ISSN 2347-1921

The Best Approximation by Using Bernstein's Polynomial in $L_p\,[0,1]$

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Abstract

We estimate in this paper the degree of approximation of $f \in L_p\,[0,1]$ by using Bernstein's polynomial involving on Ditizian-Totik modulus of smoothness $W^{r}_{\varphi^{\lambda}}(f,t)$, $0 \le \lambda \le 1$. On the other hand, we consider this polynomial to obtain an equivalence approximation theorem with this modulus.

Keywords: Bernstein's polynomial; Moduli of smoothness.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics

Vol 7, No. 2
editor@cirworld.com
www.cirworld.com, member.cirworld.com

Introduction

Let $L_p[0,1]$ be the set of all Lebesgue integrable functions defined on $[0,1]$ and $0 \le p \le \infty$. For $f \in L_p[0,1]$ and $n \in N$, the Bernstein's operators is defined as:

$$B_n(f,x) = \sum_{k=0}^{\infty} P_{n,k}(x)\, f\left(\frac{k}{n}\right), \qquad \dots (1.1)$$

Where $P_{n,k}(x) = \binom{n+k-1}{k} \frac{x^K}{(1+x)^{n+k}}$.

On the other hand, we consider combinations of these operators, which have higher orders of approximation. The linear combinations of Bernstien's operators on $L_p[0,1]$ are defined as (see[3]):

$$B_{n,r}(f,x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f,x), \qquad r \in N \qquad \dots (1.2)$$

Where n_i and $C_i(n)$ satisfy

$n = n_0 < n_1 < n_2 < \cdots < n_{r-1} \le D_n$ and

$$\sum_{i=0}^{r-1} C_i(n) \le C$$

where the constants C and D are independent of n.

In [2] the Ditizian-Totik modulus of smoothness $W^r{}_{\varphi^\lambda}(f,t)$ was used for polynomial approximation and defined by:

$$W^r_{\varphi^\lambda}(f,t) = \sup_{0 \le h \le t} \sup_{x \pm (rh\varphi^\lambda(x)/2) \in [0,1]} \left|\Delta_{h\varphi^\lambda(x)}(f,x)\right| \qquad \dots (1.3)$$

where, $\varphi(x) = \sqrt{x(1-x)}$ for $x \in [0,1]$, $0 \le \lambda \le 1$

and the r –th symmetric difference of f is given by:

$$\Delta^r{}_{h\varphi^\lambda}(f,x) = \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f\left(x - \frac{rh\varphi^\lambda}{2} + ih\varphi^\lambda\right), & x \pm \left(\frac{rh\varphi^\lambda}{2}\right) \in [0,1] \\ 0, & o.w. \end{cases} \qquad \dots (1.4)$$

And our k –functional by:

$$K_{\varphi^\lambda}(f,t^r) = \inf\{ \|f-g\| + t^r \|\varphi^{r\lambda} g^{(r)}\| \} \qquad \dots (1.5)$$

$$\widetilde{K}_{\varphi^\lambda}(f,t^r) = \inf\{ \|f-g\| + t^r \|\varphi^{r\lambda} g^{(r)}\| + t^{r/(1-\lambda/2)} \|g^{(r)}\| \}$$

and $\quad K_{\varphi^\lambda}(f,t^r) \le Ct^\alpha$ for $r \le \alpha \qquad \dots (1.6)$

where the infimum is taken on the functions satisfying $g^{(r-1)}$ is absolutely continuous function, see [3]. It is well known the following equivalence see [1]

$$W^r_{\varphi^\lambda}(f,t) \sim K_{\varphi^\lambda}(f,t^r) = \widetilde{K}_{\varphi^\lambda}(f,t^r). \qquad \dots (1.7)$$

Let the Sobolev space W^r_λ be the collection of all functions f defined on [0,1], such that $f^{(r-1)}$ is absolutely continuous and we defined as:

$$W^r_\lambda = \{ f \in C_0 : f^{(r-1)} \in A.C_{10c}, \|f\|_r < \infty \} \qquad \dots (1.8)$$

where $\|f\|_r = \sup_{x \in [0,1]} \delta_n{}^{r+a(\lambda-1)}(x)\, f^{(r)}(x)|$

and $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}} \sim \max\left\{\varphi(x), \frac{1}{\sqrt{n}}\right\}$, $0 \le \lambda \le 1, r \in N$ and $0 < \alpha < r$.

2. Auxiliary Lemmas

In this section, we mention some basis results which will be used to prove the main results. If $f \in L_p[0,1]$, $r \in N$ then by [3], we know that:

$$B_n{}^{(r)}(f,x) = \frac{(n+r-1)!}{(n-1)!} \sum_{k=0}^{\infty} P_{n+r,k}(x) \Delta^r{}_{n^{-1}} f\left(\frac{k}{n}\right).$$

Lemma 2.1 [6]:

Let $f^{(r)} \in L_p[0,1]$, $r \in N$ then we have:

ISSN 2347-1921

$$\left|\varphi^{r}(x)B_n{}^{(r)}(f,x)\right| \le C\, n^{r/2}\|f\|_\infty \,. \qquad \text{... (2.1)}$$

Lemma 2.2:

Let $f \in L_p[0,1]$, $1 \le p \le \infty$ and $0 \le \lambda \le 1$ then we have:

$$\left|\varphi^{r\lambda}(x)\, B_n{}^{(r)}(f,x)\right| \le C\, \|\varphi^{r\lambda} f\|_\infty \,.$$

Proof:

By using Holder inequality, we get

$$\left|\varphi^{r\lambda}(x)\, B_n{}^{(r)}(f,x)\right| \le \left|\varphi^{r\lambda}(x)\frac{(n+r-1)!}{(n-1)!}\sum_{k=0}^{+\infty} P_{n+r,k}(x)\Delta^{r}{}_{n^{-1}} f\left(\frac{k}{n}\right)\right|$$

$$\le \frac{(n+r-1)!}{(n-1)!}\left|\varphi^{r\lambda}(x)P_{n+r,0}(x)\Delta^{r}{}_{n^{-1}} f(\,0\,) + \sum_{k=1}^{+\infty}\varphi^{r\lambda}(x)P_{n+r,k}(x)\Delta^{r}{}_{n^{-1}} f\left(\frac{k}{n}\right)\right|$$

$$= \frac{(n+r-1)!}{(n-1)!}\left|\varphi^{r\lambda}(x)\sum_{k=1}^{+\infty} P_{n+r,k}(x)\Delta^{r}{}_{n^{-1}} f\left(\frac{k}{n}\right)\right|$$

$$\le \frac{(n+r-1)!}{(n-1)!}\left|\varphi^{r\lambda}(x)\right|\sum_{k=1}^{+\infty}\left|P_{n+r,k}(x)\Delta^{r}{}_{n^{-1}} f\left(\frac{k}{n}\right)\right|$$

$$\le C\, \left\|\varphi^{r\lambda}(x)\, f\right\|_\infty \,.$$

Lemma 2.3:

Let $f \in L_p[0,1]$, $r \in N$, $1 \le p \le \infty$ and $0 \le \lambda \le 1$ then for $n > r$, we get

$$\left|\varphi^{r\lambda}(x)\, B_n{}^{(r)}(f,x)\right| \le C\, n^{r/2}\delta_n{}^{-r(1-\lambda)}(x)\|f\|_\infty \,. \qquad \text{... (2.2)}$$

Proof:

We consider $x \in \left[0,\frac{1}{n}\right]$, then we have:

$\delta_n(x) \sim \frac{1}{\sqrt{n}}$, $\varphi(x) \le \frac{2}{\sqrt{n}}$ and using

$$\frac{d}{dx}P_{n,k}(x) = n\,(P_{n+1,k-1}(x) - P_{n+1,k}(x))$$

and

$$\int_0^\infty P_{n,k}(x)\frac{d}{dx} = \frac{1}{n-1}\,.$$

On the other hand, by using (2.1) and $\varphi(x) \sim \sigma(x)$, therefore

$$\left|\varphi^{r\lambda}(x)\, B_n{}^{(r)}(f,x)\right| = \varphi^{r(\lambda-1)}(x)\left|\varphi^{r}(x)\, B_n{}^{(r)}(f,x)\right|$$

$$\le C\,\varphi^{r(\lambda-1)}(x)\, n^{r/2}\|f\|_\infty$$

$$\le C\, n^{r/2}\delta_n{}^{-r(1-\lambda)}(x)\|f\|_\infty \,.$$

Lemma 2.4 [7]:

Let $r \in N$, $0 \le \beta \le r$, $x \pm {}^{rt}/_2 \in I$ and $0 \le t \le \frac{1}{8r}$, then we have the following inequality :

$$\int_{-t/2}^{t/2}\cdots\int_{-t/2}^{t/2}\delta_n{}^{-\beta}\left(x + \sum_{j=1}^{r} u_j\right)du_1\, du_2 \cdots du_r \le C(\beta)\, t^r \delta_n{}^{-r}(x)\,. \qquad \text{... (2.3)}$$

Lemma 2.5 [3]:

For $x, t \in (0,\infty)$, $r \in N$ then we have:

$$\beta_n((t-x)^{2r}, x\,) \le Cn^{-r}\,\varphi^{2r}(x). \qquad \text{... (2.4)}$$

Lemma 2.6:

For $x, t, u \in [0,1]$, $x < u < t$, $t, r \in N$ and $\lambda \in [0,1]$ then:

$$\beta_n\left(\left|\int_{\alpha}^{t}|t-4|^{r-1}\varphi^{-r\lambda}(u)du\right|,x\right)\leq C\ n^{-r/2}\delta_n{}^{r}(x)\varphi^{-r\lambda}(x). \qquad \text{... (2.5)}$$

Proof:

When $r = 1$, then

$$\left|\int_{\alpha}^{t}\varphi^{-\lambda}(u)du\right|\leq|t-r|\{x^{-\lambda/2}(1+t)^{-\lambda/2}+(1+x)^{-\lambda/2}t^{-\lambda/2}\}. \qquad \text{... (2.6)}$$

From lemma 2.5, then

$$\beta_n((t-x)^{2r},x)\leq Cn^{-r}\varphi^{2r}(x). \qquad \text{... (2.7)}$$

Applying the Holder inequality, we get

$$\beta_n\left(\left|\int_{\alpha}^{t}\varphi^{-\lambda}(u)du\right|,x\right)\leq\{\beta_n((t-x)^4,x)\}^{1/4}\left[x^{-\lambda/2}\{\beta_n\left((1+t)^{-2\lambda/3},x\right)\}^{3/4}+(1+x)^{-\lambda/2}\{\beta_n\left(t^{-2\lambda/3},x\right)\}^{3/4}\right]$$
$$\leq C\ n^{-1/2}\delta_n(x)\left[x^{-\lambda/2}\{\beta_n\left((1+t)^{-2\lambda/3},x\right)\}^{3/4}\right.$$
$$\left.+(1+x)^{-\lambda/3}\{\beta_n\left(t^{-2\lambda/3},x\right)\}^{3/4}\right] \qquad \text{... (2.8)}$$

Also, applying the Holder inequality, we get

$$\beta_n\left(t^{-2\lambda/3},x\right)\leq\left(\sum_{k=0}^{+\infty}P_{n,k}(x)\left(\frac{k}{n}\right)^{-1}\right)^{2\lambda/3}\leq C\,x^{-2\lambda/3}. \qquad \text{... (2.9)}$$

Similarly:

$$\beta_n\left((1+t)^{-2\lambda/3},x\right)\leq C\,(1+x)^{-2\lambda/3}. \qquad \text{... (2.10)}$$

Combining (2.8) and (2.10), we obtain (2.5).

When $r > 1$, then we have

$$\frac{|t-u|^2}{\varphi^2(u)}\leq\frac{|t-x|^2}{\varphi^2(x)},\quad \text{for } t<u<x$$

$|t-u|x\leq|t-x|u$ and

$$\frac{u|t-u|}{\varphi^2(u)}\leq\frac{x|t-x|}{\varphi^2(x)}$$

Thus

$$\frac{|t-u|^{r-2}}{\varphi^{(r-2)\lambda}(u)}\leq\frac{|t-x|^{r-2}}{\varphi^{(r-2)\lambda}(x)},\quad r>2 \qquad \text{... (2.11)}$$

because

$$\frac{|t-u|}{\varphi^2(u)}\leq\frac{|t-x|}{x}\left(\frac{1}{1+x}+\frac{1}{1+t}\right) \qquad \text{... (2.12)}$$

Otherwise,

$$\frac{(u-t)}{u}\leq\frac{(x-t)}{t}$$

and

$$\frac{1}{1+u}\leq\frac{1}{1+x}+\frac{1}{1+t},$$

$$\frac{|t-u|}{\varphi^{2\lambda}(u)}\leq\frac{|t-x|}{x^{\lambda}}\left(\frac{1}{1+x}+\frac{1}{1+t}\right)^{\lambda} \qquad \text{... (2.13)}$$

By using (2.11) and (2.13), we obtain

$$\frac{|t-u|^{r-1}}{\varphi^{r\lambda}(u)}\leq\frac{|t-x|^{r-1}}{\varphi^{(r-2)\lambda}(x)}\left\{\left(\frac{1}{1+x}\right)^{\lambda}+\left(\frac{1}{1+t}\right)^{\lambda}\right\} \qquad \text{... (2.14)}$$

ISSN 2347-1921

Since $\beta_n((1+t)^{-r},x) \le C(1+x)^{-r}$, for $n \in N$, $x \in [0,\infty]$, we have

$\beta_n((1+t)^{-2\lambda},x) \le C(1+x)^{-2\lambda}$, for $\lambda \in [0,1]$. ... (2.15)

Now, by using (2.7), (2.14) and (2.15) and by using Holder inequality, then we get

$$\beta_n\left(\left|\int_\alpha^t |t-u|^{r-1}\varphi^{-r\lambda}(u)du\right|,x\right) \le \beta_n^{1/2}(|t-u|^{2r},u)\left(\varphi^{-r\lambda}(x)+x^{-\lambda}\varphi^{-(r-2)\lambda}(x)\beta_n^{1/2}((1+t)^{-2\lambda},x)\right)$$

$$\le C\ n^{-r/2}\delta_n^{\ r}(x)\varphi^{-r\lambda}(x).$$

Lemma 2.7 [3]:

Let $x \in \left[0,\frac{1}{n}\right]$, then:

$$B_n^{(r)}(f,x) = \varphi^{-2r}\sum_{i=0}^{r} q_i(n\varphi^2(x))\, n^i \sum_{k=0}^{\infty} P_{n,k}(x)\left(\frac{k}{n}-x\right)^i \Delta^r_{\frac{1}{n}} f\left(\frac{k}{n}\right) \quad ... (2.16)$$

Where $q_i(n\varphi^2(x))$ is a polynomial in $n\varphi^2(x)$ of degree $(r-i)/2$.

Lemma 2.8:

For $r \in N$ and $f \in W^r{}_\lambda$ and $0 \le \alpha \le r$ then we have

$\|B_n f\|_r \le C\ n^{r/2}\|f\|_0$ . ... (2.17)

$\|B_n f\|_r \le C\ \|f\|_r$ . ... (2.18)

Proof:

For $x \in \left[0,\frac{1}{n}\right]$ and $\delta_n(x) \sim \frac{1}{\sqrt{n}}$ according to lemma 2.3, we get

$\left|\delta_n^{\ r+\alpha(\lambda-1)}(x)B_n^{(r)}(f,x)\right| \le C\ n^{r/2}\|f\|_0$

On the other hand, for $x \in \left[\frac{1}{n},1\right]$ and $\delta_n(x) \sim \varphi(x)$ and by using lemma 2.7, (2.16) we can obtain

$$B_n^{(r)}(f,x) = \varphi^{-2r}(x)\sum_{i=0}^{r} q_i(n\varphi^2(x))\, n^i \sum_{k=0}^{\infty} P_{n,k}(x)\left(\frac{k}{n}-x\right)^i \Delta^r{}_{n^{-1}} f\left(\frac{k}{n}\right),$$

therefore

$$\left|\varphi^{-2r}(x)\, q_i(nx)\, n^i\right| \le C\left(\frac{n}{\varphi^2(x)}\right)^{(r+i)/2} \quad ... (2.19)$$

For any $x \in \left[\frac{1}{n},1\right]$ and since

$$\left|\Delta^r{}_{n^{-1}} f\left(\frac{k}{n}\right)\right| \le C\ \|f\|_0\, \varphi^{\alpha(\lambda-1)}(x), \quad ... (2.20)$$

by using the Holder inequality, we get

$$\left|\sum_{k=0}^{\infty} P_{n,k}(x)\left(\frac{k}{n}-x\right)^i \Delta^r{}_{n^{-1}} f\left(\frac{k}{n}\right)\right| \le C\left(\frac{\varphi^2(x)}{n}\right)^{i/2} \|f\|_0\, \varphi^{\alpha(\lambda-1)}(x) \quad ... (2.21)$$

From lemma 2.7, (2.16) and (2.21) we can deduce (2.17). Similarly by using lemma 2.2 we can obtain (2.18).

Lemma 2.9 [1]:

Let $f \in L_p[0,1]$, $r \in N$, $0 \le \lambda \le 1$, then

$\|f-g_n\| \le C\, W^r{}_{\varphi^\lambda}(f,S_n)$, ... (2.22)

$$S_n^{\ r}\left\|\varphi^{r\lambda} g_n^{(r)}\right\| \le C\, W^r{}_{\varphi^\lambda}(f,S_n). \quad ... (2.23)$$

3. Main Results

Theorem 3.1:

Let $f \in L_p[0,1]$, $1 \le p \le \infty$, $r \in N$ and $0 \le \lambda \le 1$, then

$$\left|B_{n,r}(f,x)-f(x)\right| \le C\, W^r_{\varphi^\lambda}\left(f,\ n^{-1/2}\delta_n{}^{1-\lambda}(x)\right), \qquad \text{... (3.1)}$$

involving the constant C dependent on.

Proof:

By using (1.5) and (1.7) and taking $\delta_n = \delta_n(x,\lambda) = n^{-1/2}\delta_n{}^{1-\lambda}(x)$, we can choose $g_n = g_{n,x,\lambda}$ for fixed x and y and by lemma 2.9, (2.22) and (2.23), then we have

$$\delta_n{}^{r/(1-(\lambda/2))}\left\|g_n{}^{(r)}\right\| \le C\, W^r_{\varphi^\lambda}(f,\delta_n). \qquad \text{... (3.2)}$$

Now,

$$\left|B_{n,r}(f,x)-f(x)\right| = \left|B_{n,r}(f,x)-B_{n,r}(g_n,x)+B_{n,r}(g_n,x)+f-g_n(x)+g_n(x)\right| \qquad \text{... (3.3)}$$

$$\le \left|B_{n,r}(f,x)-B_{n,r}(g_n,x)\right| + |f-g_n(x)| + \left|B_{n,r}(g_n,x)-g_n(x)\right|$$

$$\le C\,\|f-g_n\|_\infty + \left|B_{n,r}(g_n,x)-g_n(x)\right|.$$

By using Taylor's series

$$g_n(t) = g_n(x) + (t-x)g_n'(x) + \cdots + \frac{(t-x)^{r-1}}{(r-1)!}g_n{}^{(r-1)}(x) + R_r(g_n,t,x),$$

Where

$$R_r(g_n,t,x) = \frac{1}{(r-1)!}\int_\alpha^t (t-u)^{r-1}\, g_n{}^{r-1}(u)du\,.$$

By using (1.2), (1.3) and lemma 2.6, (2.5) we obtain

$$\left|B_{n,r}(g_n,x)-g_n(x)\right| = \left|B_{n,r}\left(\frac{1}{(r-1)!}\int_\alpha^t (t-u)^{r-1}\, g_n{}^{r-1}(u)du\ ,x\right)\right| \qquad \text{... (3.4)}$$

$$\le C\left\|\varphi^{r\lambda}g_n{}^{(r)}\right\|\left(B_{n,r}\left(\left|\int_\alpha^t \frac{|t-u|^{r-1}}{\varphi^{r\lambda}(u)}du\right|,x\right)\right)$$

$$\le C\,\varphi^{r\lambda}(x)\, n^{-r/2}\delta_n{}^r(x)\left\|\varphi^{r\lambda}g_n{}^{(r)}\right\|$$

Also, by using (1.2), (1.3) and (2.11), we get

$$\left|B_{n,r}(g_n,x)-f(x)\right| \le C\left\|\delta_n{}^{r\lambda}g_n{}^{(r)}\right\|\left|B_{n,r}\left(\int_\alpha^t \frac{|t-u|^{r-1}}{\delta_n{}^{r\lambda}(u)}du\ ,x\right)\right| \qquad \text{... (3.5)}$$

$$\le C\left\|\delta_n{}^{r\lambda}g_n{}^{(r)}\right\| n^{r\lambda/2}\left|B_{n,r}((t-x)^{2r}\ ,x)^{1/2}\right|$$

$$\le C\ n^{-r/2}\delta_n{}^r(x)\, n^{r\lambda/2}\left\|\delta_n{}^{r\lambda}g_n{}^{(r)}\right\|.$$

Then by using lemma 2.9,(3.3), (3.4) and (3.5) and for $x \in \left[0,\frac{1}{n}\right]$ and $\delta_n(x) \sim \frac{1}{\sqrt{n}}$

$$\left|B_{n,r}(f,x)-f(x)\right| \le C\left(\|f-g_n\| + \delta_n{}^r\left\|\delta_n{}^{r\lambda}g_n{}^{(r)}\right\|\right)$$

$$\le C\left(\|f-g_n\| + \delta_n{}^r\left\|\varphi^{r\lambda}g_n{}^{(r)}\right\| + \delta_n{}^r\ n^{-r\lambda/2}\left\|g_n{}^{(r)}\right\|\right)$$

$$\le C\left(\|f-g_n\| + \delta_n{}^r\left\|\varphi^{r\lambda}g_n{}^{(r)}\right\| + \delta_n{}^{r/(1-(\lambda/2))}\left\|g_n{}^{(r)}\right\|\right)$$

$$\le C\,\widetilde{K}_{\varphi^\lambda}(f,\delta_n) \le C\, W^r{}_{\varphi^\lambda}(f,\delta_n)\,.$$

Theorem 3.2:

Let $f \in L_p[0,1]$, $r \in N$, $0 < \alpha < r$, and $0 \le \lambda \le 1$, and if

$$\left\|B_{n,r}(f,x)-f(x)\right\|_p = O(\delta_n{}^r) \qquad \text{... (3.6)}$$

Then we have

$$W^r_{\varphi^\lambda}(f,t) = O(t^\alpha) \qquad \text{... (3.7)}$$

Where $\delta_n = n^{-1/2}\delta_n{}^{1-\lambda}(x)$ and $p = 0$.

Proof:

Suppose that $\|B_{n,r}(f,x)-f(x)\|_0 = O(\delta_n{}^r)$ holds.

Now, we introduce a new k –functional as:

$K_\lambda{}^\alpha(f,t^r) = \inf_{g\in W^r{}_\lambda}\{\|f-g\|_0 + t^r\|g\|_r\}$

for $g \in W^r{}_\lambda$, such that

$\|f-g\|_0 + n^{-r/2}\|g\|_r \le 2\,K_\lambda{}^\alpha(f, n^{-r/2})$ . ... (3.8)

By (3.6), we can deduce that

$\|B_{n,r}(f,x)-f(x)\|_0 \le C\, n^{-\alpha/2}$.

Thus, by using lemma 2.8 and (3.8), we obtain

$$K_\lambda{}^\alpha(f, t^r) \le \|f-B_{n,r}(f)\|_0 + t^r\|B_{n,r}(f)\|_r$$
$$\le C\, n^{-\alpha/2} + t^r\left(\|B_{n,r}(f-g,x)\|_r + \|B_{n,r}(g,x)\|_r\right)$$
$$\le C\left(n^{-\alpha/2} + t^r\left(n^{r/2}\|f-g\|_0 + \|g\|_r\right)\right)$$
$$\le C\left(n^{-\alpha/2} + \frac{t^r}{n^{-r/2}}K_\lambda{}^\alpha(f, n^{-r/2})\right)$$

Which implies that

$K_\lambda{}^\alpha(f, t^r) \le C\,t^\alpha$ . ... (3.9)

On the other hand, since $x + \left(i-\frac{r}{2}\right)t\,\varphi^\lambda(x) \ge 0$

therefore $\left|\left(i-\frac{r}{2}\right)t\,\varphi^\lambda(x)\right| \le x$

and $x + \left(i-\frac{r}{2}\right)t\,\varphi^\lambda(x) \le 2x$

so that $\delta_n\left(x + \left(i-\frac{r}{2}\right)t\,\varphi^\lambda(x)\right) \le 2\delta_n(x)$. ... (3.10)

Thus, for $f \in W^0{}_\lambda$, we get

$$|\Delta^r{}_{t\varphi^\lambda(x)}\, f(x)| \le \|f\|_0\left(\sum_{i=0}^{r}\binom{r}{i}\delta_n{}^{\alpha(1-\lambda)}\left(x + \left(i-\frac{r}{2}\right)t\,\varphi^\lambda(x)\right)\right) \quad \text{... (3.11)}$$
$$\le 2^{2r}\delta_n{}^{\alpha(1-\lambda)}(x)\|f\|_0 .$$

From lemma 2.4 and for $g \in W^r{}_\lambda$, $0 \le t\,\varphi^\lambda(x) \le \frac{1}{\delta r}$, $x \pm rt\,\varphi^\lambda(x)/2 \in [0,1]$, then we have

$$|\Delta^r{}_{t\varphi^\lambda(x)}\, g(x)| \le \left|\int_{-(t/2)\varphi^\lambda(x)}^{(t/2)\varphi^\lambda(x)} \cdots \int_{-(t/2)\varphi^\lambda(x)}^{(t/2)\varphi^\lambda(x)} g^{(r)}\left(r + \sum_{i=1}^{r}u_i\right)du_1du_2\cdots du_r\right| \quad \text{... (3.12)}$$
$$\le \|g\|_r \int_{-(t/2)\varphi^\lambda(x)}^{(t/2)\varphi^\lambda(x)} \cdots \int_{-(t/2)\varphi^\lambda(x)}^{(t/2)\varphi^\lambda(x)} \delta_n{}^{-r+\alpha(1-\lambda)}\left(1 + \sum_{i=1}^{r}u_i\right)du_1du_2\cdots du_r$$

Then we have

$|\Delta^r{}_{t\varphi^\lambda(x)}\, g(x)| \le Ct^r\delta_n{}^{-r+\alpha(1-\lambda)}(x)\|g\|_r$.

Now, using (3.9), (3.11) and (3.12) for $0 \le t\,\varphi^\lambda(x) \le \frac{1}{\delta r}$, $x \pm rt\,\varphi^\lambda(x)/2 \in [0,1]$

Then we have

$$|\Delta^r{}_{t\varphi^\lambda(x)}\, f(x)| = |\Delta^r{}_{t\varphi^\lambda(x)}\,(f-g+g)(x)|$$
$$\le |\Delta^r{}_{t\varphi^\lambda(x)}\,(f-g)(x)| + |\Delta^r{}_{t\varphi^\lambda(x)}\, g(x)|$$
$$\le C\,\delta_n{}^{\alpha(1-\lambda)}(x)K_\lambda{}^\alpha\left(f, \frac{t^r}{\delta_n{}^{(1-\lambda)}(x)}\right) \le C\,t^r .$$

ISSN 2347-1921

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