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Numerical Solutions of Sixth Order Linear and Nonlinear Boundary Value Problems

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ABSTRACT

The aim of paper is to find the numerical solutions of sixth order linear and nonlinear differential equations with two point boundary conditions. The well known Galerkin method with Bernstein and modified Legendre polynomials as basis functions is exploited. In this method, the basis functions are transformed into a new set of basis functions, which satisfy the homogeneous form of Dirichlet boundary conditions. A rigorous matrix formulation is derived for solving the sixth order BVPs. Several numerical examples are considered to verify the efficiency and implementation of the proposed method. The numerical results are compared with both the exact solutions and the results of the other methods available in the literature. The comparison shows that the performance of the present method is more efficient and yields better results.

Keywords

Galerkin method; linear and nonlinear BVP; Bernstein and Legendre polynomials.

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1. INTRODUCTION

Agarwal [1] has discussed the theorems of the conditions for the existence and uniqueness of solutions of the sixth-order BVPs thoroughly in a book, but no numerical methods are contained there in. Non-numerical techniques were developed by Baldwin [2, 3] for solving such BVPs. Moreover, Chandrasekhar [4] determined that when an infinite horizontal layer of fluid is heated from below and is under the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth-order. Generally, sixth-order boundary value problem arises in the mathematical modeling of astrophysics; the narrow convecting layers which are believed to surround A-type stars [5]. Boutayeb and Twizell [6] developed a family of numerical methods for the solution of special and general nonlinear sixth-order BVPs. Numerical methods for the solution of special and general sixth-order BVPs with application to Benard layer eigenvalue problem were introduced by Twizell and Boutayeb [7]. Glatzmaier [8] also noticed that dynamo action in some stars may be modeled by such BVPs. Siddiqi *et al.* [9] presented the Quintic spline solution of linear sixth-order BVPs. Siraj-ul-Islam *et al*. [10] used nonpolynomial splines approach to the solution of sixth-order BVPs. Siddiqi and Akram [11] developed septic spline solutions of sixth-order BVPs. On the other hand, Chawla and Katti [12] presented numerical methods of solutions implicitly, although the authors concentrated their attention on fourth order BVPs. A second order method was introduced in [13] for solving special and general sixth-order BVPs and in later work Twizell and Boutayeb [7] developed finite difference methods of order two, four, six and eight for solving such problems. Gamel *et al*. [14] used Sinc-Galerkin method for the solution of sixth-order BVPs. Wazwaz [15] developed decomposition and modified domain decomposition methods to find the solution of the sixth-order BVPs. Siddiqi and Twizell [16] solved the sixth-order BVPs using polynomial splines of degree six where spline values at the mid knots of the interpolation interval and the corresponding values of the even order derivatives were related through consistency relations. Recently, Khan and Sultana [17] used parametric quintic spline solution for sixth order two point BVPs. Fazal-i-Haq *et al.* [18] developed the solution of sixth order BVPs by collocation method using Haar wavelets. Akram and Siddiqi [19] presented the solution of sixth order BVPs using non-polynomial spline technique. Logmani and Ahmadinia [20] derived numerical solution of sixth order BVPs with sixth degree B-spline functions.

In this paper, we consider Galerkin method [21] with Bernstein and Legendre polynomials [22] as basis functions for the numerical solution of a general sixth-order linear boundary value problem given by

$$a_6\frac{d^6u}{dx^6}+a_5\frac{d^5u}{dx^5}+a_4\frac{d^4u}{dx^4}+a_3\frac{d^3u}{dx^3}+a_2\frac{d^2u}{dx^2}+a_1\frac{du}{dx}+a_0u=r\ ,\ a<x<b \tag{1}$$

Subject to the following two types of boundary conditions

Type I: $u(a)=A_0,\ u(b)=B_0,\ u'(a)=A_1,\ u'(b)=B_1,\ u''(a)=A_2,\ u''(b)=B_2$ (2a)

Type II: $u(a)=A_0,\ u(b)=B_0,\ u''(a)=A_2,\ u''(b)=B_2,\ u^{(iv)}(a)=A_4,\ u^{(iv)}(b)=B_4$ (2b)

where $A_i, B_i, i=0,1,2,4$ are finite real constants and $a_i, i=0,1,\cdots 6$ and r are all continuous and differentiable functions of x defined on the interval [*a*, *b*]. The boundary value problem (1) is to be solved with both the boundary conditions of type I and type II.

However, in section 2 of this paper, we give a short description on Bernstein and Legendre polynomials. In section 3, the formulation for solving linear sixth-order BVP by Galerkin weighted residual method with Bernstein and Legendre polynomials is described. In particular, the proposed method with the boundary conditions of type I, eqn. (2a) is presented in section 3.1 where as the proposed method with the boundary conditions of type II, eqn. (2b) is presented in section 3.2. Then we deduce similar formulation for nonlinear problems in the next section. In section 4, numerical examples for both linear and nonlinear BVPs are considered to verify the proposed formulation. Finally the conclusions of the paper are given in the last section.

2. PIECEWISE POLYNOMIALS

(a) Bernstein Polynomials

The general form of the Bernstein polynomials [22] of *n*th degree over the interval $[a,b]$ is defined by

$$B_{i,n}(x)=\binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n},\qquad a\le x\le b\qquad i=0,1,2,\ldots,n\ .$$

For example, the first 11 Bernstein polynomials of degree 10 over the interval $[0,\ 1]$ are given bellow:

$$B_0(x)=(1-x)^{10}\qquad B_4(x)=210(1-x)^6x^4\qquad B_8(x)=45(1-x)^2x^8$$
$$B_1(x)=10(1-x)^9x\qquad B_5(x)=252(1-x)^5x^5\qquad B_9(x)=10(1-x)x^9$$
$$B_2(x)=45(1-x)^8x^2\qquad B_6(x)=210(1-x)^4x^6\qquad B_{10}(x)=x^{10}$$
$$B_3(x)=120(1-x)^7x^3\qquad B_7(x)=120(1-x)^3x^7$$

Note that each of these *n*+1 polynomials having degree *n* satisfies the following properties:

(i) $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$.

(ii) $\sum_{i=0}^{n} B_{i,n}(x) = 1$

(iii) $B_{i,n}(a) = B_{i,n}(b) = 0, \quad i = 1, 2, \ldots, n-1$

For these properties, Bernstein polynomials are used in the trail functions satisfying the corresponding homogeneous form of the *essential* boundary conditions in the Galerkin method to solve a BVP.

(b) Legendre Polynomials

The general form of the Legendre polynomials [22] of degree n is defined by

$$P_n(x) = \frac{(-1)^n}{2^n (n!)} \frac{d^n}{dx^n}[(1-x^2)^n], \qquad n \ge 1$$

Now we modify above Legendre polynomials as

$$p_n(x) = \left[\frac{1}{n!} \frac{d^n}{dx^n} \left(x^2 - x\right)^n - (-1)^n \right](x-1), \quad n \ge 1$$

We write first few modified Legendre polynomials over the interval [0, 1]:

$p_1(x) = 2x(x-1)$, $p_2(x) = 6x(x-1)^2$, $p_3(x) = 2x(x-1)(10x^2 - 15x + 6)$, $p_4(x) = 20x - 110x^2 + 230x^3 - 210x^4 + 70x^5$

$p_5(x) = -30x + 240x^2 - 770x^3 + 1190x^4 - 882x^5 + 252x^6$, $p_6(x) = 42x - 462x^2 + 2100x^3 - 4830x^4 + 5922x^5 - 3696x^6 + 924x^7$

$$p_7(x) = -56x + 812x^2 - 4956x^3 + 15750x^4 - 28182x^5 + 28644x^6 - 15444x^7 + 3432x^8$$

$$p_8(x) = 72x - 1332x^2 + 10500x^3 - 43890x^4 + 106722x^5 - 156156x^6 + 135564x^7 - 64350x^8 + 12870x^9$$

$$p_9(x) = -90x + 2070x^2 - 20460x^3 + 108570x^4 - 342342x^5 + 672672x^6 - 832260x^7 + 630630x^8 - 267410x^9 + 48620x^{10}$$

$$p_{10}(x) = 110x - 3080x^2 + 37290x^3 - 244530x^4 + 966966x^5 - 2438436x^6 + 4015440x^7 - 4302870x^8 + 2892890x^9 - 1108536x^{10} + 184756x^{11}$$

$$p_{11}(x) = -132x + 4422x^2 - 64350x^3 + 510510x^4 - 2468466x^5 + 7735728x^6 - 16219632x^7 + 22972950x^8 - 21708830x^9 + 13117676x^{10} - 4585308x^{11} + 705432x^{12}$$

Since the modified Legendre polynomials have special properties at $x = 0$ and $x = 1$: $p_n(0) = 0$ and $p_n(1) = 0, \ n \ge 1$ respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous form of the *Dirichlet* boundary conditions to derive the matrix formulation of fourth order BVP over the interval [0, 1].

3. FORMULATION OF SIXTH ORDER BVP

In this section we first derived the matrix formulation for sixth order linear BVP and then we extend our idea for solving nonlinear BVP. To solve the boundary value problem (1) by the Galerkin method we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} \alpha_i \, B_{i,n}(x) \tag{3}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions and $B_{i,n}(a) = B_{i,n}(b) = 0$ for each $i = 1, 2, \ldots n$. Using (3) into eqn. (1), the weighted residual equations are

$$\int_a^b \left[a_6 \frac{d^6 \tilde{u}}{dx^6} + a_5 \frac{d^5 \tilde{u}}{dx^5} + a_4 \frac{d^4 \tilde{u}}{dx^4} + a_3 \frac{d^3 \tilde{u}}{dx^3} + a_2 \frac{d^2 \tilde{u}}{dx^2} + a_1 \frac{d \tilde{u}}{dx} + a_0 \, \tilde{u} - r \right] B_{j,n}(x) dx = 0 \tag{4}$$

3.1 Formulation I

Integrating by parts the terms up to second derivative on the left hand side of (4), we get each term after applying the conditions prescribed in type I, eqn (2a) as

$$\int_a^b a_6 \frac{d^6\tilde{u}}{dx^6} B_{j,n}(x)dx = \left[a_6 B_{j,n}(x)\frac{d^5\tilde{u}}{dx^5}\right]_a^b - \int_a^b \frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\frac{d^5\tilde{u}}{dx^5}dx$$

$$= -\left[\frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[a_6 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx \text{ [Since } B_{j,n}(a) = B_{j,n}(b) = 0\text{]}$$

$$= -\left[\frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_6 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \int_a^b \frac{d^3}{dx^3}\left[a_6 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_6 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_6 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \int_a^b \frac{d^4}{dx^4}\left[a_6 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_6 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_6 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_6 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \left[\frac{d^4}{dx^4}\left[a_6 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^5}{dx^5}\left[a_6 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \quad (5)$$

$$\int_a^b a_5 \frac{d^5\tilde{u}}{dx^5} B_{j,n}(x)dx = \left[a_5 B_{j,n}(x)\frac{d^4\tilde{u}}{dx^4}\right]_a^b - \int_a^b \frac{d}{dx}\left[a_5 B_{j,n}(x)\right]\frac{d^4\tilde{u}}{dx^4}dx$$

$$= -\left[\frac{d}{dx}\left[a_5 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[a_5 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_5 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_5 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \int_a^b \frac{d^3}{dx^3}\left[a_5 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_5 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_5 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \left[\frac{d^3}{dx^3}\left[a_5 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right] + \int_a^b \frac{d^4}{dx^4}\left[a_5 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \quad (6)$$

$$\int_a^b a_4 \frac{d^4\tilde{u}}{dx^4} B_{j,n}(x)dx = \left[a_4 B_{j,n}(x)\frac{d^3\tilde{u}}{dx^3}\right]_a^b - \int_a^b \frac{d}{dx}\left[a_4 B_{j,n}(x)\right]\frac{d^3\tilde{u}}{dx^3}dx$$

$$= -\left[\frac{d}{dx}\left[a_4 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[a_4 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_4 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}\right]_a^b + \left[\frac{d^2}{dx^2}\left[a_4 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d^3}{dx^3}\left[a_4 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \quad (7)$$

$$\int_a^b a_3 \frac{d^3\tilde{u}}{dx^3} B_{j,n}(x)dx = \left[a_3 B_{j,n}(x)\frac{d^2\tilde{u}}{dx^2}\right]_a^b - \int_a^b \frac{d}{dx}\left[a_3 B_{j,n}(x)\right]\frac{d^2\tilde{u}}{dx^2}dx$$

$$= -\left[\frac{d}{dx}\left[a_3 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}\right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[a_3 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \quad (8)$$

$$\int_a^b a_2 \frac{d^2\tilde{u}}{dx^2} B_{j,n}(x)dx = \left[a_2 B_{j,n}(x)\frac{d\tilde{u}}{dx}\right]_a^b - \int_a^b \frac{d}{dx}\left[a_2 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx = -\int_a^b \frac{d}{dx}\left[a_2 B_{j,n}(x)\right]\frac{d\tilde{u}}{dx}dx \quad (9)$$

Substituting eqns. (5), (6), (7), (8) and (9) into eqn. (4) and using approximation for $\tilde{u}(x)$ given in eqn. (3) and after rearranging the terms for the resulting equations we get a system of equations in the matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j=1,2,\ldots,n \tag{10a}$$

where

$$\begin{aligned}
D_{i,j} = \int_a^b \Bigg\{ \Bigg[& -\frac{d^5}{dx^5}\big[a_6 B_{j,n}(x)\big] + \frac{d^4}{dx^4}\big[a_5 B_{j,n}(x)\big] - \frac{d^3}{dx^3}\big[a_4 B_{j,n}(x)\big] + \frac{d^2}{dx^2}\big[a_3 B_{j,n}(x)\big] - \frac{d}{dx}\big[a_2 B_{j,n}(x)\big] + a_1 B_{j,n}(x) \Bigg] B'_{i,n}(x) \\
& + a_0 B_{i,n}(x) B_{j,n}(x) \Bigg\} dx + \left[\frac{d}{dx}\big[a_6 B_{j,n}(x)\big] B^{(iv)}_{i,n}(x)\right]_{x=a} + \left[\frac{d^2}{dx^2}\big[a_6 B_{j,n}(x)\big] B'''_{i,n}(x)\right]_{x=b} - \left[\frac{d}{dx}\big[a_6 B_{j,n}(x)\big] B^{(iv)}_{i,n}(x)\right]_{x=b} \\
& - \left[\frac{d^2}{dx^2}\big[a_6 B_{j,n}(x)\big] B'''_{i,n}(x)\right]_{x=a} - \left[\frac{d}{dx}\big[a_5 B_{j,n}(x)\big] B'''_{i,n}(x)\right]_{x=b} + \left[\frac{d}{dx}\big[a_5 B_{j,n}(x)\big] B'''_{i,n}(x)\right]_{x=a}
\end{aligned} \tag{10b}$$

$$\begin{aligned}
F_j = \int_a^b \Bigg\{ & r B_{j,n}(x) + \Bigg[\frac{d^5}{dx^5}\big[a_6 B_{j,n}(x)\big] - \frac{d^4}{dx^4}\big[a_5(x) B_{j,n}(x)\big] + \frac{d^3}{dx^3}\big[a_4 B_{j,n}(x)\big] - \frac{d^2}{dx^2}\big[a_3 B_{j,n}(x)\big] + \frac{d}{dx}\big[a_2 B_{j,n}(x)\big] - a_1 B_{j,n}(x) \Bigg] \theta'_0(x) \\
& - a_0 \theta_0(x) B_{j,n}(x) \Bigg\} dx + \left[\frac{d}{dx}\big[a_6 B_{j,n}(x)\big] \theta_0^{(iv)}(x)\right]_{x=b} - \left[\frac{d}{dx}\big[a_6 B_{j,n}(x)\big] \theta_0^{(iv)}(x)\right]_{x=a} - \left[\frac{d^2}{dx^2}\big[a_6 B_{j,n}(x)\big] \theta_0'''(x)\right]_{x=b} \\
& + \left[\frac{d^2}{dx^2}\big[a_6 B_{j,n}(x)\big] \theta_0'''(x)\right]_{x=a} + \left[\frac{d^3}{dx^3}\big[a_6 B_{j,n}(x)\big]\right]_{x=b} \times B_2 - \left[\frac{d^3}{dx^3}\big[a_6 B_{j,n}(x)\big]\right]_{x=a} \times A_2 - \left[\frac{d^4}{dx^4}\big[a_6 B_{j,n}(x)\big]\right]_{x=b} \times B_1 \\
& + \left[\frac{d^4}{dx^4}\big[a_6 B_{j,n}(x)\big]\right]_{x=a} \times A_1 + \left[\frac{d}{dx}\big[a_5 B_{j,n}(x)\big] \theta_0'''(x)\right]_{x=b} - \left[\frac{d}{dx}\big[a_5 B_{j,n}(x)\big] \theta_0'''(x)\right]_{x=a} - \left[\frac{d^2}{dx^2}\big[a_5 B_{j,n}(x)\big]\right]_{x=b} \times B_2 \\
& + \left[\frac{d^2}{dx^2}\big[a_5 B_{j,n}(x)\big]\right]_{x=a} \times A_2 + \left[\frac{d^3}{dx^3}\big[a_5 B_{j,n}(x)\big]\right]_{x=b} \times B_1 - \left[\frac{d^3}{dx^3}\big[a_5 B_{j,n}(x)\big]\right]_{x=a} \times A_1 + \left[\frac{d}{dx}\big[a_4 B_{j,n}(x)\big]\right]_{x=b} \times B_2 \\
& - \left[\frac{d}{dx}\big[a_4 B_{j,n}(x)\big]\right]_{x=a} \times A_2 - \left[\frac{d^2}{dx^2}\big[a_4 B_{j,n}(x)\big]\right]_{x=b} \times B_1 + \left[\frac{d^2}{dx^2}\big[a_4 B_{j,n}(x)\big]\right]_{x=a} \times A_1 + \left[\frac{d}{dx}\big[a_3 B_{j,n}(x)\big]\right]_{x=b} \times B_1 \\
& - \left[\frac{d}{dx}\big[a_3 B_{j,n}(x)\big]\right]_{x=a} \times A_1
\end{aligned} \tag{10c}$$

Solving the system (10a), we find the values of the parameters α_i and then substituting these parameters into eqn. (3), we get the approximate solution of the desired BVP (1).

3.2 Formulation II

In the same way of section (3.1), integrating by parts the terms consisting fifth, fourth, third, and second derivatives on the left hand side of (4), and applying the conditions prescribed in eqn. (2b), we get a system of equations in the matrix form as

$$\sum_{i=1}^{n} D_{i,j}\,\alpha_i = F_j,\ j=1,2,\ldots,n \tag{11a}$$

where

$$\begin{aligned}
D_{i,j} = \int_a^b \Bigg\{ \Bigg[& -\frac{d^5}{dx^5}\big[a_6 B_{j,n}(x)\big] + \frac{d^4}{dx^4}\big[a_5 B_{j,n}(x)\big] - \frac{d^3}{dx^3}\big[a_4 B_{j,n}(x)\big] + \frac{d^2}{dx^2}\big[a_3 B_{j,n}(x)\big] - \frac{d}{dx}\big[a_2 B_{j,n}(x)\big] + a_1 B_{j,n}(x) \Bigg] B'_{i,n}(x) \\
& + a_0 B_{i,n}(x) B_{j,n}(x) \Bigg\} dx + \left[\frac{d^2}{dx^2}\big[a_6 B_{j,n}(x)\big] B'''_{i,n}(x)\right]_{x=b} - \left[\frac{d^2}{dx^2}\big[a_6 B_{j,n}(x)\big] B'''_{i,n}(x)\right]_{x=a} + \left[\frac{d^4}{dx^4}\big[a_6 B_{j,n}(x)\big] B'_{i,n}(x)\right]_{x=b}
\end{aligned}$$

$$-\left[\frac{d^4}{dx^4}\left[a_6B_{j,n}(x)\right]B'_{i,n}(x)\right]_{x=a}-\left[\frac{d}{dx}\left[a_5B_{j,n}(x)\right]B'''_{i,n}(x)\right]_{x=b}+\left[\frac{d}{dx}\left[a_5B_{j,n}(x)\right]B'''_{i,n}(x)\right]_{x=a}-\left[\frac{d^3}{dx^3}\left[a_5B_{j,n}(x)\right]B'_{i,n}(x)\right]_{x=b}$$

$$+\left[\frac{d^3}{dx^3}\left[a_5B_{j,n}(x)\right]B'_{i,n}(x)\right]_{x=a}+\left[\frac{d^2}{dx^2}\left[a_4B_{j,n}(x)\right]B'_{i,n}(x)\right]_{x=b}-\left[\frac{d^2}{dx^2}\left[a_4B_{j,n}(x)\right]B'_{i,n}(x)\right]_{x=a}-\left[\frac{d}{dx}\left[a_3B_{j,n}(x)\right]B'_{i,n}(x)\right]_{x=b}$$

$$+\left[\frac{d}{dx}\left[a_3B_{j,n}(x)\right]B'_{i,n}(x)\right]_{x=a} \tag{11b}$$

$$F_j=\int_a^b\left\{rB_{j,n}(x)+\left[\frac{d^5}{dx^5}\left[a_6B_{j,n}(x)\right]-\frac{d^4}{dx^4}\left[a_5B_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[a_4B_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[a_3B_{j,n}(x)\right]+\frac{d}{dx}\left[a_2B_{j,n}(x)\right]-a_1B_{j,n}(x)\right]\theta'_0(x)\right.$$

$$\left.-a_0\theta_0(x)B_{j,n}(x)\right\}dx+\left[\frac{d}{dx}\left[a_6B_{j,n}(x)\right]\right]_{x=b}\times B_4-\left[\frac{d}{dx}\left[a_6B_{j,n}(x)\right]\right]_{x=a}\times A_4+\left[\frac{d^3}{dx^3}\left[a_6B_{j,n}(x)\right]\right]_{x=b}\times B_2$$

$$-\left[\frac{d^3}{dx^3}\left[a_6B_{j,n}(x)\right]\right]_{x=a}\times A_2+\left[\frac{d}{dx}\left[a_5B_{j,n}(x)\right]\theta'''_0\right]_{x=b}-\left[\frac{d}{dx}\left[a_5B_{j,n}(x)\right]\theta'''_0\right]_{x=a}-\left[\frac{d^3}{dx^3}\left[a_5B_{j,n}(x)\right]\theta'_{i,n}\right]_{x=b}$$

$$+\left[\frac{d^3}{dx^3}\left[a_5B_{j,n}(x)\right]\theta'_{i,n}\right]_{x=b}-\left[\frac{d^2}{dx^2}\left[a_5B_{j,n}(x)\right]\right]_{x=b}\times B_2+\left[\frac{d^2}{dx^2}\left[a_5B_{j,n}(x)\right]\right]_{x=a}\times A_2-\left[\frac{d^2}{dx^2}\left[a_4B_{j,n}(x)\right]\theta'_{i,n}\right]_{x=b}$$

$$+\left[\frac{d^2}{dx^2}\left[a_4B_{j,n}(x)\right]\theta'_{i,n}\right]_{x=a}+\left[\frac{d}{dx}\left[a_4B_{j,n}(x)\right]\right]_{x=b}\times B_2-\left[\frac{d}{dx}\left[a_4B_{j,n}(x)\right]\right]_{x=a}\times A_2+\left[\frac{d}{dx}\left[a_3B_{j,n}(x)\right]\theta'_{i,n}\right]_{x=b}$$

$$-\left[\frac{d}{dx}\left[a_3B_{j,n}(x)\right]\theta'_{i,n}\right]_{x=a} \tag{11c}$$

Solving the system (11a), we find the values of the parameters α_i and then substituting these parameters into eqn. (3), we get the approximate solution of the desired BVP (1).

For nonlinear BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (11). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

4. NUMERICAL EXAMPLES

To test the applicability of the proposed method, we consider two linear and one nonlinear problems consisting of both types of boundary conditions. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by *MATLAB* 10. The convergence of linear BVP is calculated by

$$E=\left|\tilde{u}_{n+1}(x)-\tilde{u}_n(x)\right|<\delta,$$

where $\tilde{u}_n(x)$ denotes the approximate solution using n-th polynomials and δ depends on the problem which varies from 10^{-7} to 10^{-9}. In addition, the convergence of nonlinear BVP is assumed when the absolute error of two consecutive iterations, δ satisfies

$$\left|\tilde{u}_n^{N+1}-\tilde{u}_n^N\right|<\delta,$$

where N is the Newton's iteration number and δ varies from 10^{-11} to 10^{-13}.

Example 1: Consider the linear differential equation [9, 11, 17, 18, 19]

$$\frac{d^6u}{dx^6}-u=-6e^x,\ 0\le x\le 1 \tag{12a}$$

subject to boundary conditions of type I in eqn. (2a):

$$u(0)=1,\ u(1)=0,\ u'(0)=0, u'(1)=-e, u''(0)=-1, u''(1)=-2e. \tag{12b}$$

The analytic solution of the above problem is, $u(x)=(1-x)e^{x}$.

Using the method illustrated in the previous section and different number of polynomials, the maximum absolute errors and the previous results obtained so far, are summarized in Table 1.

Table 1: Observed maximum absolute errors of example 1.

Number of Polynomial used	Present method		Reference results
	Bernstein	Legendre	
10	1.216×10^{-13}	7.877×10^{-12}	1.4821×10^{-9}, Siddiqi *et al* [9] 7.46×10^{-9}, Siddiqi and Akram [11] 9.39×10^{-11}, Khan and Sultana [17] 4.3404×10^{-11}, Fazal-i-Haq *et al* [18] 2.55×10^{-9}, Akram and Siddiqi [19]
11	2.331×10^{-15}	1.185×10^{-13}	
12	2.220×10^{-16}	5.951×10^{-14}	
13	2.220×10^{-16}	8.771×10^{-15}	

Example 2: Consider the linear differential equation [10, 17, 20]

$$\frac{d^6u}{dx^6}+u=6(2x\cos x+5\sin x),\ -1\le x\le 1 \tag{13a}$$

subject to boundary conditions of type II in eqn. (2b):

$$u(-1)=u(1)=0,\ u''(-1)=-4\cos(-1)+2\sin(-1),\ u''(1)=4\cos 1+2\sin 1, \tag{13b}$$

$$u^{(iv)}(-1)=8\cos(-1)-12\sin(-1),\ u^{(iv)}(1)=-8\cos 1-12\sin 1. \tag{13c}$$

The analytic solution of the above BVP is, $u(x)=(x^2-1)\sin x$.

Using the method mentioned in this paper, the maximum absolute errors, shown in Table 2, are listed to compare with existing results obtained so far.

Table 2: Observed maximum absolute errors of example 2.

Number of Polynomial used	Present method		Reference results
	Bernstein	Legendre	
11	5.905×10^{-11}	5.905×10^{-11}	9.45×10^{-11}, Siraj-ul-Islam *et al* [10] 3.47×10^{-9}, Khan and Sultana [17] 5.801×10^{-5}, Loghmani and Ahmadinia [20]
12	7.655×10^{-14}	6.928×10^{-14}	
13	7.661×10^{-14}	6.932×10^{-14}	
14	2.776×10^{-16}	7.400×10^{-15}	

Example 3: Consider the nonlinear differential equation [15]

$$\frac{d^6u}{dx^6}=u^2e^{x},\ 0\le x\le 1 \tag{14a}$$

consisting of boundary conditions of type I defined in eqn. (2a)

$$u(0)=1,\ u(1)=e^{-1},\ u'(0)=-1, u'(1)=-e^{-1}, u''(0)=1, u''(1)=e^{-1}. \tag{14b}$$

The exact solution of this BVP is, $u(x)=e^{-x}$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x)=\theta_0(x)+\sum_{i=1}^{n}a_i\,B_{i,n}(x),\ n\ge 1 \tag{15}$$

Here $\theta_0(x) = 1 - x(1 - e^{-1})$ is specified by the essential boundary conditions in (14b). Also $B_{i,n}(0) = B_{i,n}(1) = 0$ for each $i = 1, 2, \ldots, n$.

Using (15) into equation (14a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^6 \tilde{u}}{dx^6} - \tilde{u}^2 e^x \right] B_{k,n}(x) dx = 0, k = 1, 2, \cdots, n \tag{16}$$

Integrating first term of (16) by parts, we obtain

$$\int_0^1 \frac{d^6 \tilde{u}}{dx^6} B_{k,n}(x)\, dx = \left[B_{k,n}(x) \frac{d^5 \tilde{u}}{dx^5} \right]_0^1 - \int_0^1 \frac{dB_{k,n}(x)}{dx} \frac{d^5 \tilde{u}}{dx^5} dx$$

$$= -\left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 \tilde{u}}{dx^4} \right]_0^1 + \int_0^1 \frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^4 \tilde{u}}{dx^4} dx \text{ [Since } B_{k,n}(1) = B_{k,n}(0) = 0 \text{]}$$

$$= -\left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 \tilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 - \int_0^1 \frac{d^3 B_{k,n}(x)}{dx^3} \frac{d^3 \tilde{u}}{dx^3} dx$$

$$= -\left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 \tilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^3 B_{k,n}(x)}{dx^3} \frac{d^2 \tilde{u}}{dx^2} \right]_0^1 + \int_0^1 \frac{d^4 B_{k,n}(x)}{dx^4} \frac{d^2 \tilde{u}}{dx^2} dx$$

$$= -\left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 \tilde{u}}{dx^4} \right]_0^1 + \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 \tilde{u}}{dx^3} \right]_0^1 - \left[\frac{d^3 B_{k,n}(x)}{dx^3} \frac{d^2 \tilde{u}}{dx^2} \right]_0^1 + \left[\frac{d^4 B_{k,n}(x)}{dx^4} \frac{d\tilde{u}}{dx} \right]_0^1$$

$$- \int_0^1 \frac{d^5 B_{k,n}(x)}{dx^5} \frac{d\tilde{u}}{dx} dx \tag{17}$$

Putting (17) into equation (16) and using approximation for $\tilde{u}(x)$, we obtain

$$\sum_{i=1}^{n} \Bigg[\int_0^1 \left[-\frac{d^5 B_{k,n}(x)}{dx^5} \frac{dB_{i,n}(x)}{dx} - 2\theta_0 e^x B_{i,n}(x) B_{k,n}(x) - \sum_{j=1}^{n} a_j (B_{i,n}(x) B_{j,n}(x) B_{k,n}(x)) e^x \right] dx$$

$$- \left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 B_{i,n}(x)}{dx^4} \right]_{x=1} \left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 B_{i,n}(x)}{dx^4} \right]_{x=0} + \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 B_{i,n}(x)}{dx^4} \right]_{x=1} - \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 B_{i,n}(x)}{dx^3} \right]_{x=0} \Bigg] \alpha_i$$

$$= \int_0^1 \left[\frac{d^5 B_{k,n}(x)}{dx^5} \frac{d\theta_0}{dx} + \theta_0^2 e^x B_{k,n}(x) \right] dx + \left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 \theta_0}{dx^4} \right]_{x=1} - \left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 \theta_0}{dx^4} \right]_{x=0} - \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 \theta_0}{dx^3} \right]_{x=1}$$

$$+ \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 \theta_0}{dx^3} \right]_{x=0} + \left[\frac{d^3 B_{k,n}(x)}{dx^3} \right]_{x=1} \times e^{-1} - \left[\frac{d^3 B_{k,n}(x)}{dx^3} \right]_{x=0} + \left[\frac{d^4 B_{k,n}(x)}{dx^4} \right]_{x=1} \times e^{-1} - \left[\frac{d^4 B_{k,n}(x)}{dx^4} \right]_{x=0} \tag{18}$$

The above equation (18) is equivalent to matrix form

$$(D + B)A = G \tag{19a}$$

where the elements of *A, B, D, G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k} = \int_0^1 \left[-\frac{d^5 B_{k,n}(x)}{dx^5} \frac{dB_{i,n}(x)}{dx} - 2\theta_0 e^x B_{i,n}(x) B_{k,n}(x) \right] dx - \left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 B_{i,n}(x)}{dx^4} \right]_{x=1}$$

$$+ \left[\frac{dB_{k,n}(x)}{dx} \frac{d^4 B_{i,n}(x)}{dx^4} \right]_{x=0} + \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 B_{i,n}(x)}{dx^3} \right]_{x=1} - \left[\frac{d^2 B_{k,n}(x)}{dx^2} \frac{d^3 B_{i,n}(x)}{dx^3} \right]_{x=0} \tag{19b}$$

$$b_{i,k} = -\sum_{j=1}^{n} a_j \int_0^1 (B_{i,n}(x)B_{j,n}(x)B_{k,n}(x))e^x\,dx \tag{19c}$$

$$g_k = \int_0^1 \left[\frac{d^5B_{k,n}(x)}{dx^5}\frac{d\theta_0}{dx} + \theta_0{}^2 e^x B_{k,n}(x)\right]dx + \left[\frac{d^3B_{k,n}(x)}{dx^3}\right]_{x=1} \times e^{-1} - \left[\frac{d^3B_{k,n}(x)}{dx^3}\right]_{x=0} + \left[\frac{d^4B_{k,n}(x)}{dx^4}\right]_{x=1} \times e^{-1} - \left[\frac{d^4B_{k,n}(x)}{dx^4}\right]_{x=0}$$
$$+\left[\frac{dB_{k,n}(x)}{dx}\frac{d^4\theta_0}{dx^4}\right]_{x=1} - \left[\frac{dB_{k,n}(x)}{dx}\frac{d^4\theta_0}{dx^4}\right]_{x=0} - \left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=1} + \left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=0} \tag{19d}$$

The initial values of these coefficients α_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (14a). That is, to find initial coefficients we will solve the system

$$DA = G \tag{20a}$$

where the matrices are constructed from

$$d_{i,k} = \int_0^1 \left[-\frac{d^5B_{k,n}(x)}{dx^5}\frac{dB_{i,n}(x)}{dx}\right]dx - \left[\frac{dB_{k,n}(x)}{dx}\frac{d^4B_{i,n}(x)}{dx^4}\right]_{x=1} + \left[\frac{dB_{k,n}(x)}{dx}\frac{d^4B_{i,n}(x)}{dx^4}\right]_{x=0} + \left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^3B_{i,n}(x)}{dx^3}\right]_{x=1}$$
$$-\left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^3B_{i,n}(x)}{dx^3}\right]_{x=0} \tag{20b}$$

$$g_k = \int_0^1 \left[\frac{d^5B_{k,n}(x)}{dx^5}\frac{d\theta_0}{dx} + \theta_0{}^2 e^x B_{k,n}(x)\right]dx + \left[\frac{d^3B_{k,n}(x)}{dx^3}\right]_{x=1} \times e^{-1} - \left[\frac{d^3B_{k,n}(x)}{dx^3}\right]_{x=0} + \left[\frac{d^4B_{k,n}(x)}{dx^4}\right]_{x=1} \times e^{-1} - \left[\frac{d^4B_{k,n}(x)}{dx^4}\right]_{x=0}$$
$$+\left[\frac{dB_{k,n}(x)}{dx}\frac{d^4\theta_0}{dx^4}\right]_{x=1} - \left[\frac{dB_{k,n}(x)}{dx}\frac{d^4\theta_0}{dx^4}\right]_{x=0} - \left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=1} + \left[\frac{d^2B_{k,n}(x)}{dx^2}\frac{d^3\theta_0}{dx^3}\right]_{x=0} \tag{20c}$$

Once the initial values of the α_i are obtained from eqn. (20a), they are substituted into eqn.(19a) to obtain new estimates for the values of α_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (15), we obtain an approximate solution of the BVP (14).

The maximum absolute errors, for different number of polynomials, are shown in Table 3 with 7 iterations.

Table 3: Observed maximum absolute errors of example 3 using 7 iterations.

Number of Polynomial used	Present method		Reference results
	Bernstein	Legendre	
6	2.145×10^{-7}	2.143×10^{-7}	1.389×10^{-6}, Wazwaz [15]
8	2.242×10^{-7}	2.240×10^{-7}	
10	3.114×10^{-7}	3.114×10^{-7}	
11	9.542×10^{-8}	9.540×10^{-8}	

Example 4: Consider the nonlinear boundary value problem [10]

$$\frac{d^6u}{dx^6} = 20e^{-36u} - 40(1+x)^{-6},\ 0 \le x \le 1 \tag{21a}$$

with boundary conditions type II, defined in eqn. (2b)

$$u(0)=0,\ u(1)=\frac{1}{6}\ln 2,\ u''(0)=-\frac{1}{6}, u''(1)=-\frac{1}{24}, u^{(iv)}(0)=-1, u^{(iv)}(1)=-\frac{1}{16}. \tag{21b}$$

The exact solution of this BVP is, $u(x) = \frac{1}{6}\ln(1+x)$.

Following the proposed method in this paper and as in example 3; the maximum absolute errors for this problem are summarized in Table 4.

Table 4: Observed maximum absolute errors of example 4 using 5 iterations.

Number of Polynomial used	Present method		Reference results
	Bernstein	Legendre	
7	8.915×10^{-7}	8.915×10^{-7}	2.68×10^{-11}, Siraj-ul-Islam *et al* [10]
8	8.908×10^{-8}	8.905×10^{-8}	
9	9.870×10^{-9}	9.864×10^{-9}	
10	4.794×10^{-11}	4.790×10^{-11}	

5. CONCLUSIONS

In this paper, Galerkin method has been applied for the approximate solution of sixth-order BVPs using Bernstein and Legendre polynomials as basis functions with two different types of boundary conditions. We have concentrated our attention not only on the performance of the results but also on the formulation. Some numerical examples of both linear and nonlinear BVPs have been demonstrated to verify the efficiency of the proposed method. We have found a good agreement with the exact solutions and some results are better than the results obtained by the previous methods so far. The proposed method can be coded easily and may be extended for numerical solutions of any even higher order BVPs as well.

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