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On the average order of the number of divisors of a positive integer n, and the number of distinct prime divisors of n

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Abstract: Let τ (n) denote the number of divisors of a positive integer n, and let ω (n) is the number of distinct prime divisors of n. De Koninck and Ivic [1] have been proved the asymptotic formula for $\chi \to \infty$

$$\sum_{n\leq x}\tau(n)\omega(n) = 2x\log x\log\log x + Ax\log x + Bx\log\log x + o(x)$$

Keywords:

Multiplicate function; the number of divisors of a positive integer n; the number of distinct prime divisors of n; Mean value; Asymptotic formula.

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Introduction and Preliminary Results

Let τ (n) denote the number of divisors of a positive integer n, and let ω (n) is the number of distinct prime divisors of n. De Koninck and Ivic [1] have been proved the asymptotic formula for $\chi \to \infty$

(1)- $$\sum_{n\le x}\tau(n)\omega(n) = 2x\log x \log\log x + Ax\log x + Bx\log\log x + o(x)$$

In the work of De Koninck and Katai [2] was improved an error term in (1). They obtained also a nontrivial estimate for the sum of τ (n) ω (n) where n run over short interval.

In 2005 Prosyanjuk and Varbanets [6] studies the average order $\tau(\alpha)\omega(\alpha)$ in the ring of the Gaussian integers Z [i].

In our present we consider the summatory function for f(n) ω (n) where f(n) belong to the special class M(a) of multiplicate function which we define the following way:

$f(n) \in M(a)$ if

(i) f(n) is a multiplicate function;

(ii) there exists the finite (or empty) set P_0 of prime numbers constant a, such that:

$$f(p) = \begin{cases} o & \text{if } p=P_o \\ a & \text{if } p\equiv 1 \ (\text{mod}4), p\notin P_o, \text{ or } p=2 \\ b & \text{if } p\equiv 3 \ (\text{mod}4), p\notin P_{o,}\ b=a \text{ or } 0; \end{cases}$$

(iii) $|f(p^k)| \le c$ for any prime p and each positive k, c is a fixed constant;

(iv) for $\chi^x \to \infty$ we have

(2)- $$= \sum_{n\le x} f(n) = A_0 x P(\log x) + R(x),$$

where

$R(x) = O(x^{\theta})\ \theta < 1,\ A_o > 0, P(u) = a_1 u + a_0$

Moreover, for a nonnegative function f(n) $\in$ M(a) we obtain an asymptotic formula for the sum:

$$A_k(x) = \sum_{\substack{n\le x \\ \omega(n)=k}} f(n)$$

We shall use following assertions.

Lemma1.: There exists a positive constant C_1 such that in region Re $s \ge 1 - \acute{n}_1(\log T)^{-\frac{2}{3}}(\log\log T)^{-1}$, $|\text{Im}\, S| \le T$,

the following estimate

$$\log\zeta(s) << (\log(|t|+10))^{\frac{1}{3}}(\log\log(|t|+10))$$

Here constant in symbol ,,<<´´ is absolute.

Let m be a positive integer. We denote

$$D(x,m) := \sum_{\substack{n\le x\\(n,m)=1}} f(n),$$

$$D(x) := D(x,1) = \sum_{n\le x} f(n).$$

Lemma 2:. Let the multiplicate function f(n) satisfies conditions (2).

Then for each prime p we have for $x^x \to \infty$

(3)- $D(x;p) = A_p x + O(x^{\theta})$

where $A_p = A\sum_{k=0}^{\infty}\frac{f(p^k)}{p^k}$

Proof:. First we obtain for a prime p.

$$\sum_{\substack{n=1\\(n,p)=1}}\frac{f(n)}{n^s} = (1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2s}}+\ldots)\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = g_p(s)F(s), \quad (\text{Re } s>1),$$

say.

Hence, by Perron's formula we have for c >1.

$$D(x;p) = \frac{1}{2\pi i}\int_{c-i}^{c+i} g_r(s)F(s)\frac{x^s}{s}ds + O(\frac{x^c}{T(c-1)}) + O(\frac{x^{1+\varepsilon}}{T})$$

Define N from the condition $p^{Nc}=T$, $N=\frac{\log T}{c\log p}$.

Them for $c=1+\varepsilon$, $\varepsilon>0$, T=x, we infer

$$D(x;p) = \sum_{k=0}^{N} f(p^k)\frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s)(\frac{x}{p^k})^s\frac{ds}{s} + O(\frac{x^{1+\varepsilon}}{T})$$

$$= \sum_{k=0}^{N} f(p^k)(D(\frac{x}{p^x}) + O(\frac{x^{1+\varepsilon}}{Tp^k})) + O(\frac{x^{1+\varepsilon}}{T})$$

$$= Ax\sum_{k=0}^{N}\frac{f(p^k)}{p^k} + O(\frac{x^{1+\varepsilon}}{T}) + O(\sum_{k=0}^{N}(\frac{x}{p^k})^{\theta}\frac{1}{p^k}) = A_p x + O(x^{\theta})$$

Corollary:. For any real $h\le x$ and each primer:p we have

$$D(x, x+h; p) := \sum_{x<n\le x+h} f(n) = \sum_{k=0}^{N} f(p^k) D(\frac{x}{p^k}, \frac{x+h}{p^k}) + O(1),$$

where N=$\left[\frac{\log x}{\log p}\right]$.

Lemma3: α_5 $x^x \to \infty$,

$$\sum_{p\le x} \frac{1}{p} = \log\log x + c_1 + O(\frac{1}{\log^2 x}),$$

$$\sum_{p\le x}, \sum_{p\le x} \frac{\log p}{p} = \log x - d_1 + O(\frac{1}{\log x})$$

where

$$c_1 = \gamma + \sum_p (\log(1-\frac{1}{p}) + \frac{1}{p}),$$

$$d_1 = \gamma + \sum_p \frac{\log p}{p(p-1)},$$

γ is the Euler's constant.

Moreover, for $2 \le y \le x$,

(6)- $$\sum_{x<p\le x+y} 1 \le \frac{2y}{\log y},$$

(7)- $$\sum_{x<p\le x+y} 1 = \frac{y}{\log x} + O(\frac{y}{\log^2 x}), \text{ if } x^{\frac{7}{12}+\varepsilon} \le y \le x.$$

Proof. The relations (5) are well-known (see,[7]), the inequality was proved Montgomery and Vaughan [5], and (7) proved M. Huxley [3].

__**Lemma 4:.**__ Let α be a complex number, Re α >0. Then for $x^x \to \infty$ and any positive number M

$$\gamma(\alpha, x) := \int_0^x \exp(-u) u^{\alpha-1} du = r(\alpha) - x^{\alpha-1}\exp(-x)[1 + \sum_{m=1}^{\mu} \frac{r(\alpha+m)}{r(\alpha)} (-1)^m x^{-m} + O(x^{-\mu})]$$ where r(α)

is the Euler's gamma-function; the constant in the symbol "O´´ can depend only of M.

(It is a well known estimate of the incomplete gamma- function through the complete gamma-function).

__**2. Main Results.**__

We put

(8)- $$F(x) := \sum_{n\le x} f(n)\omega(n).$$

For each positive integer n$\ge$2 there exist primes $p_1<p_2<\ldots<p_r$, writhe r=ω(n), such that.

n= $$p_1^{a_1}\ldots p_2^{a_2} = p_1^{a_1} m_1 = p_2^{a_2} m_2 = \ldots = p_r^{a_r} m_r,$$

whis some positive integers $a_1,\ldots,a_r$; $(m_i, p_i)=1$, $i=1,\ldots,r$.

It is clear that each n has $\omega(n)$ the representations as $n=p^a m$, $(m,p)=1$. Hence,

(9)- $$\sum_{n\le x} f(n)\omega(n) = \sum_{\substack{a,p\\ p^a\le x}} \sum_{\substack{m\le \frac{x}{p^a}\\ (m,p)=1}} f(p^a m)$$

Now we shall prove the following theorem.

Theorem 1:.

Let the multiplicate function f(n) belongs t_0 M(a). Then for $\chi \to \infty$

$$\sum_{n\le x} f(n)\omega(n) = B_0 x\log x\log\log x + B_1 x\log x + B_2 x\log\log x + B_3 x + O(\frac{x}{\log x})$$

where $B_0 = A_0 a_1 \frac{a+b}{2}, B_1 = A_0 a_0 \frac{a+b}{2}$, and B_2, B_3 are the computable constants; the parameters A_0, a, b take from the definition of the class M(a), and a_1, a_0 are coefficient of the polynomial P(u) in the relation (2).

From (8),(9), by Lemma 2 we have:

$$F(x) = \sum_{\substack{a,p\\ p^a\le x}} f(p^a) D(\frac{x}{p^a}; p)$$

$$\sum_{p\le x} f(p)(A_p \frac{x}{p}(a_1 \log\frac{x}{p} + a_0) + O(x^\theta)) +$$

$$+ \sum_{\substack{a\ge 2\\ p^a\le x}} f(p^a)(A_p \sum_{j=0}^{\infty} \frac{f(p^j)}{p^j} \frac{x}{p^a}(a_1 \log\frac{x}{p^a} + a_0) + O((\frac{x}{p^a})^\theta) =$$

$$x\sum_{p\le x}(\frac{f(p)}{p} A_p (a_1 \log\frac{x}{p} + a_0) + O(\frac{x}{p})^\theta) +$$

$$+ \sum_{\substack{p^a\le x\\ a\ge 2, j\ge 0}} A_p \frac{f(p^a) f(p^j)}{p^a + j} (b_1 \log\frac{x}{p^a} + a_0) + O(x^\theta \sum_{\substack{a,b\\ a\ge 2}} \frac{|f(p^a)|}{p^{a\theta}}) =$$

$$= A_0 x \sum_{p\le x} \frac{f(p)}{p}(a_1(\log x - \log p) + a_0) +$$

$$A_0 x \sum_{p} \frac{f^2(p)}{p^2}(a_1(\log x - \log p) + a_0) +$$

$$+ A_0 x \sum_{a\ge 2} \sum_{j\ge 0} \sum_{p^a\le x} \frac{f(p^a) f(p^j)}{p^{a+j}} (a_1(\log x - \log p^a) + a_0) + O(x^\theta) =$$

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$$= A_0 a_1 \frac{a+b}{2} x \log x \log\log 2 + A_0 a_0 \frac{a+b}{2} \log x + C_2 x \log\log x + C_3 x + O(\frac{x}{\log x}) =$$

$$= B_0 x \log x \log\log x + B_1 x \log x + B_2 x \log\log x + B_3 x + O(\frac{x}{\log x}),$$

where $B_0 = A_0 a_1 \frac{a+b}{2}$, $B_1 = A_0 a_0 \frac{a+b}{2}$, **$B_2$, B_3 are the computable constants, moreover, B_0=0 only if a_1=0,and B_1=0 only if a_0=0,** $B_0^2 + B_1^2 > 0$.

Remark:.

For the calculation of the coefficients c_0, c_1, c_2, c_3 we take into account the following asympic estimates.

$$\sum_{p \equiv \pm 1 (\bmod 4)} \frac{1}{p} = \frac{1}{2} \log\log x + C_1^{'} + O(\frac{1}{\log^2 x}),$$

$$\sum_{p \equiv \pm 1 (\bmod 4)} \frac{\log p}{p} = \frac{1}{2} \log x - d_1^{'} + O(\frac{1}{\log x}),$$

Moreover, the constants $c_1^{'}, d_1^{'}$, are analogical to the constants c_1, d_1 from

Lemma 3.

Let f(n) $\in$ M(a) and let z $\in$ C, $|z| = 1$. We define the function

$$F(s,z) = \sum_{n=1}^{\infty} \frac{z^{\omega(n)} f(n)}{n^s}, \quad \text{Re } s > 1.$$

Since $z^{\omega(n)} f(n)$ is a multiplicate function we infer

$$F(s,z) = \prod_p (1 + \frac{zf(p)}{p^s} + \frac{zf(p^2)}{p^{2s}} + \ldots) =$$

$$\prod_{\substack{p \\ p \equiv 1 (\bmod 4)}} (1 + \frac{az}{p^s} + \frac{zf(p^2)}{p^{2s}} + \ldots). \prod_{p \equiv 3 (\bmod 4)} (1 + \frac{z}{p^s} + \frac{zf(p^2)}{p^{2s}} + \ldots) G(s,z)$$

where:. $G(s,z) = \prod_{p \in P_0} (1 + \frac{zf(p^2)}{p^{2s}} + \ldots)(1 + \frac{zf(p)}{p^s} + \frac{zf(p^{2s})}{p^{2s}} + \ldots)^{-1} (1 + \frac{zf(2)}{2^s} + \frac{zf(4)}{2^{2s}} + \ldots)$

this we have

(11) $$F(s,z) = \zeta(s)^{az} G_0(s,z)$$

Here $G_0(s,z)$ **is a function defined by the Dirichlet's series** $\sum_{n=1}^{\infty} b_n(z) n^{-s}$ **, which converges in the region Re s>1/2, moreover**

$b_n(z)$<< n (uniformly on z, |z|=1)

Introduce the following notation

$$A_k(x)=\sum_{\substack{n\le x\\ \omega(n)=k}} f(n),\ A_k(x,h)=\sum_{\substack{x<n\le x+h\\ \omega(n)=k}} f(n)$$

$$B(x,z)=\sum_{n\le x} z^{\omega(n)} f(n),\ B(x,h,z)=\sum_{x<n\le x+h} z^{\omega(n)} f(n)$$

Theorem 2:.

Let z is a complex number, |z|=1. Assume that f(n) $\in$ M(a), 0<a$\le$1, and

$f(n)\ge 0$ for all n$\in$N. Then

$$B(x,z)=\frac{x}{(\log x)^{1-az}}\left[\frac{2\psi_0(z)}{r(az)}+\frac{\gamma_1(z)}{\log x}+\frac{\gamma_2(z)}{(\log x)^2}\right]+O(xe^{-c\frac{(\log x)^{3/5}}{\log\log x}})$$

Where $\psi_0(z)=\lim_{s\to 1}(F(s,z)\times(s-1)^{az}), and \gamma_1(z),\gamma_2(z)$ define by (21)-(22), moreover, $\psi_0(z),\gamma_1(z),\gamma_2(z)$ are regular functions for |z|≤ 2.

Proof:. We shall use the classic schema of E. Landau . By the relation

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^{s+k}}{s(s+1)...(s+k)}ds=\begin{cases}\frac{1}{k!}(y-1)^k, & \text{if } y\ge 1,\\ 0 & \text{if } 0<y<1\end{cases}$$

and taking into account that the series for F(s, z) converges for Re s= $\sigma=2$, we obtain

(12)- $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}F(s,z)\frac{x^{s+1}}{s(s+1)}ds=\sum_{n=1}^{\infty}z^{\omega(n)}nf(n)\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{(xn^{-1})^{s+1}}{s(s+1)}ds=$$

$$=\sum_{n\le x}z^{\omega(n)}(x-n)f(n):=S(x,z)$$

Since $B(x,1)=\sum_{n\le x}f(n)=A_0xP(\log x)+O(x)^{\theta}$ we can assume that |z|=1, z$\ne$1.

Take T>3. We set

(13)- $$\delta(t)=\frac{c_1}{2}(\log|t|+10)^{-2/3}(\log\log(|t|+10))^{-1},\quad |t|\le T,\delta_0=\delta(T)$$

(here c_1 takes from Lemma 1).

Let J= J 1+ J2+J3 +J 4+Jo

where J_1 consists with points s= σ +it for which

$\sigma=1-\delta(t),t>T;$

J_2 consists with points s= σ +it for which

$\sigma=1-\delta(t),t<-T$

J_3 (accorolingly, J_4) consists with that s= σ +it for which

$\sigma=1-\delta_0,0<t\le T$ (accorolingly, $-T\le t<0$);

J_0 consists of the interval (1-δ_0, 1-ρ) going in straight and back direction, an and the circle of the radius ρ with the centre in s=1.

Thus we have

$$\int_{P'} F(s,z)\frac{x^{s+1}}{s(s+1)}ds = 0$$

Hence,

$$S(x,z) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s,z)\frac{x^{s+1}}{s(s+1)}ds = \frac{1}{2\pi i}\int_{P} F(s,z)\frac{x^{s+1}}{s(s+1)}ds$$

From (11) we have on the contour J

(14)- $$\frac{1}{s(s+1)}F(s,z) = \frac{1}{(s-1)^{az}}G_1(s,z),$$

where $G_1(s, z)$ is regular on P.

From the condition Re z<1 we obtain that the integral on circle of radius ρ tends to 0 if $\rho \to 0$. Hence, we can consider P_0 as the interval (1-δ_0, 1) which pass in straight and back direction. Thus we obtain

$$(s-1)^{-az} = \begin{cases} e^{-\pi iaz}(1-s)^{-az} \\ e^{\pi iaz}(1-s)^{-az} \end{cases}$$

on back direction

So, we have

(15)- $$I = \frac{1}{2\pi i}\int_{P} F(s,z)\frac{x^{s+1}}{s(s+1)}ds = \frac{1}{2\pi i}\left(\int_{P_1} + \int_{P_2} + \int_{P_3} + \int_{P_4} + \int_{P_0}\right) := I_1 + I_2 + I_3 + I_4 + I_0$$

Applying Lemma 1 we easy infer for T= $\exp(c_2(\log x)^{3/5})$, $c_2>0$,

(16)- $$I_1 + I_2 + I_3 + I_4 \le x^2 \exp(-c_3(\log x)^{3/5}(\log\log x)^{-1}),\ c_3>0$$

Moreover,

$$I_0 = \frac{1}{2\pi i}\int_{P_0} F(s,z)\frac{x^{s+1}}{s(s+1)}ds = \frac{1}{2\pi i}\int_{1-\delta_0}^{1}(e^{\pi iaz} - e^{-\pi iaz})\frac{G_1(s,z)}{(1-s)^{az}}x^{s+1}ds$$

We set

(17)- $$G_1(s,z) = \psi_0(z) + (1-s)G_2(s,z), \quad \psi_0(z) = G_1(1,z)$$

$G_2(s, z)$ is analytic function (as function on s), moreover, $G_2(s, z)$ for $s \in [1-\delta_0, 1]$ is uniformly bound on z, |z|=1. Consider $G_2(s, z)$ as function of variable s and continue periodically (with period δ_0) on all real axis.

Then we have

$$I_0 = \frac{x^2 \sin \pi az}{\pi}\left[\int_{1-\delta_0}^{1}(1-s)^{-az}x^{s-1}\psi_0(z)ds + \int_{1-\delta_0}^{1}(1-s)^{1-az}G_2(s,z)x^{s-1}ds\right] =$$

$$= \frac{x^2 \sin \pi az}{\pi} \frac{\psi_0(z)}{(\log x)^{1-az}} \int_0^{\delta_0 \log x} e^{-u} u^{(1-az)-1} du + \frac{x^2 \sin \pi az}{\pi (\log x)^{2-az}} \int_0^{\delta_0 \log x} e^{-u} u^{1-z} G_3(u,z) du,$$

Where $G_3(u, z)$ is a periodic function of u with period $\delta_0 \log x$ and bounds by absolute constant.

Now by Lemma 4 for M=1 we have

$$I_0 = \frac{x^2 \sin \pi az}{\pi (\log x)^{1-az}} \psi_0(z) \left[r(1-z) + O(\delta_0 x^{-\delta_0} \log x) \right] + \frac{x^2 \sin \pi az}{\pi (\log x)^{2-az}} \times$$

$$\times \left(\int_0^{\infty} e^{-u} u^{1-az} G_3(u,z) du + O(\delta_0^2 x^{-\delta_0} \log^2 x) \right) = \frac{x^2 \psi_0(z)}{(\log x)^{1-az} r(az)} +$$

$$+ \frac{x^2 \gamma_0(z) \sin \pi az}{\pi (\log x)^{2-az}} + O(x^2 \exp(-c_4 (\log x)^{3/5} (\log \log x)^{-1})), \ (c_4>0)$$

where

(19)- $$\gamma_0(z) = \int_0^{\infty} e^{-u} u^{1-az} G_3(u,z) du, |\gamma_1(z)| \le const$$

for $|z|=1$.

Collecting our previous estimates we get

(20)- $$S(x,z) = \frac{x^2}{(\log x)^{1-az}} \frac{\psi_0(z)}{2r(az)} + \frac{\gamma_1(z) \sin \pi az x^2}{\pi (\log x)^{2-az}} + O(x^2 \exp(-c_5 (\log x)^{3/5} (\log \log x)^{-1})),$$

where $c_5 = \min(c_3, c_4)$.

For each u, 0<u<x, we have

(21)- $$S(x+u,z) - S(x,z) = \int_x^{x+u} \frac{d}{dy}(S(y,z)) dy = \int_x^{x+u} B(y,z) dy$$

From this, taking into account that $f(n) \ge 0$ we have, after a simple computations,

(22)- $$B(x,z) = \frac{x}{(\log x)^{1-az}} \left[\frac{2\psi_0(z)}{r(az)} + \frac{\gamma_1(z)}{\log x} + \frac{\gamma_2(z)}{(\log x)^2} \right] + O\left(xe^{-\frac{c(\log x)^{3/5}}{\log\log x}}\right),$$

$c=1/2c_5$

where $$\gamma_1(z) = \frac{(az-1)\psi_0(z)}{2r(az)} + \frac{2\gamma_0(z) \sin \pi az}{\pi},$$

(23)- $$\gamma_2(z) = \frac{1}{\pi}(az-2)\gamma_0(z) \sin \pi az$$

<u>Theorem 3</u>:. Let c(x) is a real function tending to ∞ slow than $\sqrt{\log \log x}$

Than .

$$A_k(x)=\sum_{\substack{\omega(n)=k\\ n\le x}} f(n)=\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}\frac{\psi_0(\frac{k-1}{\log\log x})}{r(1+\frac{a(k-1)}{\log\log x})}+O(k^{3/2}(\log\log x)^{-1}),$$

where ψ_0 defined in Theorem 2.

Proof. By definion B(x, z) we have

$$B(x,z)=1+\sum_{k=1}^{\infty}A_k(x)z^k$$

Obviously, that $A_k(x)=0$ if $k>2\log x$. Hence, B(x, z) is analytic function of z.

Then by formulas C_0 we have

$$A_k(x)=\frac{1}{2\pi i}\int_c B(x,z)z^{-(k+1)}dz,$$

where c is the circle with centre z=0 of radius $r=\frac{k-1}{\log\log x}$. For k=1 we put c

|z|=1/2. By Theorem 2 we infer

$$A_1(x)=\frac{1}{2\pi i}\int_{|z|=1/2}\frac{B(x,z)}{z^2}dz=\frac{2xa}{\log x}\frac{1}{2\pi i}\int_{|z|=1/2}\frac{\psi_0(z)(\log x)^{az}}{r(1+az)z}ds+\frac{x}{(\log x)^2}\frac{1}{2\pi i}\times$$

(24)- $$\times\int\frac{\gamma_1(z)+\gamma_2(z)\log^{-1}x(\log x)^{az}}{z^2}dz+O(e^{-c(\log x)^{3/5}(\log\log x)^{-1}}\int_{|z|=1/2}\frac{|dz|}{|z|^2})=$$

$$=\frac{2ax}{\log x}\psi_0(0)+O(\exp(-c(\log x)^{3/5}(\log\log x)^{-1}))$$

For k>1 Theorem 2 gives

(25)- $$A_k(x)=\frac{x}{\log x}\frac{1}{2\pi i}\int_c\frac{\psi_0(z)(\log x)^{az}}{r(az)z^{k+1}}dz+O(\frac{x}{(\log x)^2}\int_c\frac{|\log x|}{|z|^{k+1}}|dz|)+$$

$$+O(x\exp(-c(\log x)^{3/5}(\log\log x)^{-1})\int_c\frac{|dz|}{|z|^{k+1}})=I_1+I_2+I_3$$

we have:.

(26)
$$I_3<<x\exp(-c(\log x)^{3/5}(\log\log x)^{-1})(\frac{1}{k-1}\log\log x)^k$$
$$<<x\exp(\frac{-c}{2}(\log x)^{3/5}(\log\log x)^{-1})$$

$$I_2<<\frac{x}{r^k(\log x)^2}\int_{-\pi}^{\pi}(\log x)^{r\cos\alpha}d\alpha<<\frac{x}{r^k(\log x)^2}\int_{-\pi}^{\pi}e^{r\cos\alpha\log\log x}d\alpha<<$$

(27)- $$\ll \frac{x}{(\log x)^2}\left(\frac{\log\log x}{k-1}\right)^k \sum_{n=0}^{\infty}\frac{(r\log\log x)^n}{n!}\int_0^1(\cos 2\pi\alpha)^n\,d\alpha \ll \frac{x}{(\log x)^2}\frac{(\log\log x)^k e^k}{(k-1)\sqrt{k-1}}$$

$$\ll \frac{x}{(\log x)^2}\frac{(\log\log x)^k}{(k-1)!}\frac{\log\log x}{k-1};$$

(here we used the sterling's formulas for (k-1)!=r(k)).

Denote

$$g(z)=\psi_0(z)(z\Gamma(az))^{-1}=a\psi_0(z)(\Gamma(1+az))^{-1}$$

The function g(z) is analytic for |z|<3/2, and thus

$$g(z)=g(r)+g'(r)(z-r)+O(|g''(z)||z-r|^2),\quad |g''(r)|<\text{const}.$$

Hence,

(28) $$\tau_1=\frac{x}{\log x}\frac{1}{2\pi i}\int_c g(z)\frac{(\log x)^z}{z^k}dz=\frac{xg(r)}{z^k}\frac{1}{2\pi i}\int_c\frac{(\log x)^z}{z^k}dz+\frac{xg'(r)}{\log x}\frac{1}{2\pi i}\int_c\frac{(z-r)(\log x)^z}{z^k}dz$$

$$+O\left(\int_c\frac{|z-r|}{|z|^k}\right)|(\log x)^k||dz|$$

Now, as in [ ] we obtain

(29)- $$I_1=\frac{xg(r)}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}+O(k^{3/2}(\log\log x)^{-2})=$$

$$=\frac{\psi_0((k-1)(\log\log x)^{-1})}{r(1+a(k-1)(\log\log x)^{-1})}\frac{x(\log\log x)^{k-1}}{(k-1)!\log x}+O(k^{3/2}(\log\log x)^{-2})$$

The relations (25)-(29) accomplish the proof our theorem. Using our Theorems 2 and 3, and also Lemma 3, and Theorems 1 and 2 of Katai [4], we immediately obtain the following assertions.

Theorem 4:.

Let the conditions of Theorem 2 satisfy. Let, furthermore, $x^{7/12+\varepsilon}\le h\le x^{2/3-\varepsilon}$, where ε is arbitrary positive constant. Then

$$B(x,h;z)=\frac{h}{(\log x)^{1-az}}\left(\psi_0(z)+O\left(\frac{1}{\log x}\right)\right)$$

Theorem 5:.

Let the conditions of Theorem 3 satisfy. Then for each h, $x^{7/12+\varepsilon}\le h\le x^{2/3-\varepsilon}$, $\varepsilon>0$, the following asymptotic formula.

$$A_k(x,h)=\frac{h}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!\log x}\frac{\psi_0((k-1)(\log\log x)^{-1})}{r(1+(k-1)(\log\log x)^{-1})}+$$

$$+O(k^{3/2}(\log\log x)^{-2})+O\left(\frac{h}{\log x(l\log\log x)^{1/4}}\right);$$

holds.

3. Conclusion:.

The condition $0 < a \leq 1$ of Theorem 3 and 4 can expand on

$0 < a \leq 2$. Then obviously that for the multiplication functions $\tau(n)$ and ¼ r(n)

(the number of representations of n by sun of two squares) the appropriate

asymptotic formulas of theorem 2-5 are hold, moreover, we easy can write the

function $\psi_0(z), \gamma_1(z), \gamma_2(z)$. For example, if f(n)=1/4 r(n) we have

$$\psi_0(z) = \left(\frac{\pi}{4}\right)^z \prod_{p\equiv 1(\mathrm{mod}\,4)} (1-\frac{1}{p})^z (1+\frac{z}{p-1}) \prod_{p\equiv 3(\mathrm{mod}\,4)} (1-\frac{1}{p^2})^z (1-\frac{1}{2})^z (1+z)$$

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