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Approximation to the Mean and Variance of the Modified Moments Estimator of the Shape Parameter of Weibull Distribution

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Abstract

In this paper we consider the Weibull distribution of two parameters , since has been widely used as a model in many areas of applications. Properties of the distribution are introduced .Estimation of the distribution parameters is obtained by the modified moments method

The estimator of the shape parameter has so complicated form and it is difficult to find the properties of estimator so approximation to the mean and variance of the estimator is made theoretically by utilizing Taylor series expansion up to second order derivative.

Keyword: Weibull distribution; modified moments estimators; Approximation to the mean and variance of estimator .

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1 Introduction

The Weibull distn. has been widely used as a model in many areas of applications, specifically in the studies of failure components and as a model for product life. It has also been used as the distn. of strength of certain materials. It is named after the Swedish scientist Weibull who first proposed the distn. in connection with his studies on strength of materials [1]. One reason for its popularity is that it has a great variety of shapes, which make it extremely flexible in fitting many kinds of empirical data. Kao [2] used it as a model for vacuum tube failure [3], Mann gave a variety of situations in which the distn. is used for other types of failure data, Whitmore and Altschalerf used it in studies on the time interval to the occurrence of tumors in human population. Bain and Antle [4] used a Maximum Likelihood method to obtain two simple estimators of parameters for Weibull distn. . Ishioka and Nonaka [5] presented a stable technique for obtaining the maximum Likelihood estimate of Weibull parameters of the life distn.'s of two components that form a series system. Al-Ali [6] studied some estimators of parameters and reliability function for Weibull distn. and suggested four methods to estimate the shape parameter when the scale parameter is known..

Definition (1.1) [7]:

A continuous r.v. X is said to have a Weibull distn. with parameters α and θ, denoted by $X \sim W(\alpha,\theta)$, if X has the following p.d.f.

$$f(x;\alpha,\theta) = \alpha\theta x^{\alpha-1} e^{-\theta x^{\alpha}}, 0 < x < \infty \qquad (1)$$
$$= 0 \qquad , e.w. \quad ; where\ \alpha,\theta > 0.$$

We note that the Weibull distn. reduces to the Exponential distn. as a special case when $\alpha = 1$, and it reduce to Rayleigh distn. when $\alpha = 2$, and similar to Normal curve when $(3 \le \alpha \le 4)$.

The Weibull distn. depends on two parameters α and θ which are called shape and scale parameters respectively. The variety of p.d.f. shapes can be generated by fixing the values of α once and letting θ vary and fixing the values of θ and letting α vary. The professional MATLAB computer software is used to give a graphical representation of Weibull p.d.f.'s. Figure (1) and Figure (2) show respectively some Weibull p.d.f.'s for fixed θ with α varying and for fixed α with θ varying as follows:

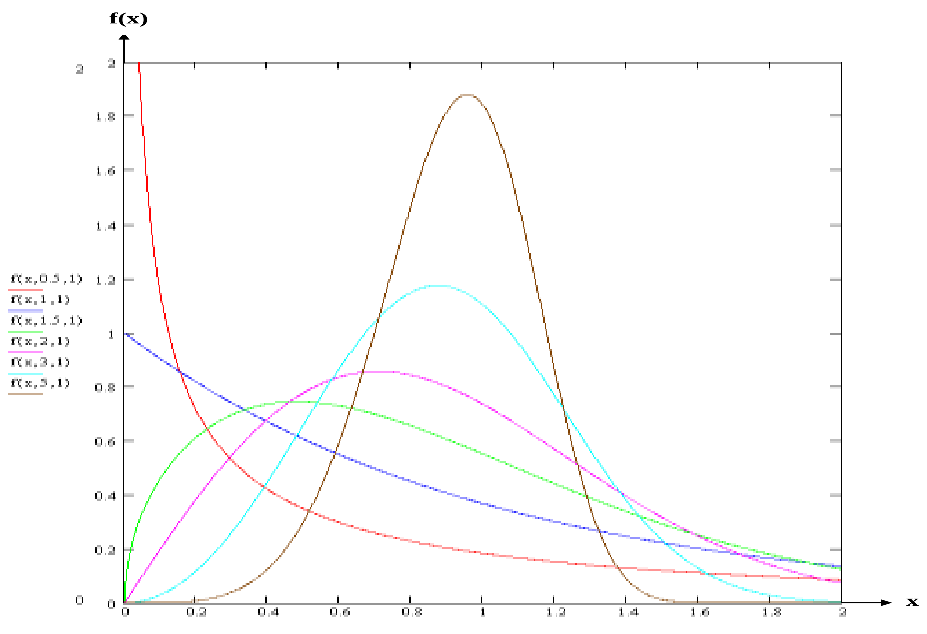

Figure (1.1)
Weibull p.d.f.'s with θ=1 and α=0.5,1,1.5,2,3,5

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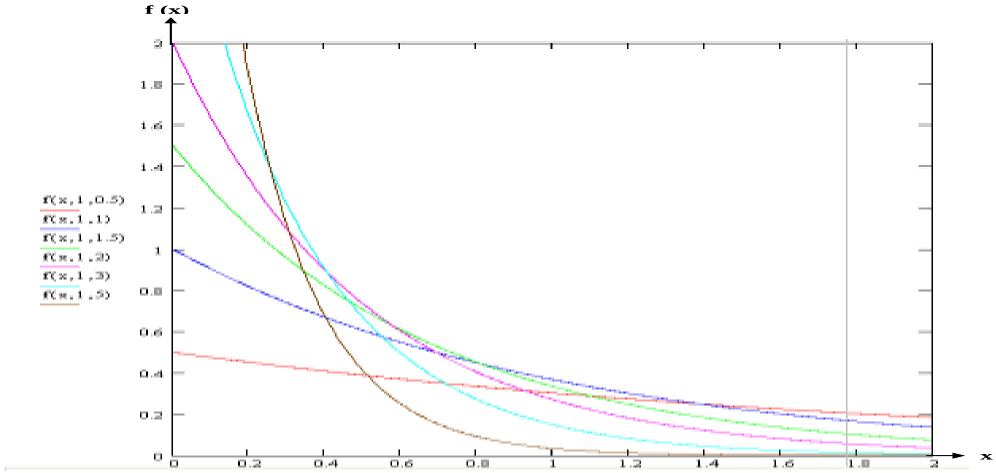

Figure (1.2)
Weibull p.d.f.'s with $\alpha=1$ and $\theta=0.5,1,1.5,2,3,5$

In general we note that the Weibull distn. have the following properties :

1- Have the x-axis as a horizontal asymptote.

2- Increasing for $0<x<\left(\frac{\alpha-1}{\alpha\theta}\right)^{\frac{1}{\alpha}}$ and decreasing for $\left(\frac{\alpha-1}{\alpha\theta}\right)^{\frac{1}{\alpha}}<x<\infty$.

3- Have a maximum point at $x=\left(\frac{\alpha-1}{\alpha\theta}\right)^{\frac{1}{\alpha}}$.

4- Have two inflection points at $x=\left(\frac{3\alpha+\sqrt{5\alpha^2-6\alpha+1}-3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}$ and $x=\left(\frac{3\alpha-\sqrt{5\alpha^2-6\alpha+1}-3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}$.

5- Concave up for $0<x<\left(\frac{3\alpha-\sqrt{5\alpha^2-6\alpha+1}-3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}$ and $\left(\frac{3\alpha+\sqrt{5\alpha^2-6\alpha+1}-3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}<x<\infty$ and Concave down for $\left(\frac{3\alpha-\sqrt{5\alpha^2-6\alpha+1}-3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}<x<\left(\frac{3\alpha+\sqrt{5\alpha^2-6\alpha+1}-3}{2\alpha\theta}\right)^{\frac{1}{\alpha}}$.

1.2.1 The Cumulative Distribution Function

The Weibull c.d.f. is defined as

$$F(x)=\int_{-\infty}^{x} f(t;\alpha,\theta)\,dt=\int_{0}^{x}\alpha\theta t^{\alpha-1}e^{-\theta t^{\alpha}}\,dt \quad \text{implies} \quad F(x)=\Pr(X\le x)=\begin{cases}0, & x<0\\ 1-e^{-\theta x^{\alpha}}, & 0<x<\infty\\ 1, & x\to\infty\end{cases}$$

(2)

1.4.1 Mean and Variance,[7]:

The mean and variance are respectively obtained from eq.(1.5) by setting $r = 1, 2$.

(i) Mean:

$E(X) = \mu = \mu'_1$ is called the mean of r.v. X (or distn.). It is a measure of central tendency.

$$\mu = \frac{1}{\theta^{\frac{1}{\alpha}}}\Gamma(1+\frac{1}{\alpha}) \qquad (3)$$

***(ii)* Variance:**

$Var(X) = \delta^2 = E\left[(X-\mu)^2\right] = E(X^2) - \mu^2$ is called the variance of r.v. X (or distn.). It is a measure of dispersion, where $\mu'_2 = E(X^2) = \frac{1}{\theta^{\frac{2}{\alpha}}}\Gamma(1+\frac{2}{\alpha})$

Hence, $Var(X) = \delta^2 = \frac{1}{\theta^{\frac{2}{\alpha}}}\Gamma(1+\frac{2}{\alpha}) - \left\{\frac{1}{\theta^{\frac{1}{\alpha}}}\Gamma(1+\frac{1}{\alpha})\right\}^2$

And so

$$var(X) = \frac{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)}{\theta^{\frac{2}{\alpha}}} \qquad (4)$$

1.4.2 Other Moments:

(i) Mode:

A mode of a distn. is the value x of r.v. X that maximize the p.d.f. $f(x)$.

For continuous distn.'s the mode x is a solution of $\frac{df(x)}{dx} = 0$ and $\frac{d^2f(x)}{dx^2} < 0$.

The mode is measure of location. Also we note that the mode may not exist or we may have more than one mode.

For Weibull case with p.d.f.

$$f(x) = \alpha\theta x^{\alpha-1}e^{-\theta x^{\alpha}}$$

$$\Rightarrow \quad \frac{df(x)}{dx} = e^{-\theta x^{\alpha}}\left[-\left(\alpha\theta x^{\alpha-1}\right)^2 + \alpha\theta(\alpha-1)x^{\alpha-2}\right] \qquad (5)$$

Equating eq.(5) to zero, and solving for x, we have $-\alpha\theta x^{\alpha-1} + \alpha - 1 = 0$ which implies the critical point is

$$x = (\frac{\alpha-1}{\alpha\theta})^{\frac{1}{\alpha}} \qquad (6)$$

This critical point satisfy that x is the distn. mode where condition

$\frac{d^2 f(x)}{dx^2} < 0$ at $x = (\frac{\alpha - 1}{\alpha\theta})^{\frac{1}{\alpha}}$ is hold .

(ii) Median:

A median of a distn. is defined to the value x of r.v. X such that $F(x) = \Pr(X \le x) = \frac{1}{2}$. The median is measure of location.

For Weibull case,

We equate the c.d.f. given by eq.(2) to $\frac{1}{2}$, that is

$$\frac{1}{2} = 1 - e^{-\theta x^{\alpha}} \tag{7}$$

Solving for x in eq.(7) lead to the median

$$x = \left(\frac{\ln 2}{\theta}\right)^{\frac{1}{\alpha}} \tag{8}$$

1.5 Estimation of Parameters for Weibull Distribution:

1.5.3.3 Estimation of Parameters by Modified Moments Method [7] :

Let X_1, X_2, …, X_n be a r.s of size n from a distn. p.d.f. $f(x, \underset{\sim}{\theta})$ where $\underset{\sim}{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ is a vector of k unknown parameters.Let $Y_1, Y_2, ..., Y_n$ represent the arrangement of the sample set $\{X_i\}$ in a ascending order of magnitude.Let $\mu_r' = E(X^r)$ be the r^{th} sample moment about the origin, r=1,2, … .

In this method we equate $\mu_r' = E(X^r)$ with $r = 1$ and ranking $E(Y_i) = Y_i$ beginning with $i = 1$ until $i = k$ this process will gives k eq.'s to provide a unique solution for $\theta_i, i = 1, 2, ..., k$ say $\hat{\theta}_i, i = 1, 2, ..., k$ and the obtained $\hat{\theta}_i$, this method is called modified moment estimator.

For Weibull case:

we have two unknown parameters α and θ and if we take a r.s. of size n from $W(\alpha, \theta)$, we let Y_1 represent the first order statistic of the sample.

From the order statistic theory the p.d.f. of Y_1 is

$$g_1(y_1) = n(1 - F(y_1))^{n-1} f(y_1)$$

$$\Rightarrow g_1(y_1) = n\alpha\theta y_1^{\alpha-1} e^{-n\theta y_1^{\alpha}}, 0 < y_1 < \infty$$
$$= 0 \qquad , e.w. \qquad ; \alpha, \theta > 0$$

This shown that $Y_1 \sim W(\alpha, n\theta)$.

Accordingly, $E\left(Y_1\right)=\frac{1}{(n\theta)^{\frac{1}{\alpha}}}\Gamma(1+\frac{1}{\alpha})$.

Now, we apply the Modified Moment Method by setting $\mu_1'=\bar{X}$ and $E\left(Y_1\right)=Y_1$ at $\alpha=\hat{\alpha}$, $\theta=\hat{\theta}$ which leads to

$$\left(\frac{1}{\hat{\theta}}\right)^{\frac{1}{\hat{\alpha}}}\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)=\bar{X} \tag{9}$$

$$\left(\frac{1}{n\hat{\theta}}\right)^{\frac{1}{\hat{\alpha}}}\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)=Y_1 \tag{10}$$

From eq.'s (9) and (10), the estimators of α and θ are respectively

$$\hat{\alpha}=\frac{\ln\left(\frac{1}{n}\right)}{\ln\left(\frac{Y_1}{\bar{X}}\right)} \tag{11}$$

$$\hat{\theta}=\left(\frac{\left(1-\frac{1}{n^{\frac{1}{\hat{\alpha}}}}\right)\Gamma(1+\frac{1}{\hat{\alpha}})}{\bar{X}-Y_1}\right)^{\hat{\alpha}} \tag{12}$$

It is difficult to find the distribution of $\hat{\alpha}$ so we shall approximate E($\hat{\alpha}$) by considering the mean and variance of the Taylor series expansion of the function g (X,Y) at point (μ_X,μ_Y) up to second order .

Remark

The first order statistics $Y_1 \sim W(\alpha, n\theta)$

Proof:

We have the i^{th} order statistic is given by [7]

$$g(y_i)=\frac{n!}{(i-1)!\,(n-i)!}[F(y_i)]^{i-1}[1-F(y_i)]^{n-i}f(y_i) \qquad -\infty \le y_i \le \infty$$

Where $i=1,2,3,\ldots,n$

In particular for $i=1$ we have the p.d.f. of the first order statistics which is

$g(y_1)=n[1-F(y_1)]^{n-1}f(y_1)$

The c.d.f. of the first order statistics is given by

$G(y_1)=1-[1-F(y_1)]$

Using eq(1) and eq(2) we have

$$g(y_1)=n\alpha\theta y_1^{\alpha-1}e^{n\theta y_1^{\alpha}} \quad 0 \le y_1 \le \infty$$
$$=0 \qquad , e.w.$$

and hence $Y_1 \sim W(\alpha, n\theta)$

so $G(y_1) = \begin{cases} 0 & 0 < y_1 \\ 1 - e^{n\theta y_1^{\alpha}} & 0 < y_1 < \infty \\ 1 & y_1 \to \infty \end{cases}$

Remark:

1- Since the first order statistics $Y_1 \sim W(\alpha, n\theta)$ then the mean and variance of Y_1 is given by

$$\mu_y = \frac{1}{n\theta^{\frac{1}{\alpha}}}\Gamma(1+\frac{1}{\alpha}) \quad (13)$$

$$Var(Y_1) = \frac{1}{(n\theta)^{\frac{2}{\alpha}}}\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\frac{1}{(n\alpha)^{\frac{1}{\alpha}}}\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2$$

so

$$Var(Y_1) = \frac{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n^{\frac{2}{\alpha}}\theta^{\frac{2}{\alpha}}} \quad (14)$$

2- The limiting distribution of Y_1 is 0 and so Y_1 converge stochastically to 0.

Approximation to the mean and variance of obtained by modified moment method

We shall approximate the mean and variance of the estimator $\hat{\alpha}$ of the parameter α that obtained by using the modified moments method by considering the mean and variance of the Taylor series expansion of the function g (X,Y) at point (μ_X, μ_Y) up to second order which is given by, [8]:

$$E[g(X,Y)] = g(\mu_X,\mu_Y) + \frac{1}{2}Var[X]\frac{\partial^2}{\partial x^2}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}} + \frac{1}{2}Var[Y]\frac{\partial^2}{\partial Y^2}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}} + Cov[X,Y]\left[\frac{\partial^2}{\partial X\,\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right] \ldots\ldots(23)$$

$$Var[g(X,Y)] = Var[X]\left[\frac{\partial}{\partial x}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]^2 + Var[Y]\left[\frac{\partial}{\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]^2 + 2Cov[X,Y]\left[\frac{\partial}{\partial x}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right]\left[\frac{\partial}{\partial Y}g(X,Y)\Big|_{\substack{\mu_X\\ \mu_Y}}\right] \ldots\ldots(24)$$

By setting $X = \overline{X}$ *and* $Y = Y_1$ in eq.(23) and eq.(24), one can get

$$E[g(\overline{X},Y_1)] \cong g\left(\mu_{\overline{X}},\mu_{Y_1}\right) + \frac{1}{2}Var[\overline{X}]\frac{\partial^2}{\partial \overline{X}^2}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}} + \frac{1}{2}Var[Y_1]\frac{\partial^2}{\partial {Y_1}^2}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}} + Cov[\overline{X},Y_1]\left[\frac{\partial^2}{\partial \overline{X}\,\partial Y_1}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right] \quad (25)$$

and

$$Var[g(\overline{X},Y_1)] = Var[\overline{X}]\left[\frac{\partial}{\partial \overline{X}}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]^2 + Var[Y_1]\left[\frac{\partial}{\partial Y_1}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]^2 + 2Cov[\overline{X},Y_1]\left[\frac{\partial}{\partial \overline{X}}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right]\left[\frac{\partial}{\partial Y_1}g(\overline{X},Y_1)\Big|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}\right] \quad (26)$$

We have the modified method estimator $\hat{\alpha}$ of the parameter α given by eq.(11) which is:

$$\hat{\alpha} = \frac{\ln\left(\frac{1}{n}\right)}{\ln\left(\frac{Y_1}{\overline{X}}\right)}$$

Let $g(\bar{X}, Y_1) = \hat{\alpha} = \frac{\ln\left(\frac{1}{n}\right)}{\ln\left(\frac{Y_1}{\bar{X}}\right)} = \frac{-\ln n}{\ln Y_1 - \ln \bar{X}}$

It is known that

$$\mu_{\overline{X}} = \mu = \frac{1}{\theta^{\frac{1}{\alpha}}}\Gamma(1+\frac{1}{\alpha}) \quad (15)$$

and

$$Var[\overline{X}] = \frac{\sigma^2}{n} = \frac{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n\theta^{\frac{2}{\alpha}}} \quad (16)$$

From eq.(13) and eq.(15) one can have:

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$$g\left(\mu_{\overline{X}},\mu_{Y_1}\right)=\frac{-\ln n}{\ln\left[\frac{1}{(n\theta)^{\frac{1}{\alpha}}}\Gamma\left(1+\frac{1}{\alpha}\right)\right]-\ln\left[\frac{1}{\theta^{\frac{1}{\alpha}}}\Gamma\left(1+\frac{1}{\alpha}\right)\right]}=\frac{\ln\frac{1}{n}}{\ln\left[\frac{\frac{1}{(n\theta)^{\frac{1}{\alpha}}}\Gamma\left(1+\frac{1}{\alpha}\right)}{\frac{1}{\theta^{\frac{1}{\alpha}}}\Gamma\left(1+\frac{1}{\alpha}\right)}\right]}=\frac{\ln\frac{1}{n}}{\frac{1}{\alpha}\ln\frac{1}{n}}=\alpha \quad (17)$$

On the other hand

$$\frac{\partial g\left(\overline{X},Y_1\right)}{\partial\overline{X}}=\frac{\frac{-\ln n}{\bar{X}}}{[\ln Y_1-\ln\bar{X}]^{2}}$$

$$\left.\frac{\partial g\left(\overline{X},Y_1\right)}{\partial\overline{X}}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{\frac{-\ln n}{\frac{1}{\theta^{\frac{1}{\alpha}}}\Gamma(1+\frac{1}{\alpha})}}{\frac{1}{\alpha^2}(\ln n)^2}=\frac{\alpha^2\theta^{\frac{1}{\alpha}}}{\ln n\,\Gamma(1+\frac{1}{\alpha})} \quad (18)$$

$$\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial\overline{X}^2}=\frac{\frac{\ln n}{\bar{X}^2}[\ln Y_1-\ln\bar{X}]^{2}-2\frac{\ln n}{\bar{X}^2}[\ln Y_1-\ln\bar{X}]}{[\ln Y_1-\ln\bar{X}]^{4}}=\frac{\frac{\ln n}{\bar{X}^2}[\ln Y_1-\ln\bar{X}-2]}{[\ln Y_1-\ln\bar{X}]^3}$$

$$\left.\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial\overline{X}^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{\frac{\ln n\,\theta^{\frac{2}{\alpha}}}{\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}\left[\ln\left[(n\theta)^{\frac{-1}{\alpha}}\Gamma\left(1+\frac{1}{\alpha}\right)\right]-\ln\left[\theta^{\frac{-1}{\alpha}}\Gamma\left(1+\frac{1}{\alpha}\right)\right]-2\right]}{\left[\ln\left[(n\theta)^{\frac{-1}{\alpha}}\Gamma\left(1+\frac{1}{\alpha}\right)\right]-\ln\left[\theta^{\frac{-1}{\alpha}}\Gamma\left(1+\frac{1}{\alpha}\right)\right]\right]^{3}}$$

And hence

$$\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial\overline{X}^2}=\frac{\frac{\alpha^3\theta^{\frac{2}{\alpha}}}{\Gamma\left(1+\frac{1}{\alpha}\right)}\left[\frac{1}{\alpha}+\frac{2}{\ln n\,\Gamma\left(1+\frac{1}{\alpha}\right)}\right]}{\ln n}$$
(19)

$$=\frac{\alpha^3\theta^{\frac{2}{\alpha}}\left[\frac{1}{\alpha}+\frac{2}{\ln n\,\Gamma\left(1+\frac{1}{\alpha}\right)}\right]}{\Gamma\left(1+\frac{1}{\alpha}\right)\ln n}$$

$$\frac{\partial g\left(\overline{X},Y_1\right)}{\partial Y_1}=\frac{\frac{\ln n}{Y_1}}{(\ln y_1-\ln\bar{X})^{2}}$$

$$\left.\frac{\partial g\left(\overline{X},Y_1\right)}{\partial Y_1}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{n^{\frac{1}{\alpha}}\theta^{\frac{1}{\alpha}}\alpha^2}{\ln n\,\Gamma\left(1+\frac{1}{\alpha}\right)}$$

$$\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial Y_1^2}=\frac{\frac{-\ln n}{y_1^2}(\ln Y_1-\ln\bar{X}+2)}{(\ln Y_1-\ln\bar{X})^3}$$

$$\left.\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial Y_1^2}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{n^{\frac{2}{\alpha}}\theta^{\frac{2}{\alpha}}\alpha^3\left(\frac{2}{\ln n}-\frac{1}{\alpha}\right)}{\ln n\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}$$

and

$$\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial\bar{x}\partial Y_1}=\frac{\frac{2\ln n}{\bar{X}Y_1}}{(\ln Y_1-\ln\bar{X})^3}$$

$$\left.\frac{\partial^2 g\left(\overline{X},Y_1\right)}{\partial\overline{X}\partial Y_1}\right|_{\substack{\mu_{\overline{X}}\\ \mu_{Y_1}}}=\frac{-2n^{\frac{1}{\alpha}}\theta^{\frac{2}{\alpha}}\alpha^3}{(\ln n)^2\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}$$

In general $\overline{X}$ converge stochastically to $\mu_{\overline{X}}$ and we prove that Y_1 converge stochastically to 0 . So $\overline{X}Y_1$ converge stochastically to 0 , [23], therefore

$E\left[\overline{X}Y_1\right]\cong 0$

and hence

$$\mathrm{Cov}[\bar{X}, Y_1] \cong E[\bar{X}Y_1] - E[\bar{X}]E[Y_1] = -\frac{\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n^{\frac{1}{\alpha}}\theta^{\frac{2}{\alpha}}}$$

By substituting eq.^{,s}(22), (28),(29),(31), (33), (34) and (36) in eq.(25), one can get the approximation to the expected value of $\hat{\alpha}$ which is given by:

$$E[\hat{\alpha}] \cong E\left[\frac{-\ln n}{\ln Y_1 - \ln \bar{X}}\right] = \alpha + \frac{1}{2}\left[\frac{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n\theta^{\frac{2}{\alpha}}}\right]\left[\frac{\alpha^3\theta^{\frac{2}{\alpha}}\left[\frac{1}{\alpha} + \frac{2}{\ln n\,\Gamma\left(1+\frac{1}{\alpha}\right)}\right]}{\Gamma\left(1+\frac{1}{\alpha}\right)\ln n}\right] +$$

$$\frac{1}{2}\left[\frac{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n^{\frac{2}{\alpha}}\theta^{\frac{2}{\alpha}}}\right]\left[\frac{n^{\frac{2}{\alpha}}\theta^{\frac{2}{\alpha}}\alpha^3\left(\frac{2}{\ln n} - \frac{1}{\alpha}\right)}{\ln n\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}\right] + \left[-\frac{\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n^{\frac{1}{\alpha}}\theta^{\frac{2}{\alpha}}}\right]\left[\frac{-2n^{\frac{1}{\alpha}}\theta^{\frac{2}{\alpha}}\alpha^3}{(\ln n)^2\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}\right]$$

Moreover

$$\lim_{n\to\infty} E(\hat{\alpha}) = \alpha.$$

This shows that $\hat{\alpha}$ is asymptotically unbiased estimator of α.

By substituting eq.^{,s}(28), (30), (22), (310) and (36) in eq.(26), one can obtain the approximation to variance of $\hat{\alpha}$ which is given by:

$$\mathrm{Var}[\hat{\alpha}] \cong \mathrm{Var}\left[\frac{-\ln n}{\ln Y_1 - \ln \bar{X}}\right]$$

$$= \left[\frac{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n\theta^{\frac{2}{\alpha}}}\right]\left[\frac{\alpha^2\theta^{\frac{1}{\alpha}}}{\ln n\ \Gamma(1+\frac{1}{\alpha})}\right]^2 + \left[\frac{\Gamma\left(1+\frac{2}{\alpha}\right) - \left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n^{\frac{2}{\alpha}}\theta^{\frac{2}{\alpha}}}\right]\left[\frac{n^{\frac{1}{\alpha}}\theta^{\frac{1}{\alpha}}\alpha^2}{\ln n\ \Gamma\left(1+\frac{1}{\alpha}\right)}\right]^2 +$$

$$2\left[-\frac{\left[\Gamma\left(1+\frac{1}{\alpha}\right)\right]^2}{n^{\frac{1}{\alpha}}\theta^{\frac{2}{\alpha}}}\right]\left[\frac{\alpha^2\theta^{\frac{1}{\alpha}}}{\ln n\ \Gamma(1+\frac{1}{\alpha})}\right]\left[\frac{n^{\frac{1}{\alpha}}\theta^{\frac{1}{\alpha}}\alpha^2}{\ln n\ \Gamma\left(1+\frac{1}{\alpha}\right)}\right]$$

and hence

$$\mathrm{Var}[\hat{\alpha}] = \frac{\alpha^4}{n\,(\ln n)^2}\left[\frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\left\{\Gamma\left(1+\frac{1}{\alpha}\right)\right\}^2} - 1\right] + \frac{\alpha^4}{(\ln n)^2}\left[\frac{\Gamma\left(1+\frac{2}{\alpha}\right)}{\left\{\Gamma\left(1+\frac{1}{\alpha}\right)\right\}^2} - 1\right] - \frac{2\alpha^4}{(\ln n)^2}$$

Conclusions

We can conclude from our study the following

1-The approximated mean and variance become more accurate if the higher order of approximation is used.

2-The estimator of the shape parameter is asymptotically unbiased estimator.

3-The estimator of the shape parameter has small variance even for small sample size.

4-This technique can be use for general form for the distn. with three parameters.

References

[1] Weibull W. (1939), “A Statistical Theory of Strength of Materials”, Ingenious Vetenskaps Academies Hand linger, No.151, Generalstabens Li Tografiska, Furiag, and Stockholm, Sweden.

[2] Mann N.R., “Point and Interval Estimation Procedures For The Two－Parameter Weibull and Extreme Value Distribution”, Technometrics, Vol.10, (1968), pp. 231－256.

[3] Kao J.H., “A Graphical Estimation of Mixed Weibull Parameters in Life Testing of Electron Tubes”, Technometrics, Vol.1, (1959), pp. 384－407.

[4] Lee J. B. and Charles E.A., “Estimation of Parameters in The Weibull Distribution”, Technometrics, Vol. 9, No.4, (Nov.,1967), pp. 621－627.

[5] Tsurenori I. (Ricoh Co. Ltd. Tokyo) and Yasuo N. (Science University of Tokyo, Tokyo), “Maximum likelihood Estimation of Weibull parameters For Two Independent Competing Risks”, IEEE Transaction on Reliability, Vol.40, No.1, (April 1991).

[6] Al－Ali A.R. (1991), “Some Estimators of Primary Test for Parameters and Reliability Function of Weibull Distribution”, M.Sc. Thesis, College of Science, Al－Mustansiriyah University, Baghdad.

[7] Mood A.M., Graybill F.A. and Boess D.C., “Introduction to The Theory of Statistics”, Third Edition, McGraw－Hill, London (1974) .

[8] Levy H., "Approximating Expected Utility by a Function of Mean and Variance", American Economic Review, (1979), 69:308–317.