ISSN 2347-1921

Some Universal Constructions For I- Fuzzy Topological Spaces II

E. H. Hamouda
Department of Basic Sciences, Industrial education college, Beni-Suef University, Egypt
ehamouda70@gmail.com

ABSTRACT

Hamouda E. H. [J. of Advances in Math. Vol 6, **2**(2014), 973-941] has introduced some universal constructions in the category **FTOP**, whose objects are the I-fuzzy topological spaces (X, μ, F) where X is an ordinary set, μ is a fuzzy set in X and F is a family of fuzzy sets in X satisfying some axioms. In this paper we introduce the dual universal constructions, namely, fuzzy co-products, fuzzy co-equalizers and fuzzy pushouts for I-fuzzy topological spaces. Also we discuss some results concerning all such universal objects.

Keywords: Fuzzy sets; I -fuzzy topological spaces; Fuzzy co-equalizers; Fuzzy pushouts; Fuzzy co-products.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics

Vol 7, No. 2
editor@cirworld.com
www.cirworld.com, member.cirworld.com

ISSN 2347-1921

1. INTRODUCTION

Zadeh [15] introduced the notion of a fuzzy set as a function from the given set to the unit interval. The first categorical definition of fuzzy sets was introduced by J. A. Goguen [5]. In the case of fuzzy topology, there are various interesting categories of fuzzy topological spaces. The collection of all fuzzy topological spaces and fuzzy continuous functions form a category. Since C. Chang, R. Lowen and J. Goguen have defined fuzzy topology in different ways, each of them defines a different category of fuzzy topological spaces [11,13]. Geetha S. [4,5,6] introduced a new category **FTOP**, the objects are I - fuzzy topological spaces (X, μ, F) where X is an ordinary set, μ is a fuzzy set in X and F is a family of fuzzy sets in X satisfying some axioms. Some applications of category theory in fuzzy topology are presented in [2, 8, 14]. Behera [2] introduced the concepts of fuzzy equalizers , fuzzy pullbacks and their duals for fuzzy topological spaces in the sense of Chang. Hamouda E. H. [8] introduced some universal objects for I - fuzzy topological spaces (X, μ, F). In this paper the dual concepts, namely, fuzzy co-products, fuzzy co-equalizers and fuzzy pushouts for I-fuzzy topological spaces are investigated. Also we discuss some results concerning all such universal objects.

2. PRELIMINARIES

As usual I denotes the closed unit interval [0,1]. A fuzzy set A in a set X is a function on X into the closed unit interval [0, 1] of the real line. The fuzzy sets in X taking on respectively the constant values 0 and 1 are denoted by 0_X and 1_X respectively. For two fuzzy sets A, B in X, we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$. For a collection of fuzzy sets $\{A_i : i \in J\}$, the union $C = \bigcup_{i\in J} A_i$ and the intersection $D = \bigcap_{i\in J} A_i$ are defined by

$$C(x) = \bigvee_{i\in J} A_i(x), \quad \text{for all } x \in X,$$

$$D(x) = \bigwedge_{i\in J} A_i(x), \quad \text{for all } x \in X.$$

If $f: X \to Y$ is a function, and A, B are fuzzy sets in X, Y respectively, then the fuzzy set $f^{-1}(B)$ in X is defined by $f^{-1}(B) = B \circ f$, and $f(A): Y \to I$ is defined as follows [3]:

$$f(A)(y) = \begin{cases} \bigvee \{A(x) : x \in f^{-1}(y)\}, & if\ f^{-1}(y) \neq \emptyset; \\ 0 & if\ f^{-1}(y) = \emptyset. \end{cases}$$

Definition 2.1 [5] *Let X be a set , $\mu: X \to I$ be a fuzzy set in X and F be a family of fuzzy sets in X satisfying the following conditions:*

(1) $A \in F$ implies that $A(x) \leq \mu(x)$ for all $x \in X$,

(2) If $A_i \in F$, $i \in J$, then $\bigcup_{i\in J} A_i \in F$,

(3) If $A, B \in F$, then $A \cap B \in F$,

(4) $0_X, \mu \in F$.

The triple (X, μ, F) is called an $I -$ fuzzy topological space or I - fts. The members of F are called $I -$ fuzzy open sets and their complements are called $I -$ fuzzy closed sets.

Remark 2.2 when $= 1_X$, an *$I -$ fuzzy topological space is nothing but a fuzzy topology in the sense of Chang[3].*

Definition 2.3 [4] Let (X_1, μ_1, F_1) and (X_2, μ_2, F_2) be two *$I -$ fuzzy topological spaces. A function $f: (X_1, \mu_1, F_1) \to (X_2, \mu_2, F_2)$ is fuzzy continuous if:*

i. $\mu_1(x) \leq \mu_2(f(x))$, $\forall\, x \in X$,

ii. $\mu_1 \cap f^{-1}(U) \in F_1$, $\forall\, U \in F_2$.

The notion **FTOP** will denote the category of I − fuzzy topological spaces and fuzzy continuous functions. We shall use the categorical terminology of [1]. For more information about the category **FTOP**, the reader could consult [4].

3. UNIVERSAL CONSTRUCTIONS IN FTOP

In this section we discuss fuzzy co-products, fuzzy co-equalizers and fuzzy pushouts for $I -$ fuzzy topological spaces. By remark 2.2, some results in [2] are considered as a special case of the results below. The word "map" will always mean a continuous function, but the word "function" does not imply continuity.

The concept of fuzzy co-product has introduced in [4, 7,10,12].The following theorem emphasizes the universal property of fuzzy co-product in **FTOP**. From now on, J is referred to as the index set and the word "fuzzy spaces" means $I -$ fuzzy topological spaces.

Theorem 3.1 *For a given fuzzy spaces (X_i, μ_i, F_i), $i \in J$, the following hold:*

(1) There exists a fuzzy space (S,μ,F) and fuzzy maps $f_i:(X_i,\mu_i,F_i)\to(S,\mu,F)$ for each $i\in J$.

(2) For any fuzzy space (X,γ,H) with fuzzy maps $\varphi_i:(X_i,\mu_i,F_i)\to(X,\gamma,H)$, there is a unique fuzzy map $\theta:(S,\mu,F)\to(X,\gamma,H)$ such that $\theta\circ f_i=\varphi_i$ for each $i\in J$.

Proof. (1) For the given fuzzy spaces (X_i,μ_i,F_i) , we consider the disjoint union $\coprod_{i\in J}(X_i,\mu_i,F_i)$ to be the fuzzy space (S,μ,F), where $S=\coprod_{i\in J}X_i=\bigcup_{i\in J}(X_i\times\{i\})$ is the disjoint union of ordinary sets X_i with inclusion maps $f_i:X_i\to S$, defined by $f_i(x)=(x,i)$, μ is a fuzzy set in S defined by $\mu(x,i)=\mu_i(x)$ for each $i\in J$ and $\mathcal{U}_i=\{U:S\to I|f_i^{-1}(U)\cap\mu_i\in F_i, i\in J\}$. It is easily seen that $\mathcal{U}_i$ is the finest fuzzy topology making f_i fuzzy continuous and the intersection $F=\bigcap_{i\in J}\mathcal{U}_i$ is the finest fuzzy topology making all the functions f_i fuzzy continuous [10]. In equivalent words, $U\in\mathcal{U}_i$ if and only if $f_i^{-1}(U)\cap\mu_i\in F_i, i\in J$.

(2) Define $\theta:(S,\mu,F)\to(X,\gamma,H)$ by $\theta(x,i)=\varphi_i(x)$ for all $x\in X_i, i\in J$. With the definition of θ, we have $\theta\circ f_i=\varphi_i$. For θ to be fuzzy continuous, $\theta^{-1}(B)\cap\mu$ must belong to F for each $B\in H$ and $\mu(x,i)\le\gamma(\theta(x,i))$. First, since φ_i is a fuzzy map for each $i\in J$, then $\gamma\big(\theta(x,i)\big)=\gamma\big(\varphi_i(x)\big)\ge\mu_i(x)=\mu(x,i)$ for each $x\in X_i$. Let B be a fuzzy open set belonging to H and $x\in X_i$, then

$$\begin{aligned}(f_i^{-1}(\theta^{-1}(B)\cap\mu)\cap\mu_i)(x)&=(f_i^{-1}(\theta^{-1}(B)\cap\mu)(x)\wedge\mu_i(x)\\&=(\theta^{-1}(B)\cap\mu)(f_i(x))\wedge\mu_i(x)\\&=(\theta^{-1}(B)\cap\mu)(x,i)\wedge\mu_i(x)\\&=B\big(\theta(x,i)\big)\wedge\mu(x,i)\wedge\mu_i(x)\\&=B(\varphi_i)(x)\wedge\mu_i(x)\\&=({\varphi_i}^{-1}(B)\cap\mu_i)(x).\end{aligned}$$

Hence, $f_i^{-1}(\theta^{-1}(B)\cap\mu)\cap\mu_i={\varphi_i}^{-1}(B)\cap\mu_i$ belongs to F_i for each $i\in J$. Therefore, $\theta^{-1}(B)\cap\mu\in F$, proving the fuzzy continuity of θ .This verifies the existence of the universal property. The uniqueness of θ is an immediate consequence of the definition . □

Proposition 3.2 *Let $(X_i,\mu_i,F_i), i\in J$, be a collection of fuzzy spaces, and give $\coprod_{i\in J}X_i$ the fuzzy co-product topology. Then the fuzzy co- product is unique up to fuzzy homeomorphism.*

Proof. Let $S=Q=\coprod_{i\in J}X_i$ with fuzzy maps $f_i:(X_i,\mu_i,F_i)\to(S,\mu,F)$ and $g_i:(X_i,\mu_i,F_i)\to(Q,\gamma,H)$ respectively. Then the universal property of the fuzzy co-product S implies that there is a unique fuzzy map θ: $:(S,\mu,F)\to(Q,\gamma,H)$ such that $\theta\circ f_i=g_i$ for each $i\in J$. In similar way, there exist a unique fuzzy map $\varphi:(Q,\gamma,H)\to(S,\mu,F)$ such that $\varphi\circ g_i=f_i$ for each i ∈ J. Thus $f_i=\varphi\circ g_i=\varphi\circ\theta\circ f_i=id_S\circ f_i$ for each i ∈ J. Form the uniqueness condition of theorem 3.1, it follows that $\varphi\circ\theta=id_S$. Similarly, we have $\theta\circ\varphi=id_Q$. Therefore, S and Q are fuzzy homeomorphic [3]. □

For the sake of simplicity, we shall use the symbol (μ_X,F_X) for the $I-$ fuzzy topological space (X,μ,F). The following theorem states the definition [1] and proves the existence of fuzzy co-equalizers in **FTOP**.

Theorem 3.3 Let $f,g:(\mu_X,F_X)\to(\mu_Y,F_Y)$ be fuzzy maps, then

(1) There exists a fuzzy space (μ_E,F_E) and a fuzzy map $q:(\mu_Y,F_Y)\to(\mu_E,F_E)$ such that $q\circ f=q\circ g$.

(2) For any fuzzy space (μ_A,F_A) with a fuzzy map $\varphi:(\mu_Y,F_Y)\to(\mu_A,F_A)$ satisfying $\varphi\circ f=\varphi\circ g$, there exists a unique fuzzy map $h:(\mu_E,F_E)\to(\mu_A,F_A)$ such that $\varphi=h\circ q$.

Proof. (1) Define Q to be $\{\big(f(x),g(x)\big):x\in X\}\subseteq Y\times Y$. It is known that E need not be an equivalence relation on Y. Let R be the smallest equivalence relation on Y containing Q, that is the intersection of all equivalence relations on Y containing Q. Let $E=Y/R$ be the usual quotient set and $q:Y\to E, y\mapsto[y]$, be the usual quotient map. Now we define the quotient fuzzy topology on E as follows: define μ_E to be the image of μ_Y under q , $q(\mu_E)$, and $F_E=\{V:E\to I|q^{-1}(V)\cap\mu_Y\in F_Y\}$. Then (μ_E,F_E) is the desired fuzzy space, and $q:(\mu_Y,F_Y)\to(\mu_E,F_E)$ is a fuzzy map[4]. Since for any $x\in X$, $[f(x)]=[g(x)]$, it follows that $q\big(f(x)\big)=[f(x)]=[g(x)]=q\big(g(x)\big)$, i.e., $q\circ f=q\circ g$.

(2) We then have to verify the universal property. For any fuzzy space (μ_A,F_A), we define $h:(\mu_E,F_E)\to(\mu_A,F_A)$ by $h([y])=\varphi(y), y\in Y$, and this implies that $\varphi=h\circ q$. We must show that this is well-defined. Given $[y_1]=[y_2]$ in E, for $y_1,y_2\in Y$, then $q(y_1)=q(y_2)$ such that $(y_1,y_2)\in R$. Define the relation $R_\varphi=\{(y,z)\in Y\times Y:\varphi(y)=\varphi(z)\}$ on Y. It is an easy matter to check that R_φ is an equivalence relation on Y, moreover, the relation Q is a subset of R_φ, being $\varphi\big(f(x)\big)=\varphi(g(x)$ for every $x\in X$ and this implies that $R\subseteq R_\varphi$. Thus, $(y_1,y_2)\in R\subseteq R_\varphi$ and we have $\varphi(y_1)=\varphi(y_2)$. It follows that h is well-defined. Now to complete the existence of h , we have to show that h is a fuzzy map.

Since $\varphi = h \circ q$ is a fuzzy map, then the fuzzy continuity of h comes directly from the well known result: h is fuzzy continuous if and only if $h \circ q$ is fuzzy continuous[4]. The uniqueness of h comes directly from the definition. □

We now define ***fuzzy pushouts***. Suppose we have fuzzy maps $f: (\mu_X, F_X) \to (\mu_Y, F_Y)$ and $g: (\mu_X, F_X) \to (\mu_Z, F_Z)$. Then the fuzzy pushout of these fuzzy maps is a fuzzy space (μ_D, F_D) together with fuzzy maps $\alpha: (\mu_Y, F_Y) \to (\mu_D, F_D)$ and $\beta: (\mu_Z, F_Z) \to (\mu_D, F_D)$ such that $\alpha \circ f = \beta \circ g$, and such that the following *universal property* holds: Suppose that (μ_C, F_C) is a fuzzy space and that $\acute{\alpha}: (\mu_Y, F_Y) \to (\mu_C, F_C)$ and $\acute{\beta}: (\mu_Z, F_Z) \to (\mu_C, F_C)$ are fuzzy maps with $\acute{\alpha} \circ f = \acute{\beta} \circ g$. Then there is a unique fuzzy map $\varphi: (\mu_D, F_D) \to (\mu_C, F_C)$ with $\varphi \circ \alpha = \acute{\alpha}$ and $\varphi \circ \beta = \acute{\beta}$. We then call (μ_D, F_D) a fuzzy pullback of f and g.

In the following theorem, we show that fuzzy pushouts exist in the category **FTOP** by constructing them as fuzzy co-products.

Theorem 3.4 Let $f: (\mu_X, F_X) \to (\mu_Y, F_Y)$ and $g: (\mu_X, F_X) \to (\mu_Z, F_Z)$ be any fuzzy maps in the category **FTOP**. Then there exists a fuzzy pushout (μ_D, F_D).

Proof. Define D to be the quotient of the disjoint union $Y \coprod Z = Y \times \{1\} \cup Z \times \{2\}$ by the equivalence relation generated by the relation $(f(x), 1) \sim (g(x), 2)$ for each $x \in X$, then $D = Y \coprod Z / \sim$. Let the functions $\alpha: Y \to D$ and $\beta: Z \to D$ be defined by $\alpha(y) = [(y, 1)]$, $\beta(z) = [(z, 2)]$. Now we define the fuzzy topology on D as follows: $\mu_D([(y, 1)]) = \mu_Y(y)$ for each $y \in X$, $\mu_D(z, 2) = \mu_Z(z)$ for each $z \in Z$, and by the definition of fuzzy co-product, D is assigned the finest fuzzy topology so that the functions α and β are fuzzy continuous.

Lemma. A fuzzy set $U: D \to I$ is fuzzy open, $U \in F_D$, iff $\alpha^{-1}(U) \cap \mu_Y \in F_Y$ and $\beta^{-1}(U) \cap \mu_Z \in F_Z$.

Proof. consider the quotient map $q: Y \coprod Z \to D$, and set $\alpha = q|_Y$ and $\beta = q|_Z$. Let $U: D \to I$ be a fuzzy open set belongs to F_D. Because q is a fuzzy map, $q^{-1}(U) \cap \mu_{Y \coprod Z} \in F_{Y \coprod Z}$. Then we have

$$\left(q^{-1}(U) \cap \mu_{Y \coprod Z}\right)(y, 1) = q^{-1}(U)(y, 1) \wedge \mu_{Y \coprod Z}(y, 1) = U\big(q(y, 1)\big) \wedge \mu_Y(y) = U\big(\alpha(y)\big) \wedge \mu_Y(y) = (\alpha^{-1}(U) \cap \mu_Y)(y).$$

But this implies that $\alpha^{-1}(U) \cap \mu_Y \in F_{Y \coprod Z}$. By the definition of fuzzy co-product , we conclude that $\alpha^{-1}(U) \cap \mu_Y \in F_Y$. In similar way and by replacing $(y, 1)$ with $(z, 2)$, we have $\beta^{-1}(U) \cap \mu_Z \in F_Z$. Conversely, suppose that $U: D \to I$ is a fuzzy set for which $\alpha^{-1}(U) \cap \mu_Y \in F_Y$ *and* $\beta^{-1}(U) \cap \mu_Z \in F_Z$. The above calculations implies that $\big(q^{-1}(U) \cap$ *μY∐Zy,1=α−1U∩μYy* and *q−1U∩μY∐Zz,2=β−1U∩μZz,2*, proving the belonging of *q−1U∩μY∐Z* to *FY∐Z*. but this means that U is fuzzy open by the definition of the fuzzy quotient topology.

It is clear that $\alpha\big(f(x)\big) = [\big(f(x)\big), 1)] = [(g(x), 2)] = \beta\big(g(x)\big), x \in X$, this implies that $\alpha \circ f = \beta \circ g$. To show that D satisfies the universal property, given a fuzzy space (μ_C, F_C) with fuzzy maps $\acute{\alpha}: (\mu_Y, F_Y) \to (\mu_C, F_C)$ and $\acute{\beta}: (\mu_Z, F_Z) \to (\mu_C, F_C)$ such that $\acute{\alpha} \circ f = \acute{\beta} \circ g$, we define $\varphi: (\mu_D, F_D) \to (\mu_C, F_C)$ by $\varphi([(y, 1)]) = \acute{\alpha}(y)$, and $\varphi([(z, 2)]) = \acute{\beta}(z)$. It is easy to check that this is well-defined. Also, it is clear that $\varphi \circ \alpha = \acute{\alpha}$ and $\varphi \circ \beta = \acute{\beta}$. For the uniqueness property, consider the quotient map $q: Y \coprod Z \to D$ and define $\bar{\varphi}: Y \coprod Z \to C$ by $\bar{\varphi}|_Y = \acute{\alpha}$ and $\bar{\varphi}|_Z = \acute{\beta}$. We can verify that $\bar{\varphi}$ factors through the map q to yield the map $\varphi: (\mu_D, F_D) \to (\mu_C, F_C)$. Further, φ is unique because it is required to satisfy $\varphi \circ q = \bar{\varphi}$ (by the uniqueness condition in theorem 3.1). Also the last equation guarantees the fuzzy continuity of φ. □

Example 3.5

(1). Let (X, μ, F) be a fuzzy space , $(\{z\}, 1_{\{z\}}, F_{\{z\}})$ be a terminal object in **FTOP** [4] and $A \subset$ X. Let

$\mu_A = \mu|_A$, $F_A = \{U|_A : U \in F\}$, then (A, μ_A, F_A) is called a subspace of (X, μ, F)[4]. The fuzzy pushout of the diagram

$$(\{z\}, 1_{\{z\}}, F_{\{z\}}) \leftarrow (A, \mu_A, F_A) \to (X, \mu, F),$$

Is the fuzzy quotient $(X/A, \gamma, \mathcal{U})$,

(2). Let (X, μ, F) be a fuzzy space, $A, B \subset$ X so that $X = A \cup B$. Let $\mu_A = \mu|_A$, $F_A = \{U|_A : U \in F\}$ and

$\mu_B = \mu|_B$, $F_B = \{U|_B : U \in F\}$, then (A, μ_A, F_A) and (B, μ_B, F_B) are fuzzy spaces considered as

subspaces of (X, μ, F). Consider the diagram of fuzzy maps (inclusions)

$$(B, \mu_B, F_B) \leftarrow (A \cap B, \mu_{A \cap B}, F_{A \cap B}) \to (A, \mu_A, F_A),$$

where $(A \cap B, \mu_{A \cap B}, F_{A \cap B})$ is a subspace of (A, μ_A, F_A) and (B, μ_B, F_B). Then the fuzzy pushout (μ_D, F_D) is the fuzzy space (X, μ, F) .

Proposition 3.6 Given any fuzzy maps $f: (\mu_X, F_X) \to (\mu_Y, F_Y)$ and $g: (\mu_X, F_X) \to (\mu_Z, F_Z)$, the fuzzy pushout (μ_D, F_D) is unique up to fuzzy homeomorphism.

Proof. Suppose $(\mu_{D'}, F_{D'})$, with fuzzy maps $f^{'}: (\mu_Y, F_Y) \to (\mu_{D'}, F_{D'})$ and $g^{'}: (\mu_Z, F_Z) \to (\mu_{D'}, F_{D'})$, is another fuzzy pushout. Take $(\mu_C, F_C) = (\mu_{D'}, F_{D'})$; we find a fuzzy map $\varphi: (\mu_D, F_D) \to (\mu_{D'}, F_{D'})$ such that with $\varphi \circ \alpha = \acute{f}$ and $\varphi \circ \beta = \acute{g}$. By reversing the roles of (μ_D, F_D) and $(\mu_{D'}, F_{D'})$, we find a fuzzy map $\varphi^{'}: (\mu_{D'}, F_{D'}) \to (\mu_D, F_D)$ such that $\varphi^{'} \circ \acute{f} = \alpha$ and $\varphi^{'} \circ \acute{g} = \beta$. Then $\varphi^{'} \circ \varphi \circ \alpha = \alpha$, and similarly $\varphi^{'} \circ \varphi \circ \beta = \beta$. Now take $(\mu_C, F_C) = (\mu_D, F_D), \alpha = \alpha^{'}$ and $\beta = \beta^{'}$. We have two fuzzy maps, $\varphi^{'} \circ \varphi: (\mu_D, F_D) \to (\mu_D, F_D)$ and id_D that satisfy the conditions of fuzzy pushout; by the uniqueness in definition of fuzzy pushout, $\varphi^{'} \circ \varphi = id_D$. Similarly, $\varphi \circ \varphi^{'} = id_{D'}$, so that φ and $\varphi^{'}$ are inverse fuzzy homeomorphisms. □

We close this section by investigation the relationship among the universal constructions mentioned above.

Theorem 3.7 In **FTOP**, fuzzy pushouts exist if and only if fuzzy co-equalizers exist.

Proof. (*Co-equalizers* $\Rightarrow$ *Pushouts*) Consider the arbitrary fuzzy maps $f: (\mu_X, F_X) \to (\mu_Z, F_Z)$ and $g: (\mu_Y, F_Y) \to (\mu_Z, F_Z)$. Let $(\mu_{Y\coprod Z}, F_{Y\coprod Z})$ be the fuzzy co-product of the fuzzy spaces (μ_Y, F_Y) and (μ_Z, F_Z) with the fuzzy maps $e_1: Y \to Y\coprod Z$ and $e_2: Z \to Y\coprod Z$ defined by $e_1(y) = (y, 1)$, and $e_2(z) = (z, 2)$. Let (μ_E, F_E) together with a fuzzy map $q: (\mu_{Y\coprod Z}, F_{Y\coprod Z}) \to (\mu_E, F_E)$ be the fuzzy co-equalizer of fuzzy maps $e_1 \circ f, e_2 \circ g: (\mu_X, F_X) \to (\mu_{Y\coprod Z}, F_{Y\coprod Z})$ such that $q \circ (e_1 \circ f) = q \circ (e_2 \circ g)$, by theorem 3.3. Now we prove that (μ_E, F_E) together with fuzzy maps $q \circ e_1: (\mu_Y, F_Y) \to (\mu_E, F_E)$ and $q \circ e_2: (\mu_Z, F_Z) \to (\mu_E, F_E)$ is the pushout of the fuzzy maps f and g. Let (μ_C, F_C) be a fuzzy space with fuzzy maps $\alpha: (\mu_Y, F_Y) \to (\mu_C, F_C)$ and $\beta: (\mu_Z, F_Z) \to (\mu_C, F_C)$ so that $\alpha \circ f = \beta \circ g$. By theorem 3.1, there exists a unique fuzzy map $\theta: (\mu_{Y\coprod Z}, F_{Y\coprod Z}) \to (\mu_C, F_C)$ such that $\theta \circ e_1 = \alpha$, $\theta \circ e_2 = \beta$. Since (μ_E, F_E) is the fuzzy co-equalizer of fuzzy maps $e_1 \circ f, e_2 \circ g: (\mu_X, F_X) \to (\mu_{Y\coprod Z}, F_{Y\coprod Z})$ and $\theta \circ (e_1 \circ f) = (\theta \circ e_1) \circ f = \alpha \circ f = \beta \circ f = (\theta \circ e_2) \circ g = \theta \circ (e_2 \circ g)$, then theorem 3.3 implies that there exists a unique fuzzy map $h: (\mu_E, F_E) \to (\mu_C, F_C)$ such that $h \circ q = \theta$. Hence $h \circ (q \circ e_1) = \theta \circ e_1 = \alpha$ and $h \circ (q \circ e_2) = \theta \circ e_2 = \beta$. Thus the required fuzzy pushout is obtained.

(*Pushouts* $\Rightarrow$ *Co-equalizers*) Given any two fuzzy maps $f, g: (\mu_X, F_X) \to (\mu_Y, F_Y)$. Consider the function $f \sqcup g: X\coprod X \to Y$ defined by $(f \sqcup g)(x, 1) = f(x)$ and $(f \sqcup g)(x, 2) = g(x)$, in similar way, we define $id_X \sqcup id_X: X \coprod X \to X$. Now we show that $f \sqcup g$ is fuzzy map. Since f and g are fuzzy maps, then

$$\mu_Y\big((f \sqcup g)(x, 1)\big) = \mu_Y\big(f(x)\big) \geq \mu_X(x) = \mu_{X\coprod X}(x, 1),$$

$$\mu_Y\big((f \sqcup g)(x, 2)\big) = \mu_Y\big(g(x)\big) \geq \mu_X(x) = \mu_{X\coprod X}(x, 2).$$

Let U be a fuzzy open set in F_Y, $f_1, f_2: X \to X\coprod X$ be fuzzy maps defined by $f_1(x) = (x, 1)$ and $f_2(x) = (x, 2)$, then

$$\begin{aligned}\mu_X \cap f_1^{-1}(\mu_{X\coprod X} \cap (f \sqcup g)^{-1}(U)) &= \mu_X \cap \mu_X \cap f_1^{-1}(\mu_{X\coprod X}) \cap f_1^{-1}\big((f \sqcup g)^{-1}(U)\big)\\ &= \mu_X \cap f_1^{-1}(\mu_{X\coprod X}) \cap f_1^{-1}(\mathrm{U}(f \coprod g))\\ &= \mu_X \cap \mu_X \cap U(f) = \mu_X \cap (f^{-1}(U) \in F_X.\end{aligned}$$

This implies that $\mu_{X\coprod X} \cap (f \sqcup g)^{-1}(U) \in F_{X\coprod X}$, proving the fuzzy continuity of $f \sqcup g: X\coprod X \to Y$. Let (μ_D, F_D) be the fuzzy pushout of $f \sqcup g: X \coprod X \to Y$and $id_X \coprod id_X: X \coprod X \to X$ so that $\alpha: (\mu_Y, F_Y) \to (\mu_D, F_D)$ and $\beta: (\mu_X, F_X) \to (\mu_D, F_D)$ are fuzzy maps with $\alpha \circ (f \sqcup g) = \beta \circ (id_X \sqcup id_X)$. We claim that

$$(\mu_X, F_X) \underset{g}{\overset{f}{\rightrightarrows}} (\mu_Y, F_Y) \overset{\alpha}{\to} (\mu_D, F_D)$$

is a fuzzy co-equalizer diagram. It is an easy work to check that $(f \sqcup g) \circ f_1 = f$, $(f \sqcup g) \circ f_2 = g$. Thus $\alpha \circ f = \alpha \circ (f \sqcup g) \circ f_1 = \beta \circ (id_X \sqcup id_X) \circ f_1 = \beta$, and $\alpha \circ g = \alpha \circ (f \sqcup g) \circ f_2 = \beta \circ (id_X \sqcup id_X) \circ f_2 = \beta$ and hence $\alpha \circ f = \alpha \circ g$. For any fuzzy map (μ_C, F_C) and $\theta: (\mu_Y, F_Y) \to (\mu_C, F_C)$ with $\theta \circ f = \theta \circ g$, so that $\theta \circ (f \sqcup g) = \theta \circ f = \theta \circ g = \theta \circ f \circ (id_X \sqcup id_X)$. By theorem 3.4, there exists a fuzzy map $\varphi: (\mu_D, F_D) \to (\mu_C, F_C)$ such that $\theta = \varphi \circ \alpha$ and thus the universal property of the co-equalizer $(\mu_Y, F_Y) \overset{\alpha}{\to} (\mu_D, F_D)$ is satisfied .□

REFERENCES

[1] Adàmek, J., Herrlich, H. and Strecker, G. 1990 Abstract and concrete categories. Wiley, New York

[2] Behera, A. 2000 Universal constructions for fuzzy topological spaces. Fuzzy Sets and Systems (39), 271-276.

[3] Chang, C.L. 1968 Fuzzy topological spaces. J. Math. Anal. Appl. (24), 182-190.

[4] Geetha, S. 1992 Studies on fuzzy topological semigroups and related areas. Doctoral Thesis, University of Cochin.

[5] Geetha, S. 1993 On L- fuzzy topological semigroups. J. Math. Anal. Appl. (174), 147-152.

[6] Geetha, S. 1993 On L- fuzzy semitopological semigroups. J. Math. Anal. Appl. (175), 1-9.

[7] Ghanim, M.H., Kerre, E.E. and Mashhour, A.S. 1984 Separation axioms, subspaces and sums in fuzzy topology. J. Math. Anal. Appl. (102), 189-202.

[8] Hamouda,E. H. 2014 On some universal constructions for I- fuzzy topological spaces. J. of Advances in Math. Vol. 6, No.2,973-941

[9] Liu, Y., Zhang, D. and Luo, M.. 2000 Some categorical aspects of fuzzy topology1. J. Math. Anal. Appl. (251), 649–668

[10] Lowen, R. 1977 Initial and final topologies and the fuzzy Tychonoff theorem. . Math. Anal. Appl. (58), 11-21

[11] Shostak, A. P. 1989 Two decades of fuzzy topology: basic idea, notions and results. Russ. Math. Surveys (6), 125-186.

[12] Wong, C.K. 1974 Fuzzy topology: Product and quotient theorems. J. Math. Anal. Appl. (45), 517- 521.

[13] Wong, C.K. 1976 Categories of fuzzy sets and fuzzy topological spaces. J. Math. Anal. Appl. (53), 704-714.

[14] Wuyts, P. and Lowen, R. 1988 Reflectors and coreflectors in the category of fuzzy topological spaces. Comput. Math. Applic (16), 823-836

[15] Zadeh, L.A. 1965 Fuzzy sets. Inform. And Control (8), 338-353.