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Hyers-Ulam Stability Of Abstract First-Order Dynamic Equations On Time Scales

Alaa E. Hamza
Department of Mathematics, Faculty of Science. Cairo University
hamzaaeg2003@yahoo.com
Nesreen A.Yaseen
Department of information systems Higher institute for computers science. El-Shoruk academy
yehya_73@yahoo.com

ABSTRACT

In this paper we investigate the Hyers-Ulam Stability of the abstract dynamic equation of the form

$$x^{\Delta}(t) = A(t)x(t) + f(t), \qquad t \in \mathbb{T}, \qquad x(t_0) = x_0 \in \mathbb{X},$$

where $A: \mathbb{T} \to L(\mathbb{X})$ (The space of all bounded linear operators from a Banach space $\mathbb{X}$ into itself) and f is rd-continous from a time scale $\mathbb{T}$ to $\mathbb{X}$. Some examples illustrate the applicability of the main results.

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1. INTRODUCTION

In the past decades, stability analysis of dynamic systems has been become an important topic both theoretically and practically because dynamic systems occur in many areas such as mechanics, physics, and economics.

In 1998, Ger and Alsina [11] were the first authors who investigated the Hyers-Ulam stability of differential equations.

Since then a significant interest in Hyers-Ulam stability, especially in relation to ordinary differential equations, was introduced in [9,10,12,13,14,15,17,18,20,22]. Also of interest, many articles are dealing with Hyers-Ulam stability edited by Rassias [21]. In 2005 Popa [19] proved the Hyers-Ulam stability of a linear recurrence with constant coefficient, Also Wang, Zhou and Sun introduced the Hyers-Ulam stability of linear differential equations of first order [24], see also the more recent [6,9].

In 2012, Douglas R. Anderson, BenGates, and Dylan Heuer [4] introduced a lemma, which establishes the Hyers-Ulam stability of linear first order delta dynamic equations.

In this paper we generalize and extend the work of Douglas R. Anderson, BenGates and Dylan Heuer [4] to investigate the Hyers-Ulam stability of the abstract dynamic equation of the form :

$$x^{\Delta}(t) = A(t)x(t) + f(t) \ , t \in \mathbb{T}, t > t_0, \ x(t_0) = x_0 \ \in \mathbb{X}, \tag{1.1}$$

where $A \in C_{rd}(\mathbb{T}, L(X))$, the space of all rd-continuous from a time scale $\mathbb{T}$ to $L(X)$ and $f \in C_{rd}(\mathbb{T}, \mathbb{X})$.

It is well known [3, 5] that if $A \in C_{rd}(\mathbb{T}, L(X))$ and $f \in C_{rd}(\mathbb{T}, \mathbb{X})$ such that

1) $sup_t \|A(t)\| < \infty$,

2) A is regressive i.e. $(I + \mu(t)A(t))$ is invertible $\forall t \in \mathbb{T}$,

then equation (1.1) has the unique solution $x(t) = e_A(t, t_o)\, x_0$, where e_A is the operator exponential function. For more details about properties of e_A see [2].

2. Preliminaries

We need the following definitions and notations from [5] in proving our main results in Section 3.

Definition 2.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Definition 2.2. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = inf\{s \in \mathbb{T}: s > t\}$, and $\rho(t) = sup\{s \in \mathbb{T}: s < t\}$ are called the jump operators.

Definition 2.3. A point $t \in \mathbb{T}$ is said to be right–dense if $\sigma(t) = t$, right-scattered $if\ \sigma(t) > t$, left-dense $if\ \rho(t) = t$, left-scattered $if\ \rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$, and dense $if\ \rho(t) = t = \sigma(t)$.

Definition 2.4. Let $t \in \mathbb{T}$. The graininess function

$\mu: \mathbb{T} \to [0, \infty[$ is defined as $\mu(t) = \sigma(t) - t$.

Definition 2.5. A function $f: \mathbb{T} \to \mathbb{X}$ is called rd-continuous provided

(i) f is continuous at every right –dense point;

(ii) $lim_{s \to t^-} f(s)$ exists (finite) at left-dense points in $\mathbb{T}$.

The set of rd-continuous functions $f: \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.6. (The Delta Derivative). A function $f: \mathbb{T} \to \mathbb{X}$ is called Δ-differentiable at $t \in \mathbb{T}^k$ if there exists an element $f^{\Delta}(t) \in \mathbb{X}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that:

$$\left\|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right\| \le \varepsilon|\sigma(t) - s|, s \in (t - \delta, t + \delta) \cap \mathbb{T}.$$

In this case $f^{\Delta}(t)$ is called the delta derivative of f at t, provided it exists and we have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}. \tag{2.1}$$

If $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$, we say that f is delta differentiable on $\mathbb{T}^k$, where

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & if \sup\mathbb{T} < \infty \\ \mathbb{T} & if \sup\mathbb{T} = \infty. \end{cases}$$

The set of all functions $f: \mathbb{T} \to \mathbb{X}$ that are differentiable and whose derivatives are rd-continuous is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{X})$.

Definition 2.7. A function $f: \mathbb{T} \to \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.8. Let $f: \mathbb{T} \to \mathbb{X}$ be regulated function. Any function F which satisfies $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C, \tag{2.2}$$

where C is an arbitrary constant. We define the Cauchy integral of f by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r), \quad r, s \in \mathbb{T}. \tag{2.3}$$

Definition 2.9. We say that a function $p: \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0, \qquad for\, all\, t \in \mathbb{T}.$$

The set of all regressive functions $f: \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.10 (The Generalized Exponential Function).

If $p \in \mathcal{R}$, then we define the exponential function $e_p(t,s)$ by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(t)}\left(p(\tau)\right)\Delta\tau\right), \qquad for\; s, t \in \mathbb{T},$$

where

$$\xi_{\mu(s)}\left(p(s)\right) = \begin{cases} \dfrac{1}{\mu(s)} log(|1 + \mu(s)p(s)| + i\, Arg(1 + \mu(s)p(s))), & for\; \mu(s) > 0 \\ p(s) & \mu(s) = 0 \end{cases}$$

Definition 2.11 [23]. We say that the equation

$$x^{\Delta}(t) = F(t,x), \quad t \in \mathbb{T}. \tag{2.4}$$

has Hyers-Ulam stability if for every $\varepsilon > 0$ and $u \in C^1_{rd}(\mathbb{T}, \mathbb{X})$ satisfies

$$\left\|u^{\Delta}(t) - F(t, u(t))\right\| < \varepsilon, \qquad t \in \mathbb{T},$$

there exists a solution $x\; of$ (2.4) such that:

$$\|u(t) - x(t)\| < L\varepsilon, t \in \mathbb{T} \;\; \text{for some } L > 0.$$

3. HYERS-ULAM STABILITY FOR ABSTRACT FIRST ORDER LINEAR DYNAMIC EQUATIONS.

This section contains two parts. We begin the first part by assuming that $A \in C_{rd}\left(\mathbb{T}, \mathrm{L}(\mathbb{X})\right), f \in C_{rd}(\mathbb{T}, \mathbb{X})$. In addition we assume that A satisfies the following conditions:

(1) $sup_t\|A(t)\| < \infty$

(2) A is regressive,

which implies that the following abstract initial value problem $(\mathrm{I.V.P})$:

$$x^{\Delta}(t) = A(t)x(t) + f(t), \ \ t \in \mathbb{T}, \qquad t > t_{0},$$

$$x(t_0) = x_0 \in \mathbb{X}, \tag{3.1}$$

has the unique solution

$$x(t) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A\big(t,\sigma(s)\big)f(s)\Delta s. \tag{3.2}$$

For more details about the properties of the operator exponential function, see [2, 3]. The main result of this part is the following

Theorem 3.1. If the function

$$F(t) = \int_{t_0}^{t} \big\|e_A\big(t,\sigma(s)\big)\big\|\Delta s, t_0 \in \mathbb{T} \tag{3.3}$$

is bounded, then the equation (3.1) has Hyers-Ulam stability, That is, whenever for every $\varepsilon > 0$, $g \in C_{rd}^{1}(\mathbb{T},\mathbb{X})$ satisfies:

$\|g^{\Delta}(t) - A(t)g(t) - f(t)\| \leq \varepsilon,\ t \in \mathbb{T},\ t \geq t_{0,}$

there exists a solution $\boldsymbol{v} \in \boldsymbol{C}_{\boldsymbol{rd}}^{\Delta}(\mathbb{T},\mathbb{X})$ $\boldsymbol{of}$ $(\mathbf{3.1})$ such that:

$\|\boldsymbol{g(t)} - \boldsymbol{v(t)}\| \leq \boldsymbol{L\varepsilon}, \boldsymbol{t} \in \mathbb{T}, \boldsymbol{t} \geq \boldsymbol{t_0}$ for some constant $\boldsymbol{L} > 0$.

Proof. Given $\boldsymbol{\varepsilon}>0$, suppose there exists $\boldsymbol{g} \in \boldsymbol{C}_{\boldsymbol{rd}}^{\mathbf{1}}(\mathbb{T},\mathbb{X})$ that satisfies

$\left\|\boldsymbol{g}^{\Delta}(\boldsymbol{t}) - \boldsymbol{A}(\boldsymbol{t})\boldsymbol{g}(\boldsymbol{t}) - \boldsymbol{f}(\boldsymbol{t})\right\| \leq \varepsilon,\ \ \boldsymbol{t} \in \mathbb{T}, \mathbf{t} \geq \mathbf{t_0}.$

Set

$$h(t) := g^{\Delta}(t) - A(t)g(t) - f(t), \qquad g(t_0) = g_0 \in \mathbb{X}. \tag{3.4}$$

By [2], g is given by

$$g(t) = \ e_A(t,t_0)g_0 + \int_{t_0}^{t} e_A\big(t,\sigma(s)\big)[f(s) + h(s)]\Delta s. \tag{3.5}$$

The unique solution of the initial value problem

$$v^{\Delta} - Av - f = 0,\ \ v(t_0) = g_0 \tag{3.6}$$

is given by

$$v(t) = e_A(t,t_0)g_0 + \int_{t_0}^{t} e_A\big(t,\sigma(s)\big)f(s)\Delta s\ , t \in \mathbb{T}. \tag{3.7}$$

Since $\|h(t)\| \leq \varepsilon,\ t \geq t_{0,}$ then

$$\|g(t) - v(t)\| = \int_{t_0}^{t} \|e_A\ (t,\sigma(s))\ h(s)\|\Delta s$$

$$\leq \varepsilon sup_{t\in\mathbb{T}}F(t),\ \ t \geq t_{0.}$$

This completes the proof.

As a direct consequence, in view of the boundedness of $F(t) = \int_{t_0}^{t} \|e_A(t,\sigma(s))\|\Delta s \ \mathrm{on}[a,b]_{\mathbb{T}}$,

we obtain the following result

Corollary 3.2. The equation

$$x^{\Delta}(t) = A(t)x(t) + f(t), \qquad t \in [a,b]_{\mathbb{T}}, t > t_0, \tag{3.8}$$
$$x(t_0) = x_0$$

has Hyers-Ulam stability.

The previous result yields the result in [4], when both of f and A are scalar functions.

The second part is devoting to studying the Hyers-Ulam stability of first order dynamic equations when the linear operator A is non-regressive and time invariant. We assume that A is the generator of a C_0–semigroup $\{T(t)\}_{t\in\mathbb{T}}$. Here $\mathbb{T} \in \mathbb{R} \geq 0$ is a time scale semigroup in the sense that, $a - b \in \mathbb{T}$, for all $a, b \in \mathbb{T}$ with $a > b$. We refer the reader to [1,16].

As usual a C_0 –semigroup T on $\mathbb{T}$ is a family of bounded linear operators $\{T(t): t \in \mathbb{T}\} \subset L(\mathbb{X})$, satisfying

1. $T(0) = I$, (I is the identity operator on $\mathbb{X}$).
2. $T(t+s) = T(t)T(s)$ for every $t, s \in \mathbb{T}$.
3. $\lim_{t\to 0} T(t)x = x$ ($i.e. T(.)x : \mathbb{T} \to \mathbb{X}$ is continous 0 for each $x \in \mathbb{X}$).

Definition 3.1 [1]. Let T be a C_0 –semigroup on $\mathbb{T}$. We say that a linear operator A is the generator of T if

$$Ax = \lim_{s\to 0} \frac{T\big(\mu(t)\big)x - T(s)x}{\mu(t) - s}, \quad x \in D\,(A),$$

where the domain $D(A)$ of A is the set of all $x \in \mathbb{X}$ for which the above limit exists uniformly in t.

Theorem3.3. Let A be the generator of a C_0–semigroup $\{T(t)\}_{t\in\mathbb{T}}, f \in C_{rd}(\mathbb{T},\mathbb{X})$ such that the function $F(t) = \int_{t_0}^{t} \|T(t - \sigma(s)\|\Delta s$, $t_0 \in \mathbb{T}$ is bounded. Then the I.V.P

$$x^{\Delta}(t) = Ax(t) + f(t), \; t \in \mathbb{T}, \qquad t > t_0,$$
$$x(t_0) = x_0 \in D(A), \tag{3.9}$$

has Hyers-Ulam stability.

Proof. Given $\varepsilon > 0$, suppose there exist $g \in C_{rd}^{1}(\mathbb{T},\mathbb{X})$ that satisfies

$$\left\| g^{\Delta}(t) - A\,g(t) - f(t) \right\| \leq \varepsilon, t \in \mathbb{T}, \quad t \geq t_0.$$

Set $\qquad h(t) := g^{\Delta}(t) - A\,g(t) - f(t),\; t \in \mathbb{T}, g(t_0) = g_0.$

Since, the linear dynamic equation (3.1) including a non-regressive operator A, and despite of the exponential function does not exist, the dynamic equation has a solution given in terms of the semigroup generating A. see [1]. So, g is given by,

$$g(t) = T\,(t - t_0)\,g_0 + \int_{t_0}^{t} T\big(t - \sigma(s)\big)[f(s) + h(s)]\Delta s$$

Let $v \in C_{rd}^{\Delta}(\mathbb{T})$ be the unique solution of the initial value problem

$$v^{\Delta} - Av - f = 0, \qquad v(t_0) = g_0$$

is given by

$$v(t) = T(t - t_0)g_0 + \int_{t_0}^{t} T\big(t - \sigma(s)\big)f(s)\Delta s.$$

We obtain

$$\|g(t)-v(t)\| = \int_{t_0}^{t} \|T(t-\sigma(s))h(s)\|\Delta s$$

$$\leq \varepsilon \int_{t_0}^{t} \left\|T\big(t-\sigma(s)\big)\right\|\Delta s \leq \varepsilon \sup_{t\in\mathbb{T}} F(t).$$

Therefore, the equation (3.9) has Hyers-Ulam stability.

4. Illustrative examples

The following examples show the applicability of the main results.

Example 4.1.

The linear dynamic equation

$$x^{\Delta}(t) = A(t)x(t), t \in \mathbb{T}, \qquad t > 0, \quad x(0) = x_0 \in \mathbb{R}^2, \tag{4.1}$$

has Hyres-Ulam stability on $\mathbb{T} = \mathbb{R}^{\geq 0}$, where $A(t)$ is the $2\text{x}2$ matrix defined by

$$A(t) = \begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix}, \quad t \in \mathbb{T}.$$

We use the formula in [7], for $\mathbb{T} = \mathbb{R}^{\geq 0}$, to get

$$e_A\big(t,\sigma(s)\big) = e_A(t,s)$$

$$= I + \int_s^t A(y_1)\Delta y_1 + \int_s^t A(y_1)\int_s^{y_1} A(y_2)\Delta y_2\Delta y_1 + \cdots$$

$$+ \int_s^t A(y_1)\int_s^{y_1} A(y_2)\ldots\int_s^{y_{i-1}} A(y_i)\Delta y_i \ldots \Delta y_1 + \cdots$$

$$= I - \int_s^t y_1 dy_1 I + \int_s^t y_1 \int_s^{y_1} y_2 dy_2\, dy_1\, I + \ldots + (-1)^i \int_s^t y_1 \int_s^{y_1} y_2 \ldots \int_s^{y_{i-1}} y_i dy_i \ldots dy_1 I + \ldots$$

$$= I - \frac{t^2-s^2}{2} I + \frac{1}{2!}\left(\frac{t^2-s^2}{2}\right)^2 I - \frac{1}{3!}\left(\frac{t^2-s^2}{2}\right)^3 I + \ldots + (-1)^i \frac{1}{i!}\left(\frac{t^2-s^2}{2}\right)^i I + \ldots$$

$$= e^{-\frac{t^2-s^2}{2}} I$$

And

$$\|e_A(t,\sigma(s))\| = e^{\frac{-(t^2-s^2)}{2}}$$

Since $F(t) = \int_0^t e^{\frac{-(t^2-s^2)}{2}} ds$ is bounded, then by Theorem 3.1, equation (4.1) has Hyres-Ulam stability on $\mathbb{T} = \mathbb{R}^{\geq 0}$.

In the following example we see that the boundedness of the function F defined by (3.3) is essential for the Hyers-Ulam stability of equation (3.1).

Example 4.2. Consider the following linear dynamic system,

$$x^{\Delta}(t) = A(t)x(t), t > 0, t \in \mathbb{T} \qquad , x(0) = x_0, \tag{4.2}$$

where $A(t) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$. We see that A is regressive when $\mu(t) \neq \frac{1}{2}$ for all $t \in \mathbb{T}$. In this case the matrix exponential function $e_A(t,0)$ is given by

$$e_A(t,\sigma(s)) = \begin{pmatrix} 1 & 0 \\ 0 & e_{-2}(t,\sigma(s)) \end{pmatrix}$$

We see that the generalized exponential function $e_{-2}(t,\sigma(s))$ is given by:

(i) $\quad e_{-2}(t,\sigma(s)) = e^{-2(t-s)}\ if\ t \in \mathbb{T} = \mathbb{R}^{\geq 0}$

(ii) $\quad e_{-2}(t,\sigma(s)) = (1-2h)^{\frac{(t-s-h)}{h}} if\ t \in \mathbb{T} = h\mathbb{Z}^{\geq 0}, h \neq \frac{1}{2}$.

When $\mathbb{T} = \mathbb{R}^{\geq 0}$, $e_A(t,s) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2(t-s)} \end{pmatrix}$. Hence

$\| e_A(t,s)\| = \sqrt{e^{-4(t-s)}+1} \geq 1$ and $\int_0^t \|e_A(t,s)\|ds \geq t,\ \ t \in \mathbb{T}$. This implies that $\int_0^t \|e_A(t,s)\|ds$ is unbounded. Now we check that the I.V.P (4.2) does not have Hyers-Ulam stability on $\mathbb{T} = \mathbb{R}^{\geq 0}$.

Let $\varepsilon > 0$ and $g = \varepsilon \begin{pmatrix} -\frac{t}{2} \\ \frac{1}{3} \end{pmatrix}$. We have $\|g^{\Delta} - Ag\| < \varepsilon$. Now, let $v(t)$ be the solution of the equation (4.2) with the initial value $g_0 = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Then,

$v(t) = e_A(t,0)g_0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ ye^{-2t} \end{pmatrix}$.

So,

$\|g(t) - v(t)\| = \left\|\begin{pmatrix} -t\varepsilon/2 \\ \varepsilon/3 - ye^{-2t} \end{pmatrix}\right\| = \sqrt{(-t\varepsilon/2)^2 + (\varepsilon/3 - ye^{-2t})^2} \to \infty\ as\ t \to \infty$. Therefore, equation(4.2) does not have Hyers-Ulam stability on $\mathbb{T} = \mathbb{R}^{\geq 0}$.

When $\mathbb{T} = h\mathbb{Z}^{\geq 0},\ h \neq \frac{1}{2},\ e_A(t,\sigma(s)) = \begin{pmatrix} 1 & 0 \\ 0 & (1-2h)^{\frac{t-s-h}{h}} \end{pmatrix}$, and

$$\|e_A(t,\sigma(s))\| = \sqrt{1 + ((1-2h)^{\frac{(t-s-h)}{h}})^2} \geq 1,\ \ t \in \mathbb{T},$$ Hence,

$\int_0^t \|e_A(t,\sigma(s))\|\Delta s \geq t,\ \ t \in \mathbb{T}$ and $\int_0^t \|e_A(t,\sigma(s))\|\Delta s$ is unbounded. Similarly, we can show that the I.V.P (4.2) does not have Hyers-Ulam stability on $\mathbb{T} = h\mathbb{Z}^{\geq 0},\ h \neq \frac{1}{2}$. Indeed,

let $\varepsilon > 0\ and\ g = \varepsilon\begin{pmatrix} -t/2 \\ 1/3 \end{pmatrix}$. This implies that $\|g^{\Delta} - Ag\| = \left\|\begin{pmatrix} -\varepsilon/2 \\ 2\varepsilon/3 \end{pmatrix}\right\| \leq \varepsilon$.

A solution $v(t)$ of the equation (4.2) with the initial value $g_0 = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is given by

$v(t) = e_A(t,0)g_0 = \begin{pmatrix} 0 \\ y(1-2h)^{\frac{t}{h}} \end{pmatrix}$. We have

$$\|g(t) - v(t)\| = \left\|\begin{pmatrix} -t\varepsilon/2 \\ \varepsilon/3 - y(1-2h)^{\frac{t}{h}} \end{pmatrix}\right\|$$

$$= \sqrt{(-t\varepsilon/2)^2 + (\varepsilon/3 - y(1-2h)^{\frac{t}{h}})^2} \to \infty\ as\ t \to \infty$$

Therefore, equation (4.2) does not have Hyers-Ulam stability on $\mathbb{T} = h\mathbb{Z}^{\geq 0}$, $h \neq \frac{1}{2}$.

In the following example we treat with the case $h = \frac{1}{2}$.

Example 4.3s. Consider the linear dynamic equation

$$x^{\Delta}(t) = A(t)x(t), t \in \mathbb{T}, t > 0 \quad , x(0) = x_0, \tag{4.3}$$

where $A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$ with the time scale $\mathbb{T} = \left\{\frac{n}{2} : n \in \mathbb{Z}^{\geq 0}\right\}$. Here $\mu(t) = \frac{1}{2}$.

The operator $I + \mu A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible, i.e. A is non-regressive and the matrix exponential function $e_A(t, 0)$ does not exist. On the other hand A is the generator of the C_0 –semigroup $T(k) = (I + \mu A)^{2k}$, $k \in \mathbb{Z}^{\geq 0}$. See [1] and [16]. Thus $T(k) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, k \in \mathbb{Z}^{\geq 1}$.

Let $\varepsilon > 0$. Again, taking $g = \varepsilon \begin{pmatrix} -t/2 \\ 1/3 \end{pmatrix}$, we get $\|g^{\Delta}(t) - Ag(t)\| < \varepsilon$.

According to [1], any solution $v(t)$ of equation (4.3) with initial value $x_0 = \begin{pmatrix} x \\ y \end{pmatrix}$, is given by

$v(t) = T(t)x_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$. This implies $\|g(t) - v(t)\| = \sqrt{(\varepsilon t/2 + x)^2 + (\varepsilon/3)^2} \to \infty$ as $t \to \infty$. Therefore, equation (4.3) does not have Hyers-Ulam stability on $\mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}$.

Example 4.4. Consider the following non-homogeneous linear dynamic equation

$$x^{\Delta}(t) - Ax(t) - f(t) = 0, t > 0, t \in \mathbb{T} \quad , x(0) = x_0, \tag{4.4}$$

where $A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ with the time scale $\mathbb{T} = \mathbb{Z}^{\geq 0}$, and $f(t) = \begin{pmatrix} 0 \\ -2t - 2 \end{pmatrix}$.

We have that $\mu(t) = 1$. So, $I + \mu A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible i.e. A is non-regressive, and the matrix exponential function $e_A(t, 0)$ does not exist. On the other hand A is the generator of the C_0 –semigroup $T(k) = (I + \mu A)^k$, $k \in \mathbb{Z}^{\geq 0}$. We have $T(k) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, k \in \mathbb{Z}^{\geq 1}$. One can see that equation (4.4) does not have Hyers-Ulam stability on $\mathbb{T} = \mathbb{Z}^{\geq 0}$. Indeed,

let $\varepsilon > 0$, and $g(t) = \begin{pmatrix} (-\varepsilon t/2) + 1 \\ \varepsilon/3 - 2t \end{pmatrix}$. Then $\|g^{\Delta}(t) - Ag(t) - f(t)\| \leq \varepsilon$.

A solution $v(t)$ of equation (4.4) with an initial value $g_0 = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by

$v(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -2s - 2 \end{pmatrix} \Delta s = \begin{pmatrix} x \\ 0 \end{pmatrix}$. We conclude that

$\|g(t) - v(t)\| = \sqrt{(-\frac{t}{2}\varepsilon + 1 - x)^2 + (\frac{1}{3}\varepsilon - 2t)^2} \to \infty$ as $t \to \infty$.

Therefore, equation(4.4) does not have Hyers-Ulam stability.

Example 4.5. Now, we consider the linear dynamic system,

$$x^{\Delta}(t) = Ax(t), t > 0, t \in \mathbb{T} \quad , x(0) = x_0, \tag{4.5}$$

Where $A(t) = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & -2 \end{bmatrix}$ with the time scale $\mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}$. The operator

$I + \mu A = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ 0 & 0 \end{bmatrix}$ is not invertible and A is the generator of the C_0 −semigroup

$T(t) = (I + \mu A)^{2t} = (\frac{1}{4})^{2t} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}^{2t} = (\frac{3}{4})^{2t} \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$ $, t \in \mathbb{T}$. Thus $\|T(t)\| = \frac{\sqrt{10}}{3} (\frac{3}{4})^{2t}, t \in \mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}$. We conclude that $\int_{t_0}^{t} \|T(t - \sigma(s)\| \Delta s = 4\frac{\sqrt{10}}{3}.(1 - \left(\frac{3}{4}\right)^{2t})$ which is a bounded function. Therefore, by Theorem 3.3 equation (4.5) has Hyers-Ulam stability on $\mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}$.

Example 4.6. Consider the linear dynamic system,

$$x^{\Delta}(t) = Ax(t) \quad , x(0) = x_0, \ t \in \mathbb{T}, \qquad (4.6)$$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{5}{3} \end{bmatrix}$ with the time scale $\mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}$. Hence $I + \mu A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} \end{bmatrix}$ is not invertible and A is the generator of the C_0 −semigroup $T(t) = (I + \frac{1}{2}A)^{2t} = (\frac{1}{4})^{t} \begin{bmatrix} 3 & 1 \\ 1 & 1/3 \end{bmatrix}^{2t}$.

By Putzer Algorithm, [8], we have

$$T(t) = (\frac{1}{4})^{t} (\frac{10}{3})^{2t-1} \begin{bmatrix} 3 & 1 \\ 1 & 1/3 \end{bmatrix},$$

and

$$\|T(t)\| = (\frac{1}{2})^{2t} (\frac{10}{3})^{2t}$$
$$= (\frac{5}{3})^{2t}, \quad t \in \mathbb{T} = \frac{1}{2}\mathbb{Z}^{\geq 0}.$$

We have $\int_{t_0}^{t} \|T(t - \sigma(s)\| \Delta s = \sum_{s=0}^{t-\frac{1}{2}} \left\|T(t - s - \frac{1}{2})\right\| = \left(\frac{3}{2}\right) ((\frac{5}{3})^{2t} - 1),$ which is unbounded. We leave the reader to check that equation (4.6) does not have Hyers-Ulam stability.

References

[1] Alaa E. Hamza, Karima M. Oraby, Stability of Abstract Dynamic Equations on Time Scales, Advances in Difference Equations, (2012)(1) Doi:10.1186/1687-2012-143.

[2] Alaa E. Hamza, Moneer Al-Qubaty, On The Exponential Operator Functions on Time Scales, Advances in Dynamical Systems and Applications, 7(1) (2012), 57-80.

[3] M. Al-Qubaty, On Theory of Dynamic equations on Time Scales, Ph. D. Thesis, Ain Shams University, (2011).

[4] D. R. Anderson, B. Gates and D. Heuer, Hyers-Ulam stability of second -0rder linear dynamic equations on time scales, Communications in applied Analysis 16 (2012),no.3, 281-292.

[5] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[6] J. Brazdek, D. Popa and B. Xu, Remarks on stability of linear recurrence of higher order, Appl. Math. Let. 23 Issue 12 (2010), 1459-1463.

[7] J. J. Dacunha, Transition matrix and generalized matrix exponential via the Peano-Baker seies , J. Diff. Eqs. And Appl., Vol. 11, No. 15, 15 December (2005), 1245-1264.

[8] S. N. Elaydi, An introduction to difference equations, (ISBN0387230599) (3ed., Springer, 2005).

[9] P. Gavruta and L. Gavruta, A new method for the generalized Hyers-Ulam Rassias stability, Int. J. Nonlinear Anal. Appl. 1 no.2 (2010), 11-18.

[10] P. Gavruta, S. M. Jung, and Y. J. Li, Hyers-Ulam stability for second –order linear differential equations with boundary conditions, Electronic J. Diff. Equations 2011 No. 80 (2011), 1-5.

[11] R. Ger, and C. Alsina, On some inqualities and stability results related to the exponential function, J. Inequal. Appl.2 (1998), 373-380.

[12] S.-M. Jung, Hyers- Ulam stability of linear differential eq. of first order (I), International J. Appl. Math. Stat. 7N0.Fe07 (2007), 96-100.

[13] S.-M. Jung, Hyers-Ulam stability of linear differential equations of the first order (III), J. Math, Anal. Appl. 311 (2005), 139-146.

[14] S.-M. Jung, Hyers-Ulam stability of linear differential equations of first order (II), Appl. Math. Lett. 19(2006), 854-858.

[15] S.-M, Jung, Hyres-Ulam stability of first order linear differential equations with constant coefficients, J. Math. Anal. Appl. 320 (2006), 549-561.

[16] Karima M. Oraby, Asympotototic behavior of solutions of dynamic equations on times scales, M. Sc. Thesis, Cairo University, (2012).

[17] T. Miura, S. Miyajima, and S. E. Takahasi, A characterization of Hyers-Ulam stability of first order linear differential operators, J. Math. Anal. Appl. 286 Issue 1 (2003), 136-146.

[18] T. Miura. S. Miura. And S. E. Takahasi, Hyers-Ulam stability of linear differential operator with constant coefficients, Math. Nachr. 258 (2003), 90-96.

[19] D. Popa, Hyers-Ulam stability of the linear recurrence with constant coefficient, Adv. Difference equations 2005 2(2005), 101-107.

[20] D. Popa, and I. Rasa, On Hyers–Ulam stability of the linear differential equation, J. Math. Appl. 381 (2011), 530-537.

[21] M. Rassias, special editor-in chief, Special Issue on Leonhard Paul Eulers Functional Equations and Inequalities, International J. Appl. Math. Stat., 7 Number Fe07, February 2007. (21).

[22] I. A. Rus, Ulam stability of ordinary differential equations, Stud. Univ. Babes-Bolyai Math. 54 (2009), 125-134.

[23] S. M. Ulam, A Collection of the Mathematical Problems, Interscience, New York, 1960.

[24] G. Wang, M. Zhou, and L. Sun, Hyers-Ulam stability of linear differential equations of first order, Appl. Math. Lett.21 (2008), 1024-1028.