because it is very difficult process to prove or disprove it by direct method. In this paper I use alternative method to prove the conjecture. It must be noted that if any one property of the Barnette's graph is deleted graph is non Hamiltonian.

A planer graph is an undirected graph that can be embedded into the Euclidean plane without any crossings. A planer graph is called polyhedral if and only if it is three vertex connected, that is, if there do not exists two vertices the removal of which would disconnect the rest of the graph. A graph is bipartite if its vertices can be colored with two different colors such that each edge has one end point of each color. A graph is cubic if each vertex is the end point of exactly three edges. And a graph is Hamiltonian if there exists a cycle that pass exactly once through each of its vertices. Self loops and parallel edges are not allowed in these graphs. Barnett's conjecture states that every cubic polyhedral graph is Hamiltonian. P.G. Tait in (1884) conjectured that every cubic polyhedral graph is Hamiltonian; this came to be known as Tait's conjecture. It was disproved by W.T. Tutte (1946), who constructs a counter example with 46 vertices; other researchers later found even smaller counterexamples, however, none of these counterexamples is bipartite. Tutte himself conjectured that every cubic 3-connected bipartite graph is Hamiltonian but this was shown to be false by discovery of a counterexample, the Horton graph [16] .David W. Barnett (1969) proposed a weakened combination of Tait's and Tutte's conjecture, stating that every cubic bipartite polyhedral graph is Hamiltonian this conjecture first announced in [12] and later in [3]. In [10], Tutte proved that all planar 4-connected graphs are Hamiltonian ,and in [9] thomassen extended this result by showing that every planer 4-connected graph is Hamiltonian connected, that is for any pair of vertices, there is a Hamiltonian path with those vertices as endpoints.

Support for the conjecture

In [5] Holton confirmed through a combination of clever analysis and computer search that all Barnett graphs with up to and including 64 vertices are Hamiltonian. in an announcement [14,11], McKay Used computer search to extend this result to 84 vertices this implies that if Barnett conjecture is indeed false than a minimal counterexample must contain at least 86 vertices, and is therefore considerable larger than the minimal counterexample to Tait and Tuttle conjecture. This is not all we know about a possible counterexample; another interesting result is that of Fowler, who in an unpublished manuscript [15] provided a list of sub graphs that cannot appear in any minimal counterexample to Barnett's conjecture.

Goody in [2] consider proper subsets of the Barnett graphs and proved the following.

Theorem 1: Every Barnett graph which has faces consisting exclusively of quadrilaterals, and hexagons is Hamiltonian, and further more in all such graphs, any edge that is common to both a quadrilateral and a hexagon is a part of some Hamiltonian cycle.

Theorem 2 : Every Barnett graph which has faces consisting of 7 quadrilaterals, I octagon and any number of hexagons is Hamiltonian, and any edge that is common to both a quadrilateral and an octagon is a part of some Hamiltonian cycle.

In [6] Jensen and Toft reported that Barnett conjecture is equivalent to following,

Theorem 3: Barnett conjecture is true if and only if for every Barnett graph G, it is possible to partition its vertices in to two subsets so that each induced an acyclic sub graph of G. (this theorem is not correct)

Theorem 4 [8]: The edges of any bipartite graph G can be colored with Δ colors, where Δ is the minimum degree of any vertex in G.

Theorem 5[4]: Barnett conjecture holds if and only if any arbitrary edge in a Barnett graph is a part of some Hamiltonian cycle.

Theorem 6 [13]: Barnett conjecture holds if and only if for any arbitrary face in a Barnett graph there is a Hamiltonian cycle which passes through any two arbitrary edges on that face.

Theorem 7 [7]: Barnett conjecture holds if and only if for any arbitrary face in a Barnett graph and for any arbitrary edges e_1 and e_2 on that face there is a Hamiltonian cycle which passes through e_1 and avoids e_2

It is difficult to say whether any of the technique described above will aid in settling Barnett conjecture. Certainly many of them seems to be useful and worth extending. One strategy is to keep chipping away at it; if Barnett conjecture is true then Godey's result can be extended to show that successively large and large subsets of Barnett graphs are Hamiltonian.

Grin berg's Theorem [1] is not useful to find the counter example to Barnett's conjecture because all faces in Barnett graphs have even number of sides.

New results including some counter examples and definitions Support to prove the conjecture

Definition 1: Any closed sub graph H of cubic planer three connected graph G is called complete cubic planer three connected sub graph H^C if all possible edges in that sub graph H are drawn then it also becomes cubic planer three connected graph. Thus we say H^C is cubic planer 3-connected graph.
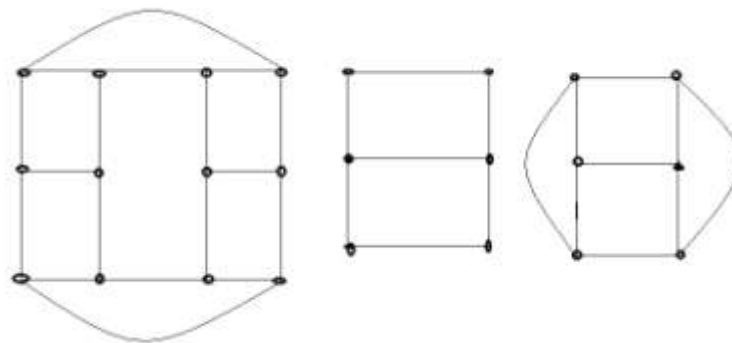
Illustrate by counter example

Let G be any cubic planer three connected graph as shown in fig 1 we take its subgraph H shown in fig 2 then we draw all possible edges in the subgraph as shown in fig 3 the subgraph graph becomes compleate cubic planer three connected H^C subgraph

Fig 1   **Fig 2**   **Fig 3**

Definition 2: Any closed sub graph H of cubic planer three connected graph G is called complete planer n-1 cubic three connected sub graph and is denoted by H^{C+} if all possible edges in that sub graph H are drawn then it becomes planer n-1 cubic three connected graph. i.e. Only one vertex has degree two and remaining graph is cubic planer three connected.

Illustrate by counter example.

Let G be any cubic planer three connected graph as shown in fig 4 H be its sub graph as shown in fig 5 we draw all possible edges in the sub graph as shown in fig 6 but still there exist a vertex having degree two only thus we say the sub graph H^{C+} be its complete planar n-1 cubic three connected graph.

Fig 4   **Fig 5**   **Fig6**
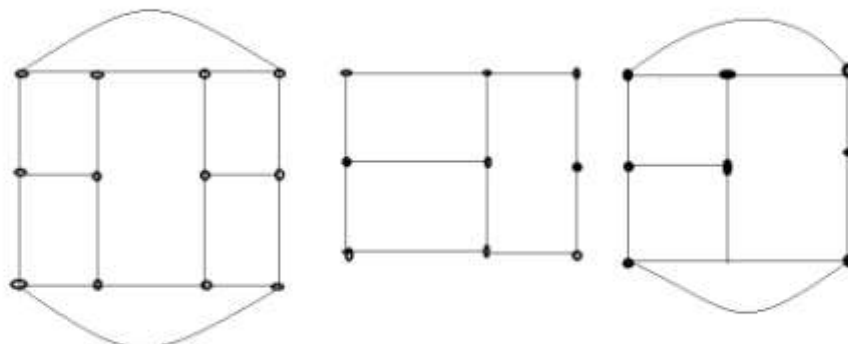
Remark 1

A vertex can not have degree one in closed cubic planer three connected sub graphs , then it should be pendent vertex which is not possible in closed graphs so the degree of remaining vertices is two and degree connot be more than three because it is the subgraph of cubic planer three connected graph so only possabulity is that degree of remaining vertex is two.

Defnition 3: A closed sub graph H of cubic planer three connected graph is called compleate planer n-r cubic and three connected if all possible edges in that subgraph H are draw then it becomes cubic planer three connected ,but it is still planer n-r cubic and three connected, I,e its "r" vertices have degree two and remaining all vertices are cubic and three connected. It can be represented by H^{Cr+}

Lemma 1

A planer bipatite three connected and n-3 cubic is non hamiltonion.In other words a planer graph which is bipatite three connected and n-3 cubic I,e only three of its vertices are of degree four and remaning graph is cubic then such a graph is non hamiltonion. (Only encircle vertices is of degree four and rest of the graph is cubic and three connected)

I shall prove this result by counter example. The main aim behind the result is to prove that if a single property is deleted in cubic planer three connected bipartite graphs then it is non- Hamiltonian. This graph can be divided in to three closed sub graphs and an isolated vertex such that these closed sub graphs are H^{C+} sub graphs. Later I use this result in the main theorem.
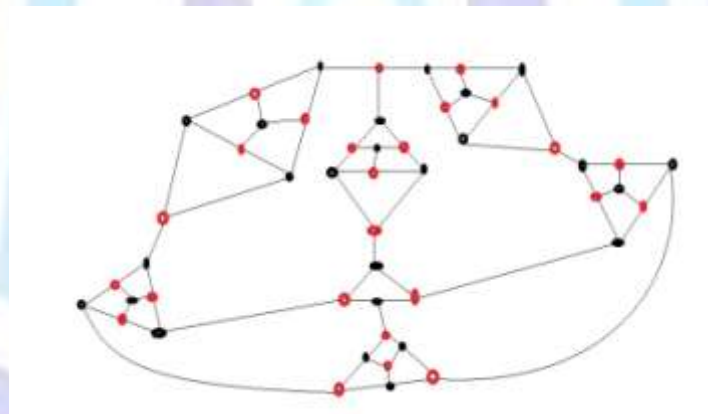
Fig. 7
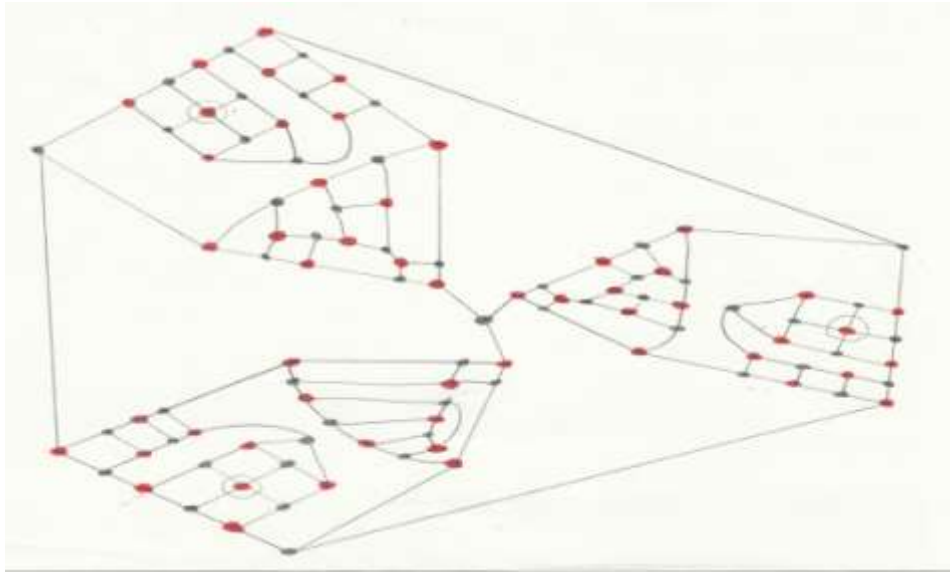
Lemma 2

A cubic planer bipartite 2-connected graph is non Hamiltonian.

It can be seen in this example. (it is not possible for me to give number of counter examples even though i can construct number of such examples) fig 8 below is the cubic planar bipartite 2-connected graph but non Hamiltonian

Remark 2

In every cubic planer three connected bipartite graph if any one of the property is deleted then the graph is non Hamiltonian

Remark 3

Let [ ] represents greatest integer function

1)   If a and b are any two positive integers then
      [a + b] = [a] + [b]

2)   If a is any positive integer and b is any positive real number then
      [a + b] = [a] + [b]

Remark 4

Let G be any graph and if G is cubic planar three connected, we know that every cubic planer three connected graph, the Degree of each vertex is exactly equal to three. Thus the sum of all the degree of the Graph is 3n that is

$$\sum_{i=1}^{n} d_i = 3n$$

Since each edge contributes two to the degrees thus the number of edges in the graph is

$$E = \frac{\sum_i^n d_i}{2}$$

$$E = \frac{3n}{2}$$

Where n is the number of vertices of the graph. Thus we conclude that if number of nodes is n number of edges is $\frac{3n}{2}$

and if number of edges is $\frac{3n}{2}$ the number of nodes is $\frac{2}{3}E = \frac{2}{3}\frac{3n}{2} = n$ thus we conclude that in any cubic planer

three connected graph edges and nodes are connected by certain relation.

Number of edges of any cubic planer three connected graph is always divisible by three if we take any planer cubic three connected graph and number of edges is not divisible by three then given graph is not H^C it is planer n-1 cubic and three connected I.e. it contain a vertex of degree two only such a graph is denoted by H^{C+}. There does not exist any two vertices of degree two because we can draw an edge between them.

Lemma 3

The number of regions in every cubic three connected planer graph and every cubic planer bipartite three connected graph of n vertices is where n is the number of vertices of the graph.

$$\frac{n+4}{2}.$$

Proof

Since in every cubic planer three connected graph and every cubic planer three connected bipartite graph the degree of each vertices is exactly equal to three as graph is cubic. Thus the sum of all the degree of the Graph is 3n that is

$$\sum_{i=1}^n d_i = 3n$$

Since each edge contributes two to the degrees thus the number of edges in these graph is

$$E = \frac{\sum_i^n di}{2}$$

$E = \frac{3n}{2}$ where n is the number of vertices of these graph.

Thus we conclude that if number of vertices is n number of edges is $\frac{3n}{2}$ and if number of edges is $\frac{3n}{2}$ the number of

vertices is $\frac{2}{3}e = \frac{2}{3}\frac{3n}{2} = n$ i.e. number of vertices and edges are connected by certain relation. We know by Euler's

theorem on planer graphs the number of regions is equal to

$$r = e - v + 2 \rightarrow I$$

Since we have a graph of n vertices as we know it is cubic planer three connected or cubic planer three connected

bipartite graph the number of edges in such graph's is $\frac{3n}{2}$ as shown above now substitute these values in equation I we

get.

$$r = e - n + 2$$

$$r = \frac{3n}{2} - n + 2$$

$$r = \frac{3n - 2n}{2} + 2$$

$$r = \frac{n}{2} + 2$$

$$r = \frac{n+4}{2} \text{ that proves the result.}$$

Thus from the above result we conclude that in every cubic planer three connected and every cubic planer bipartite three connected graph it is true that

$$e - v + 2 = \frac{n+4}{2}$$

The above result is not true for other planer graphs as we can take a counter example of three connected bipartite planar graph known as Herschel graph which contain 11 vertices and 18 edges. Contain 9 regions Does not satisfy the above result.

Note.

In every cubic planer three connected graph G and every cubic planer bipartite three connected graph G^+

a: Order of graphs G and G^+ is even.

b: Number of regions in both the graphs G and G^+ are equal to $\frac{n+4}{2}$ where n is the total number of vertices (see lemma 3)

c: Edges and vertices in both the graphs are connected by certain relation i.e. $E = \frac{3n}{2}$ and $V = \frac{2E}{3}$

d: In G odd cycles are allowed but in G^+ it is bipartite thus odd cycles are not allowed.

Necessary condition for a cubic planer 3-connected graph to be non Hamiltonian

Theorem A

Statement: - A cubic planer 3-connected graph is non Hamiltonian if the graph is divided into three closed sub graphs of any order and an arbitrary isolated vertex such that these three closed sub graphs are planer n-1 cubic three connected I.e. they are H^{C+} sub graphs in other words a planer 3-connected graph is non Hamiltonian if these three sub graphs are such that

$$\left[\frac{3n}{2} \right] \neq 0 (\bmod 3) \qquad \qquad [\ ] \text{ represents greatest integer function}$$

where $\frac{3n}{2}$ is the number of edges in these sub graphs (remark 4 above)

Proof

Let G be any cubic planer 3-connected graph of order n number of edges is $\frac{3n}{2}$

Let us suppose that all the three closed sub graphs of G are complete closed planer cubic 3-connected, i.e. H^C sub graphs then

$$\left[\frac{3n}{2} \right] \equiv o (\bmod 3)$$

Since odd cycles are allowed so we can take any closed sub graph of any order in such a way that these closed sub graphs are necessarily H^{C+} first of all we shall take order of all closed sub graphs is odd if these closed sub graphs are H^{C+} then we have to stop the process of searching as such sub graphs exist but if such closed sub graphs are not H^{C+} then we try for different orders

Let order of closed sub graph be odd i.e. n is odd say n = 2m + 1 or n = 2m - 1

$$\left[\frac{3n}{2} \right] \equiv 0 (\bmod 3)$$

$$\left\lceil \frac{3(2m+1)}{2} \right\rceil \equiv 0(\bmod 3)$$

$$\left\lceil \frac{6m+3}{2} \right\rceil \equiv (0\,\bmod 3)$$

Since in graphs number of vertices and edges represent positive integers so

$$\left\lceil \frac{6m}{2} \right\rceil + \left\lceil \frac{3}{2} \right\rceil \equiv 0(\bmod 3)$$

$$3m+1 \equiv 0(m0d3)$$

$$\frac{3}{3m+1} \quad \text{and} \quad \frac{3}{-3m}$$

$$\frac{3}{3m+1-3m}$$

$$\frac{3}{1} \qquad \text{Which is contradiction similarly if } n = 2m - 1$$

We get

$$\frac{3}{3m-1-3m}$$

$$\frac{3}{-1}$$

This again gives contradiction

Thus we conclude that $\left\lceil \dfrac{3n}{2} \right\rceil \neq 0(\bmod 3)$ (Since such sub graphs exists we shell stop our search)

Thus there exists one vertex in all the three closed sub graphs having degree 2only (the degree cannot be one are more than three discussed above remark 4) that is these three sub graphs are H^{C+} sub graphs. If two vertices are of degree two we can draw an edge between them and sub graph becomes H^C only that is these sub graphs are cubic planer three connected which is not possible. When graph satisfy these conditions we first of all delete all those edge which we have added in the sub graphs then we join these sub graphs together with the arbitrary isolated vertex (it must be noted that such graphs does not contain only one arbitrary vertex it may contain more than one arbitrary vertex) with those vertices of the sub graphs having degree 2 in such a way that graph becomes cubic planer three connected. since odd cycles are allowed when we start from any arbitrary vertex it is not possible to travel all the vertices once and reaches back at the stating vertex because an arbitrary vertex can be traveled only at once so we can travel at most two of these H^{C+} sub graphs which we have joined to make the graph cubic planer three connected thus the graph so obtained is non Hamiltonian.

Now I shall illustrate the result with following graphs and prove that this condition is satisfied by these graphs. All these graphs are cubic planer three connected and non Hamiltonian satisfy the above conditions.fig 9 t0 26 below

1) First of all let us take Tuttle graph, in which we take an encircle vertex as an isolated vertex and divide the remaining graph in three closed sub graphs not necessary of same order we shall show that these closed sub graphs are H^{C+} sub graphs.
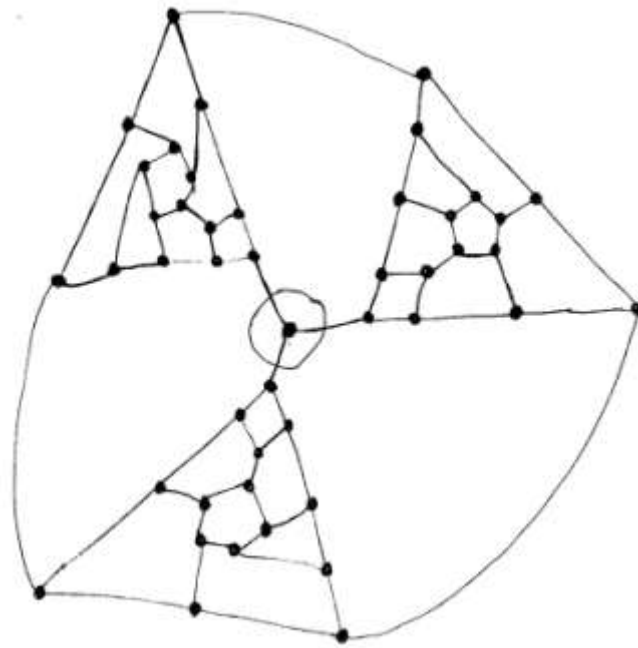
Fig 9

G
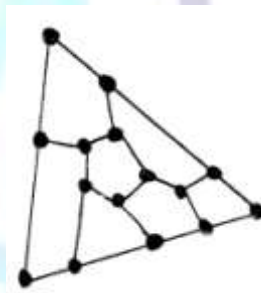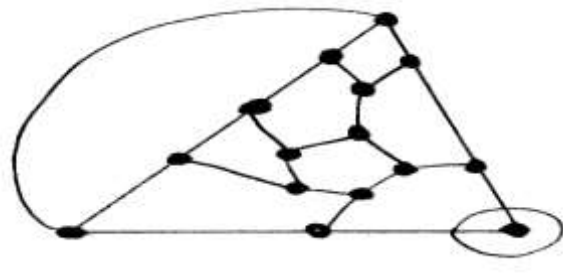
Let H be its sub graph shown below

Fig 10   H

again if we draw all possible edges in this closed subgraph the subgraph becomes planer n-1 cubic and three connected I,e H^{C+} subgrap as shown below and a vertex having degree two only has been shown by encircling the vertex.

H^{C+}

Fig 11

Since all the three closed sub graphs of this graph are of same order so other two sub graphs have same property as discussed above.

2)   Now let us take younger graph of 44 vertices which is cubic planer three connected non Hamiltonian. And isolated vertex is shown by encircle it, and remaining graph is divided into three closed sub graphs not necessarily of same order all these closed sub graphs are H^{C+}
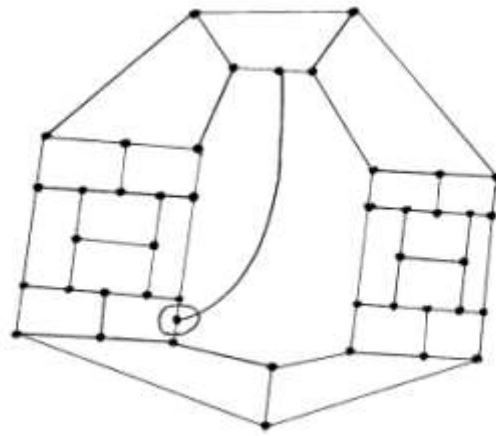
Fig 12   G
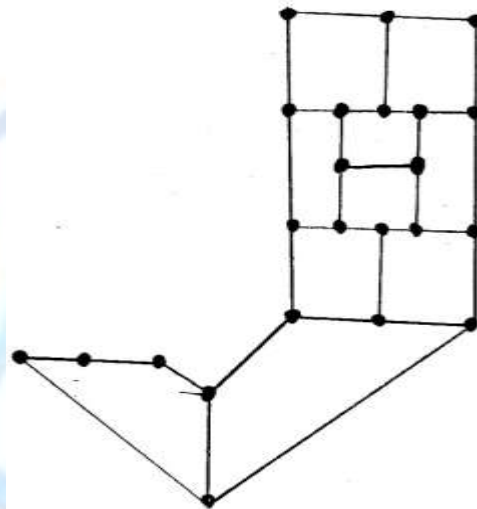
Let H₁ be its one closed sub graph as   **Fig 13**

If we draw all possible edges in this sub graph it becomes planer n-1 cubic three connected as shown below H^{C+} only encircle vertex is of degree two.
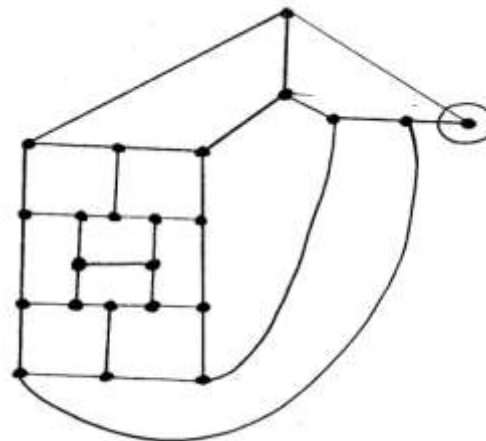
Fig 14 H^{C+}

Let another sub graph H₂ of the graph is given below

Fig 15

H_2
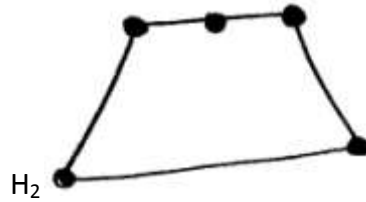
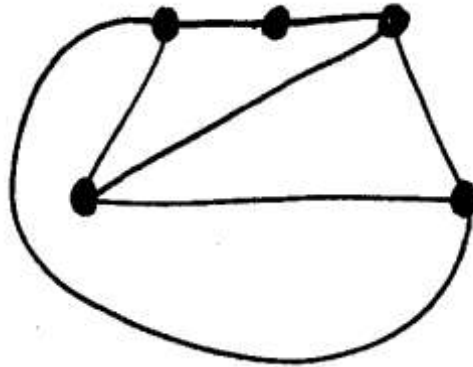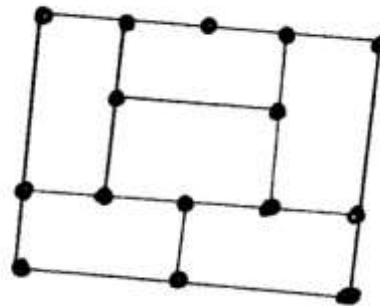If we draw all possible edges in the subgraph it also becomes H^{C+} sub graph as shown below.
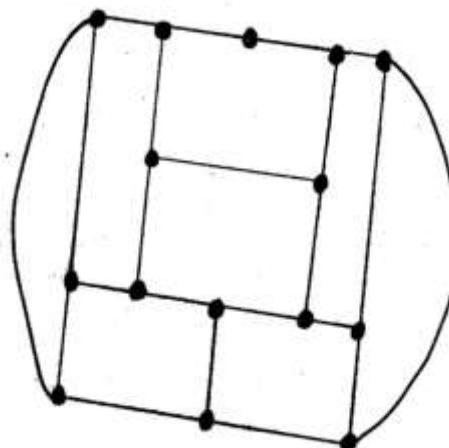

Fig 16 H^{C+}

Let another sub graph H_3 of a graph is given as **Fig 17**   H

If we draw all possible edges in the sub graph it becomes H^{C+} sub graph as shown below
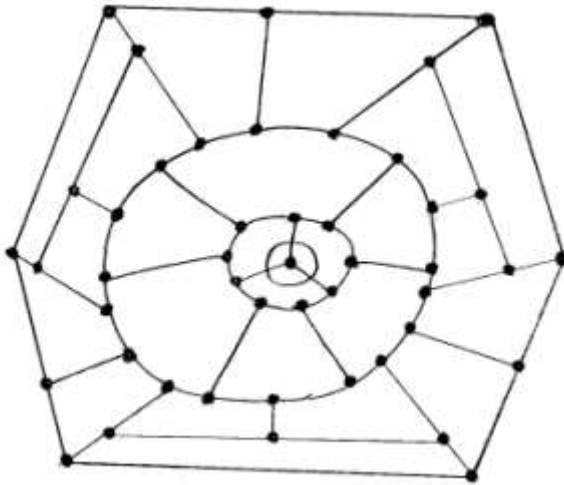

Fig 18 H^{C+}

3)   Now let us take another example of cubic planer three connected non hamiltonian graph known as Grinberg graph 0f 46 vertices as shown below in which encircle vertex is an arbiterly vertex.
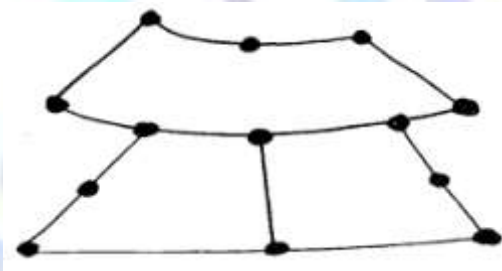
4)

Fig 19   G

Let us take its closed subgraph H$_1$ as shown below

Fig 20   H

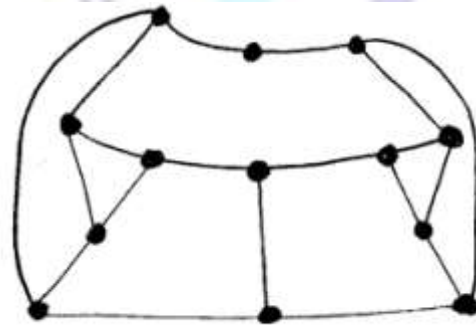Now if we draw all possible edges in the sub graph it becomes H^{C+} sub graph as shown below

Fig 21   H^{C+}

Let us take another sub graph H$_2$ of the graph given below.
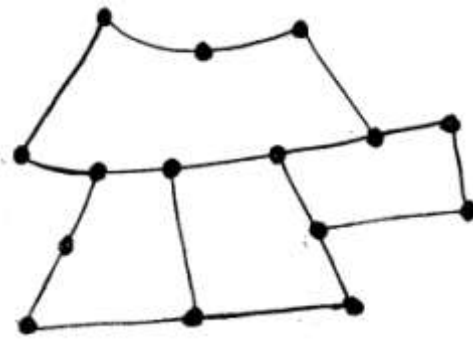
Fig 22 H_2

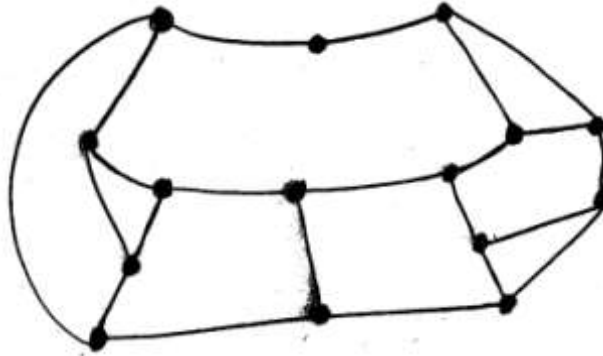If we draw all possible edges in the graph it becomes planer n-1 cubic and three connected as shown below

Fig 23 H^{C+}

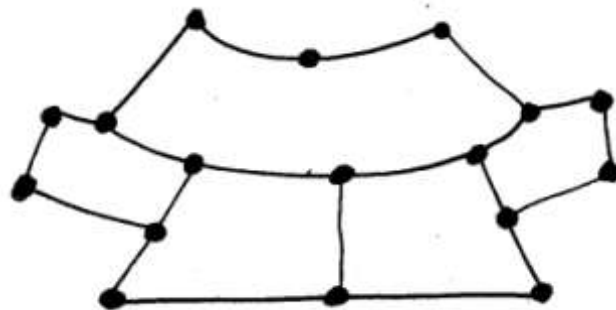Now again if we take another closed sub graph H_3 as below

Fig 24 H_3

If we again draw all possible edges in the closed sub graph it becomes H^{C+} sub graph as shown below
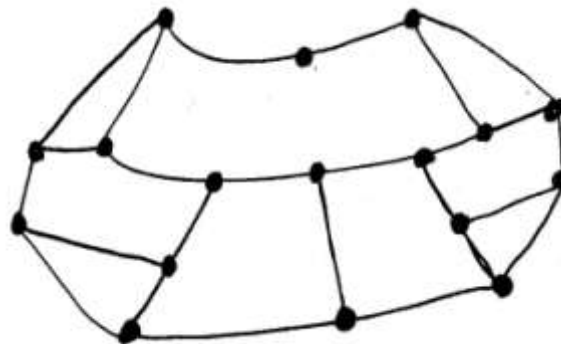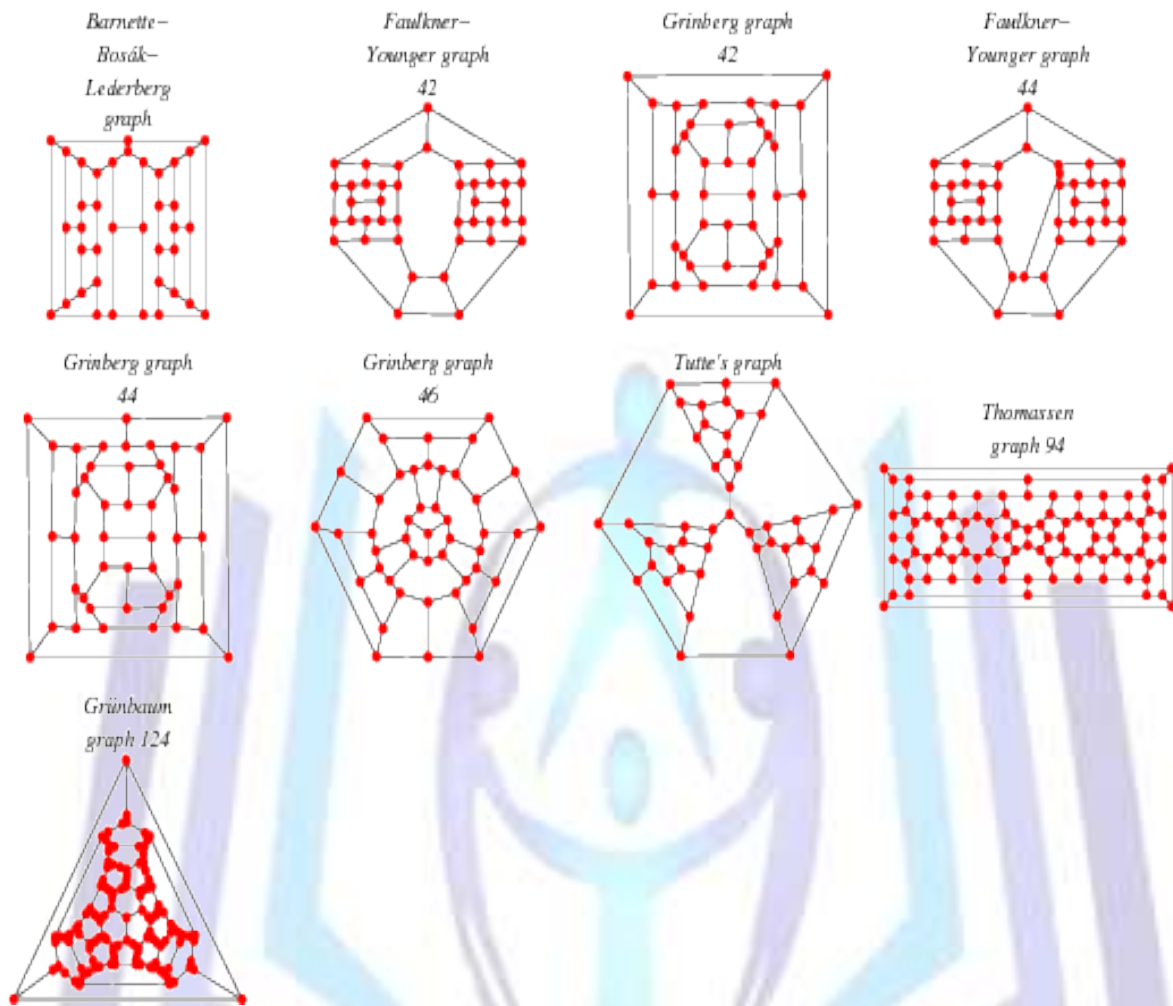

Fig 25 H^{C+}

Now all other planer cubic three connected non Hamiltonian graphs satisfy this condition these graphs are shown below. In fig   26

one of the most important thing regarding the cubic planer three connected non-Hamiltonian graphs which was proved by professor LINFAN MAO and professor YANPEI LIU in 2001 [17] there exists infinite three connected non Hamiltonian cubic maps on every surface (orient able or non-orient able) not only the above graphs but also these infinite graphs satisfy the condition which I have proved above.

Now I shall show that above result is sharp. I use counter example to prove this sharpness.

 Below example fig 27 is a graph which is cubic planer three connected contain Hamiltonian cycle start from V_1, V_2, V_3, V_4 -   -   - V_{14}, V_1 in this graph if we take V_4 as arbitrary vertex all the three closed sub graphs are not H^{C+} as shown below, it is not necessary we take V_4 as arbitrary vertex we can take any vertex as arbitrary vertex in such a way that remaining graph is divided into three closed sub graphs of any order but all such closed sub graphs are H^{C+} which is not possible in this graph.

Thus we conclude that all cubic planer three connected non Hamiltonian graphs can be divided in to three closed sub graphs of any order and an isolated vertex satisfying the property that all the three closed sub graphs are H^{C+}, But if graph is cubic planer three connected and Hamiltonian it is not necessary that all the three closed sub graphs satisfy H^{C+} property as shown below, thus the condition which I use to prove the theorem is sharp. In other words every cubic planer three connected graph which is Hamiltonian and can be divided into three closed sub graphs of any order and an isolated vertex all the three sub graphs may are may not be H^{C+} sub graphs, but if graph is non Hamiltonian all such closed sub graphs are H^{C+} (There are other examples as well but it is not possible to draw all in this paper)
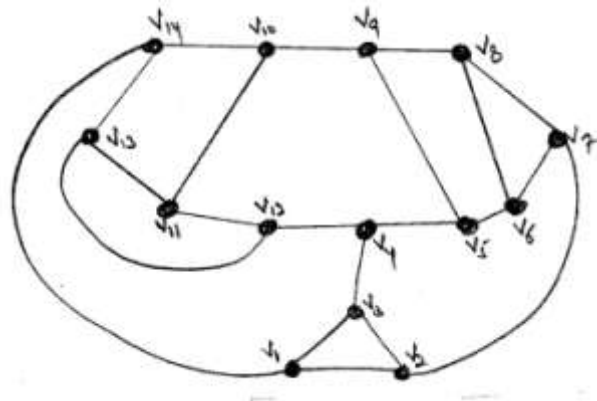
Fig 27

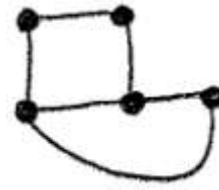Take a closed subgraph H and its H^{C+} subgraph
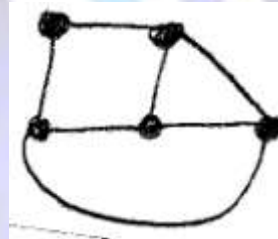
Fig 28 H

Fig 29 H^{C+}

Take another closed subgraph H and its H^{C+} subgraph

Fig 30 H

Fig 31 H^{C+}

And finally take a closed sub graph H of order three so it is not H^{C+} because we cannot draw any more edge in this sub graph (these edges are parallel edges)

Fig 32   H

Remark 5

It has been discussed above that number of regions in cubic planer three connected graphs and cubic planer three connected bipartite graphs are $\dfrac{n+4}{2}$ thus it is necessary that every cubic planer three connected bipartite graph is non Hamiltonian if it has at least one closed sub graph which is H^{C+} also in lemma 1 I have given a counter example of n-3 cubic planer three connected bipartite non Hamiltonian graph satisfy H^{C+} property.

Theorem

Statement: - every cubic planer bipartite three connected graph is Hamiltonian (Barnett's conjecture)

Proof:-

Since every bipartite graph is two colorable and thus without odd cycles so it contains only even cycles and number of vertices is also even, we cannot take any closed sub graph of odd order because it is not connected as odd cycles are not allowed, so every closed sub graph of cubic planer bipartite three connected graph is of even length. Thus in this type of graph n is always even, such graphs are non Hamiltonian only if there exist at least one sub graph H of any order which is planer n-1 cubic and three connected I.e. H^{C+} sub graph, Then conjecture is not true because counter example can be constructed to disprove it if such a condition is satisfied. (See theorem A above) and fig 7 of lemma 1

Let it is true that such graphs have at least one closed sub graph of H^{C+} then it must satisfy the following condition

$$\left[\frac{3n}{2}\right] \neq 0 \pmod 3$$

Since n is necessarily even i.e. order of every sub graph is even because graph is bipartite and odd cycles are not allowed.

Let n = 2m

$$\frac{3n}{2} = \frac{3(2m)}{2} = 3m$$

$$\left[\frac{3n}{2}\right] \equiv 0 \pmod 3$$

$$3m \equiv 0 \pmod 3$$

$$\Rightarrow \quad 3\big/ 3m \quad \text{and} \quad 3\big/ -3m$$

$$\Rightarrow \quad 3\big/ 3m - 3m$$

$$\Rightarrow \quad 3\big/ 0$$

Which is true this contradicts the given statement that

$$\left[\frac{3n}{2}\right] \neq 0 \pmod 3$$

Thus we conclude that there does not exist any sub graph H of cubic planer bipartite three connected graph G which is planer n-1 cubic and three connected I.e. which is H^{C+} thus there does not exists any counter example which proves that Barnett's conjecture does not hold, thus every cubic planer bipartite three connected graph is Hamiltonian that proves the conjecture.

The above theorem can be verified by lemma 1 above the non Hamiltonian graph G of lemma 1 can be divided into an arbitrary vertex and three closed sub graphs H^{C+} even though graph is bipartite(without odd cycles) three connected planer n-3 cubic, only three vertices are of degree four and contain only even cycles.

Below is the graph which is cubic planer three connected contains Hamiltonian cycle. This cannot be divided into an arbitrary vertex and three closed sub graphs H^{C+}
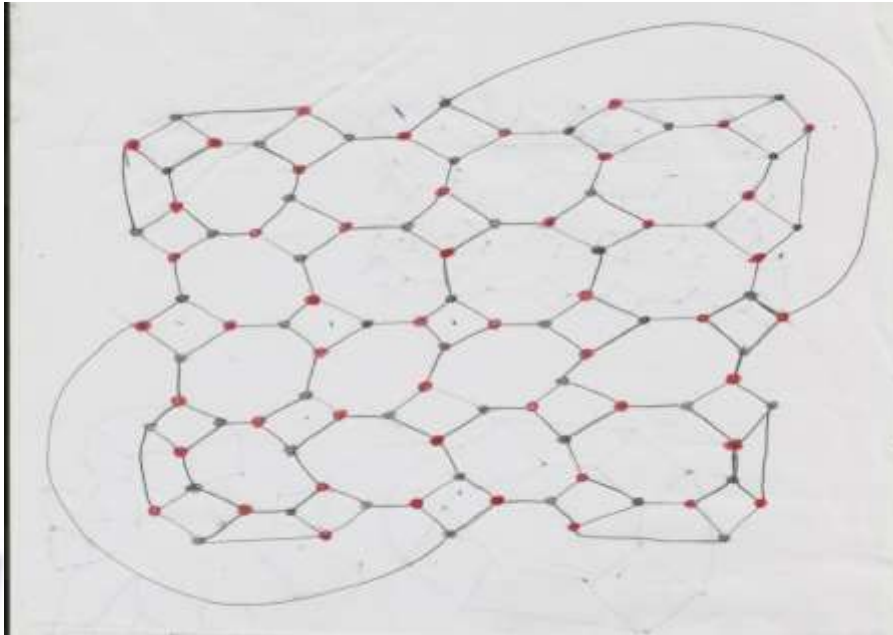

Fig 33

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