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Some Remarks on Multidimensional Fixed Point Theorems in Partially Ordered Metric Spaces

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Abstract

In this paper, we show that multidimensional (coupled, tripled, quadrupled, n-tupled) theorems can be reduced to unidimensional fixed point theorems. Our results generalize, extend and improve the coupled fixed point results of Bhaskar and Lakshmikantham, Nonlinear Analysis, Theory, Methods and Applications, vol.65, no.7, 2006, pp. 1379-1393, V. Lakshmikantham and L. Ciric, Nonlinear Analysis, Theory, Method and Applications, vol. 70, no12, 2009, pp. 4341-4349, tripled fixed point results by Berinde and Borcut, Nonlinear Analysis, Volume 74, Issue 15, October 2011, Pages 4889-4897, Quadruple fixed point theorems by E. Karapınar and V. Berinde, *Banach Journal of Mathematical Analysis*, vol. 6, no. 1, pp. 74–89, 2012 and multidimensional fixed point results by Muzeyyen Erturk and Vatan Karakaya, Journal of Inequalities and Applications 2013, 2013:196, pp. 1-19, M. Imdad, A. H. Soliman, B. S. Choudhary and P. Das, Journal of Operators, Volume 2013, Article ID 532867, pp. 1-8 and M. Paknazar, M. E. Gordji, M. D. L. Sen and S. M. Vaezpour, *Fixed Point Theory and Applications* 2013, **2013**:11 etc.

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1. Introduction:

Fixed point theory is a very useful tool in solving a variety of problems in control theory, economic theory, nonlinear analysis and global analysis. The Banach contraction principle is the most famous, simplest and one of the most versatile elementary results in fixed point theory. A huge amount of literature is witnessed on applications, generalizations and extensions of this principle carried out by several authors in different directions, *e.g.*, by weakening the hypothesis, using different setups, considering different mappings.

In [20], Bhaskar and Lakshmikantham introduced the notions of mixed monotone property and coupled fixed point for the contractive mapping $F : X \times X \to X$, where X is a partially ordered metric space and proved some coupled fixed point theorems for a mixed monotone operator. As an application of the coupled fixed point theorems, they determined the existence and uniqueness of the solution of a periodic boundary value problems. It is very natural to extend the definition of 2-dimensional fixed point (coupled fixed point), 3-dimensional fixed point (tripled fixed point) and so on to multidimensional fixed point (n-tuple fixed point). Recently, Berinde and Borcut [7] and E.Karapinar et.al [5] introduced the concept of tripled and quadrupled fixed points respectively and proved some related theorems (see also [2,26,1,5]). The last remarkable result of this trend was given by M.Imdad et al. [16] by introducing the notion of multidimensional fixed points.(see also[27,18,24,14,4].

In this paper, we have developed a method of reducing coupled, tripled and multidimensional results in partially ordered metric spaces to respective results with one variable, even obtaining more general theorems. Our results generalize, extend, unify and extend results of [2,3,7,8,18,19].

2. Defintions and preliminaries:

We consider the following definitions and results which shall be required in the sequel.

Definition 2.1 [6] Let (X, $\preceq$) be a partially ordered set and $F: X \times X \to X$ then F enjoys the mixed monotone property if F(x, y) is monotonically non-decreasing in x and monotonically non-increasing in y, that is, for any x, y $\in$X,

$$x_1, x_2 \epsilon X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y) \text{ and } y_1, y_2 \epsilon X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 2.2 [6] Let (X, $\preceq$) be a partially ordered set and $F: X \times X \to X$, then (x, y) $\in X \times X$ is called a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.3 [6] Let (X, $\preceq$) be a partially ordered set and $F: X \times X \to X$ and $g : X \to X$ then F enjoys the mixed g-monotone property if $F(x, y)$ is monotonically g-non-decreasing in x and monotonically g-non-increasing in y, that is ,for any $x, y \in X$,

$x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$,

$y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

Definition 2.4 [6] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$, then $(x, y) \in X \times X$ is called a coupled coincidence point of the maps F and g if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.5 [6] Let $(X, \preceq)$ be a partially ordered set, then $(x, y) \in X \times X$ is called a coupled fixed point of the maps $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y) = x$ and $gy = F(y, x) = y$.

Berimde and Borcut [2] studied tripled coincidence points as follows.

Definition 2.6 [7]. Let $F: X^3 \to X$ be a given map, we say that $(x, y, z) \in X^3$ is a tripled fixed point of F if

$F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, x, y) = z$.

Definition 2.7 [7] Let (X, ≤) be a partially ordered set and $F: X^3 \to X$. We say that F has the mixed monotone property if $F(x, y, z)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y , that is , for any $x, y, z \in X$

$x_1, x_1 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z)$,

$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z)$ and

$z_1, z_2 \in X, z_1 \preceq z_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2)$.

Karpinar and Loung [5] studied the quadruple case as follows:

Definition 2.8 [19] An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F: X^4 \to X$ if $F(x, y, z, w) = x$, $F(y, z, w, x) = y$, $F(z, w, x, y) = z$ and $F(w, x, y, z) = w$.

Definition 2.9 [19] Let (X, $\preceq$) be a partially ordered set and $F: X^4 \to X$. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y and w , that is , for any $x, y, z, w \in X$

$x_1, x_1 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y, z, w) \preceq F(x_2, y, z, w)$,

$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1, z, w) \succeq F(x, y_2, z, w)$,

$z_1, z_2 \in X, z_1 \preceq z_2 \Rightarrow F(x, y, z_1, w) \preceq F(x, y, z_2, w)$ and

$w_1, w_2 \in X, w_1 \preceq w_2 \Rightarrow F(x, y, z, w_1) \succeq F(x, y, z, w_2)$.

Definition 2.10 [19] An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F: X^4 \to X$ and $g: X \to X$ if

$F(x, y, z, w) = gx = x$, $F(y, z, w, x) = gy = y$, $F(z, w, x, y) = gz = z$ and $F(w, x, y, z) = gw = w$.

Definition 2.11 [19] Let (X, ≤) be a partially ordered set and $F: X^4 \to X$ and $g: X \to X$ be two maps. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y and w, that is , for any $x, y, z, w \in X$

$x_1, x_1 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1, y, z, w) \preceq F(x_2, y, z, w)$,

$y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x, y_1, z, w) \succeq F(x, y_2, z, w)$,

$z_1, z_2 \in X, gz_1 \preceq gz_2 \Rightarrow F(x, y, z_1, w) \preceq F(x, y, z_2, w)$ and

$w_1, w_2 \in X, gw_1 \preceq gw_2 \Rightarrow F(x, y, z, w_1) \succeq F(x, y, z, w_2)$.

M.Imdad et.al [12] introduced the notion of n-tupled coincidence and n-tupled fixed point (assuming n as even natural number) as follows:

Definition 2.12 [14] Let (X, ≤) be a partially ordered set and $F: \prod_{i=1}^{r} X^i \to X$ then F is said to have the mixed monotone property if F is non-decreasing in its odd position arguments and non-increasing in its even positions arguments , that is, if,

(i) For all $x_1^1, x_2^1 \in X, x_1^1 \preceq x_2^1 \Rightarrow F(x_1^1, x^2, x^3, \ldots, x^r) \preceq F(x_2^1, x^2, x^3, \ldots, x^r)$,

(ii) For all $x_1^2, x_2^2 \in X, x_1^2 \preceq x_2^2 \Rightarrow F(x^1, x_1^2, x^3, \ldots x^r) \succeq F(x^1, x_2^2, x^3, \ldots, x^r)$,

(iii) For all $x_1^3, x_2^3 \in X, x_1^3 \preceq x_2^3 \Rightarrow F(x^1, x^2, x_1^3, x^4, \ldots, x^r) \preceq F(x^1, x^2, x_2^3, x^4, \ldots, x^r)$,

…

For all $x_1^r, x_2^r \in X, x_1^r \preceq x_2^r \Rightarrow F(x^1, x^2, x^3, \ldots, x_1^r) \succeq F(x^1, x^2, x^3, \ldots, x_2^r)$.

Definition 2.13 [14] Let (X, ≤) be a partially ordered set and $F: \prod_{i=1}^{r} X^i \to X$ and g: X→X be two maps. Then F is said to have the mixed g-monotone property if F is g-non-decreasing in its odd position arguments and g-non-increasing in its even positions arguments , that is, if ,

(i) For all $x_1^1, x_2^1 \in X, gx_1^1 \preceq gx_2^1 \Rightarrow F(x_1^1, x^2, x^3, \ldots, x^r) \preceq F(x_2^1, x^2, x^3, \ldots, x^r)$,

(ii) For all $x_1^2, x_2^2 \in X, gx_1^2 \preceq gx_2^2 \Rightarrow F(, x^1, x_1^2, x^3, \ldots, x^r) \succeq F(x^1, x_2^2, x^3, \ldots, x^r)$,

(iii) For all $x_1^3, x_2^3 \in X, gx_1^3 \preceq gx_2^3 \Rightarrow F(x^1, x^2, x_1^3, \ldots, x^r) \preceq F(x^1, x^2, x_2^3, \ldots, x^r)$,

…

For all $x_1^r, x_2^r \in X, gx_1^r \preceq gx_2^r \Rightarrow F(x^1, x^2, x^3, \ldots, x_1^r) \succeq F(x^1, x^2, \ldots, x_2^r)$.

Definition 2.14 [14] Let X be a nonempty set. An element $(x^1, x^2, x^3, \ldots \ldots .., x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the mapping $F: \prod_{i=1}^{r} X^i \to X$ if

$$x^1 = F(x^1, x^2, x^3, \ldots, x^r),$$

$$x^2 = F(x^2, x^3, \ldots, x^r, x^1),$$

$$x^3 = F(x^3, \ldots, x^r, x^1, x^2),$$

…

$$x^r = F(x^r, x^1, x^2, \ldots, x^{r-1}).$$

Definition 2.15 [14] Let X be a nonempty set. An element $(x^1, x^2, x^3, \ldots, x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled coincidence point of the maps $F: \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ if

$$gx^1 = F(x^1, x^2, x^3, \ldots, x^r),$$

$$gx^2 = F(x^2, x^3, \ldots, x^r, x^1),$$

$$gx^3 = F(x^3, \ldots, x^r, x^1, x^2),$$

$$\ldots$$

$$gx^r = F(x^r, x^2, x^3, \ldots, x^{r-1}).$$

Definition 2.16 [14] Let X be a nonempty set. An element $(x^1, x^2, x^3, \ldots \ldots .., x^r) \in \prod_{i=1}^{r} X^i$ is called an r-tupled fixed point of the maps $F: \prod_{i=1}^{r} X^i \to X$ and $g : X \to X$ if

$$x^1 = gx^1 = F(x^1, x^2, x^3, \ldots, x^r),$$

$$x^2 = gx^2 = F(x^2, x^3, \ldots, x^r, x^1)$$

$$\ldots$$

$$x^r = gx^r = F(x^r, x^1, x^2, \ldots, x^{r-1}).$$

G.Bhaskar and V. Lakshmikantham [26] proved the following:

Theorem 1 [6] Let (X, $\preceq$) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t$ and $\lim_{r\to t^+}\varphi(t) < t$ for each t > 0. Further let $F: X \times X \to X$ and g: X→X be two maps such that F has the mixed g-monotone property satisfying the following conditions:

(1) $F(X \times X) \subseteq g(X)$,

(2) g is continuous and monotonically increasing,

(3) the pair (g, F) is commuting,

(4) $d\big(F(x,y), F(u,v)\big) \leq \varphi\left(\frac{d((g(x),g(u))+d((g(y),g(v))}{2}\right)$,

for all $x, y, u, v \in X$, with $g(x) \leq g(u)$, and $g(y) \geq g(v)$. Also, suppose that either

(a) F is continuous or

(b) X has the following properties:

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$ for all $n \geq 0$.

(ii) If a non-inecreasing sequence $\{y_n\} \to y$ then $y \leq y_n$ for all $n \geq 0$.

If there exist $x_0, y_0 \in X$ such that

(5) $g(x_0) \leq F(x_0, y_0)$, $g(x_0) \geq F(x_0, y_0)$.

Then F and g have a coupled coincidence point, i. e there exist $x, y \in X$ such that

$$g(x) = F(x,y), g(y) = F(y,x).$$

Many generalizations and extensions of Theorem 1 exist in the literature, see [1-9,13,17,19-23,25,28]. Recently, M. Imdad et. al [14] introduced the concept of n- tupled fixed point and established fixed point results for mappings having a mixed monotone property and satisfying a contractive condition in ordered metric spaces.

M. Imdad et. al [14] proved the following:

Theorem 2 [14] Let (X, $\preceq$) be a partially ordered set equipped with a metric d such that (X, d) is a complete metric space. Assume that there is a function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(t)<t$ and $\lim_{r\to t^+}\varphi(t) < t$ for each t > 0. Further let $F: \prod_{i=1}^{r} X^i \to X$ and g: X→X be two maps such that F has the mixed g-monotone property satisfying the following conditions:

(1) $F(\prod_{i=1}^{r} X^i) \subseteq g(X)$,

(2) g is continuous and monotonically increasing,

(3) the pair (g, F) is commuting,

(4) $d\big(F(x^1, x^2, x^3, \ldots, x^r), F(y^1, y^2, y^3, \ldots, y^r)\big) \leq \varphi\left(\frac{1}{r}\sum_{n=1}^{r} d\big(g(x^n), g(y^n)\big)\right)$,

for all $x^1, x^2, x^3, \ldots, x^r, y^1, y^2, y^3, \ldots, y^r \in X$, with $gx^1 \leq gy^1$, $gx^2 \geq gy^2$, $gx^3 \leq gy^3, \ldots, gx^r \geq gy^r$. Also, suppose that either

(c) F is continuous or

(d) X has the following properties:

(i) If a non-decreasing sequence $\{x_n\} \to x$ then $x_n \leq x$ for all $n \geq 0$.

(ii) If a non-inecreasing sequence $\{y_n\} \to y$ then $y \leq y_n$ for all $n \geq 0$.

If there exist $x_0^1, x_0^2, x_0^3, \dots, x_0^r \in X$ such that

(5) $$gx_0^1 \leq F(x_0^1, x_0^2, x_0^3, \dots, x_0^r),$$
$$gx_0^2 \geq F(x_0^2, x_0^3, \dots, x_0^r, x_0^1),$$
$$gx_0^3 \leq F(x_0^3, \dots, x_0^r, x_0^1, x_0^2),$$
$$\dots$$
$$gx_0^r \geq F(x_0^r, x_0^1, x_0^2, x_0^3, \dots, x_0^{r-1}).$$

Then F and g have a r-tupled coincidence point, i. e there exist $x^1, x^2, x^3, \dots, x^r \in X$ such that

$$gx^1 = F(x^1, x^2, x^3, \dots, x^r),$$
$$gx^2 = F(x^2, x^3, \dots, x^r, x^1),$$
$$gx^3 = F(x^3, \dots, x^r, x^1, x^2),$$
$$\dots$$
$$gx^r = F(x^r, x^1, x^2, x^3, \dots, x^{r-1}).$$

3. Main Results:

Now, we prove our main result as follows:

Remark1 Thorem 2 in [14] is not valid if n is odd.

Proof. For the sake of simplicity, we consider the case for n=3 which is very illustrative and can be identically extrapolated to the case in which n is odd. Using the initial points $x_0^1, x_0^2, x_0^3 \in X$, it is possible to construct three sequences $\{x_k^1\}, \{x_k^2\}$ and $\{x_k^3\}$ recursively defined by :

$$gx_k^1 = F\left(x_{k-1}^1, x_{k-1}^2, x_{k-1}^3\right),$$
$$gx_k^2 = F\left(x_{k-1}^2, x_{k-1}^3, x_{k-1}^1\right),$$
$$gx_k^3 = F\left(x_{k-1}^3, x_{k-1}^1, x_{k-1}^2\right) \text{ for all } k \in N, k \geq 1.$$

By assumption, we have

$$gx_0^1 \preceq F(x_0^1, x_0^2, x_0^3) = gx_1^1,$$
$$gx_0^2 \succeq F(x_0^2, x_0^3, x_0^1) = gx_1^2,$$
$$gx_0^3 \preceq F(x_0^3, x_0^1, x_0^2) = gx_1^3.$$

Then the authors affirmed that these sequences verify, for all $k \geq 1$,

$$gx_{k-1}^1 \preceq gx_k^1,$$
$$gx_{k-1}^2 \succeq gx_k^2,$$
$$gx_{k-1}^3 \preceq gx_k^3.$$

However, it is impossible to prove that $gx_1^2 \succeq gx_2^2$ because the mixed g-monotone property leads to contrary inequalities. At most, we can deduce the following properties

$$gx_1^2 \preceq gx_0^2 \Rightarrow F(x_1^2, x_0^3, x_0^1) \preceq F(x_0^2, x_0^3, x_0^1) = gx_1^2.$$

Moreover,

$$gx_0^3 \preceq gx_1^3 \Rightarrow F(x_1^2, x_0^3, x_0^1) \succeq F(x_1^2, x_1^3, x_0^1).$$

Above two inequalities gives,

$$F(x_1^2, x_1^3, x_0^1) \preceq F(x_1^2, x_0^3, x_0^1) \preceq F(x_0^2, x_0^3, x_0^1) = gx_1^2.$$

However, in the third component, the inequality is on the contrary

$$gx_0^1 \preceq gx_1^1 \Rightarrow F(x_1^2, x_1^3, x_0^1) \preceq F(x_1^2, x_1^3, x_1^1) = gx_2^2.$$

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Then we get

$F(x_1^2, x_1^3, x_0^1) \leq gx_1^2$ and $F(x_1^2, x_1^3, x_0^1) \leq gx_2^2$.

Since other possibilities gives to similar incomparable cases, we cannot get the inequality $gx_1^2 \geq gx_2^2$.

Remark 2 Also, we notice that, in the case n=3, definition (2.15)

$$gx^1 = F(x^1, x^2, x^3),$$
$$gx^2 = F(x^2, x^3, x^1),$$
$$gx^3 = F(x^3, x^1, x^2),$$

do not extend the notion of tripled coincidence point by Brinde and Borcut. Therefore their results are not extensions of well known results in tripled case and hence we can say that the odd case is not well posed.

Remark 3. Also, we see that the system of equations defined in (2.14) is not suitable to work with the classical mixed monotone property when r is odd . For example, if r = 5 and F is monotone non-decreasing in its odd arguments and monotone non-increasing in its even arguments, then the equations

$x^1 = F(x^1, x^2, x^3, x^4, x^5)$ (x^1 and x^5 are placed in non-decreasing arguments of F) and

$x^2 = F(x^2, x^3, x^4, x^5, x^1)$ (x^1 and $x^5$are placed in arguments of different monotone type of F)

Do not let us to show the existence of fixed points using the classical mixed monotone property.

To make the paper free from these flaws, we recall here the concept of multidimensional fixed point/ coincidence point introduced by Roldan et. al [27], which is an extension of Berzig and Samet's notion given in [4].

Throughout the paper , we will abbreviate MS for metric space. Let n be a positive integer. Henceforth, X will denote a non-empty set and X^n will denote the product space $X \times X \times \ldots . \times X$. Throughout this manuscript, m and k will denote non-negative integers and $i, j, s \in \{1,2, \ldots, n\}$. Unless otherwise stated, "for all m" will mean " for all $m \geq 0$ " and " for all i " will mean " for all $i \in \{1,2, \ldots, n\}$ ".

Henceforth, fix a partion $\{A, B\}$ of $A_n = \{1,2, \ldots, n\}$, that is $A_n = A \cup B$ and $A \cap B = \emptyset$ where A and B are non-empty sets. We will denote:

$\Omega_{A,B} = \{\sigma: A_n \to A_n : \sigma(A) \subseteq A \text{ and } \sigma(B) \subseteq B\}$ and

$\Omega'_{A,B} = \{\sigma: A_n \to A_n : \sigma(A) \subseteq B \text{ and } \sigma(B) \subseteq A\}$.

If $(X, \preceq)$ is a partially ordered space, $x, y \in X$ and $i \in A_n$, we will use the following notation:

$$x \preceq_i y \Leftrightarrow \begin{Bmatrix} x \preceq y, if \ i \in A, \\ x \succeq y, if \ i \in B. \end{Bmatrix}$$

Consider on the product space X^n, the following partial order:

$X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in X^n$,

$X \sqsubseteq Y \Leftrightarrow x_i \preceq_i y_i$, for all i.

We say that two points X and Y are comparable if $X \sqsubseteq Y$ or $Y \sqsubseteq X$.

Definition 3.1 [27] Let $(X, \preceq)$ be a partially ordered space with the maps $F: X^n \to X$ and $g: X \to X$. We say that F has the mixed g-monotone property ($w.r.t \ \{A, B\}$) if F is monotone g- nondecreasing in arguments of A and monotone g-nonincreasing in arguments of B, i.e, for all $x_1, x_2, \ldots, x_n, y, z \in X$ for all i,

$gy \preceq gz \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \preceq_i F(y_1, \ldots, y_{i-1}, z, y_{i+1}, \ldots, y_n)$.

Henceforth, let $\sigma_1, \sigma_2, \ldots, \sigma_n: A_n \to A_n$ be n mappings from A_n into itself and let γ be the n-tupple $(\sigma_1, \sigma_2, \ldots, \sigma_n)$. The main aim of this paper is to study the following class of multidimensional fixed points.

Definition 3.2 [27] A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a γ-fixed point of the mapping F if

$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = x_i$ for all i.

Definition 3.3 [27] A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a γ-coincidence point of the mappings $F: X^n \to X$ and $g: X \to X$ if $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i$ for all i.

Definition 3.4 [27] A point $(x_1, x_2, \ldots, x_n) \in X^n$ is called a γ-fixed point of the mappings $F: X^n \to X$ and $g: X \to X$ if $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(n)}) = gx_i = x_i$ for all i.

Remark 4 If one represent a mapping $\sigma : A_n \to A_n$ throughout its order image, that is, $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(r))$, then

(i) G-Bhaskar and Lakshmikantham's election in $n = 2$ is $\sigma_1 = \tau = (1,2)$ and $\sigma_2 = (2,1)$

(ii) Berinde and Borcut's election in $n = 3$ is $\sigma_1 = \tau = (1,2,3)$, $\sigma_2 = (2,1,2)$ and $\sigma_3 = (3,2,1)$

(iii) Karapinar's election in $n = 4$ is $\sigma_1 = \tau = (1,2,3,4)$, $\sigma_2 = (2,3,4,1)$, $\sigma_3 = (3,4,1,2)$ and $\sigma_4 = (4,1,2,3)$.

These cases consider A as the odd numbers in $\{1,2,\dots,n\}$ and B as its even numbers. However, for Berzig and Samet [14], use $A = \{1,2,\dots,m\}, B = \{m+1,\dots,n\}$ and arbitrary mappings.

Definition 3.5 An ordered MS $\{X, d, \preceq\}$ is said to have the sequential g-monotone property if it saitisfies:

(i) If $\{x_m\}$ is a non-decreasing sequence and $\{x_m\} \xrightarrow{d} x$, then $gx_m \preceq gx$ for all m.

(ii) If $\{x_m\}$ is a non-increasing sequence and $\{x_m\} \xrightarrow{d} x$, then $gx_m \succeq gx$ for all m.

If g is the identity mapping, then X is said to have sequential monotone property.

Proposition 3.1. If $X \subseteq Y$, it follows that

$(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \subseteq (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)})$ if $\in \Omega_{A,B}$,

$(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \supseteq (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)})$ if $\sigma \in \Omega'_{A,B}$.

Proof. Suppose that $x_i \preceq_i y_i$ for all i. Hence $x_{\sigma(i)} \preceq_{\sigma(i)} y_{\sigma(i)}$ for all i. Fix $\sigma \in \Omega_{A,B}$. If $i \in A$, then $\sigma(i) \in A$, so $x_{\sigma(i)} \preceq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preceq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preceq_i y_{\sigma(i)}$. If

$i \in B$, then $\sigma(i) \in B$, so $x_{\sigma(i)} \preceq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \preceq_i y_{\sigma(i)}$. In any case, if $\sigma \in \Omega_{A,B}$, then $x_{\sigma(i)} \preceq_i y_{\sigma(i)}$for all i. It follows that

$$(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \subseteq (y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)})$$

Now fix $\sigma \in \Omega'_{A,B}$. If $i \in A$, then $\sigma(i) \in B$, so $x_{\sigma(i)} \preceq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \succeq y_{\sigma(i)}$, which means that $x_{\sigma(i)} \succeq_i y_{\sigma(i)}$. If $i \in B$, then $\sigma(i) \in A$, so $x_{\sigma(i)} \preceq_{\sigma(i)} y_{\sigma(i)}$ implies that $x_{\sigma(i)} \preceq y_{\sigma(i)}$ which mean that $x_{\sigma(i)} \succeq_i y_{\sigma(i)}$

Lemma 1 Let (X, d) be a MS and define $D_n : X^n \times X^n \to [0,\infty)$, for all $A = (a_1, a_2, \dots, a_n)$,

$B = (b_1, b_2, \dots, b_n) \in X^n$, by $D_n(A, B) = \sum_{i=1}^{n} d(a_i, b_i)$.

Then D_n is a complete metric on X^n.

Theorem 3 Let $(X, d, \preceq)$ be a partially ordered MS and $F: X^n \to X$ and $g: X \to X$ be mappings. Let $\gamma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ be a n-tuple of mappings from $(1,2,\dots,n)$ into itself satisfying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma \in \Omega'_{A,B}$ if $i \in B$. Define

$F_\gamma, g_\gamma: X^n \to X^n$ by

$$F_\gamma(x_1, x_2, \dots, x_n) = \begin{pmatrix} F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \dots, x_{\sigma_1(n)}), \\ F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \dots, x_{\sigma_2(n)}), \\ \cdot \\ \cdot \\ \cdot \\ F(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \dots, x_{\sigma_n(n)}) \end{pmatrix}, \quad \text{and } g_\gamma(x_1, x_2, x_3, \dots, x_n) = (g(\sigma_1 x_1), g(\sigma_2 x_2), \dots, g(\sigma_n x_n)).$$

(1) If F has the mixed g-monotone property, then F_γ is g_γ non-decreasing $w.r.t$ the partial order $\subseteq$ on X^n.

(2) If F and g are continuous $w.r.t$ D_n, then F_γ and g_γ are also continuous $w.r.t$ D_n.

(3) A point $(x_1, x_2, \dots, x_n) \in X^n$ is a γ-coincidence point of F and g if and only if $(x_1, x_2, \dots, x_n)$ is a coincidence point of F_γ and g_γ.

Proof: Suppose that $(gx_1, gx_2, \dots, gx_n) \le (gy_1, gy_2, \dots, gy_n)$, that is $gx_i \preceq_i gy_i$ for all i. Since F has the mixed g-monotone property, it is not difficult to prove that, for all $a_1, a_2, \dots, a_n \in X$

$$F\left(a_1, a_2, \dots, a_{j-1}, x_i^{(j)}, a_{j+1}, \dots, a_n\right) \preceq F\left(a_1, a_2, \dots, a_{j-1}, y_i^{(j)}, a_{j+1}, \dots, a_n\right), \text{ if } i, j \in A \text{ or } i, j \in B,$$

$$F\left(a_1, a_2, \dots, a_{j-1}, x_i^{(j)}, a_{j+1}, \dots, a_n\right) \succeq F\left(a_1, a_2, \dots, a_{j-1}, y_i^{(j)}, a_{j+1}, \dots, a_n\right), \text{ if } i \in A, j \in B \text{ or } i \in B, j \in A.$$

Suppose that $i \in A$. Therefore $\sigma_i \in \Omega_{A,B}$, that is $\sigma_i(A) \subseteq A$ and $\sigma_i(B) \subseteq B$. Therefore $j \in A$ if and only if $\sigma_i(j) \in A$ and the same holds if we replace A by B. In this case ,

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) \preceq \qquad (either\ 1,\ \sigma_i(1) \in A\ or\ 1,\ \sigma_i(1) \in B\ )$$

$$\preceq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}) \preceq \qquad (either\ 2,\ \sigma_i(2) \in A\ or\ 2,\ \sigma_i(2) \in B\ )$$

$$\dots$$

$\preceq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}),$

that is, $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) \preceq_i F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)})$ Now suppose that $i \in B$. Therefore $\sigma_i \in \Omega'_{A,B}$, that is $\sigma_i(A) \subseteq B$ and $\sigma_i(B) \subseteq A$. Therefore $j \in A$ if and only if $\sigma_i(j) \in B$ and the same

holds if we replace A by B. In this case,

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) \succeq \quad (either\ 1 \in A,\ \sigma_i(1) \in B\ or\ 1 \in B,\ \sigma_i(1) \in A)$$

$$\succeq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}) \quad (either\ 2 \in A,\ \sigma_i(2) \in B\ or\ 2 \in B,\ \sigma_i(2) \in A)$$

$$\dots$$

$$\succeq F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}),$$

That is, $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) \preceq_i F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)})$ also holds. Hence,

$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) \preceq_i F(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)})$ for all i, and , consequently,

$F_\gamma(x_1, x_2, \dots, x_n) \sqsubseteq F_\gamma(y_1, y_2, \dots, y_n)$.

(2) It is an straightforward exercise.

(3) $(x_1, x_2, \dots, x_n) \in X^n$ is a γ-coincidence point of F and g if and only if $gx_i = F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \dots, x_{\sigma_1(n)})$ for all i, that is $F_\gamma(x_1, x_2, \dots, x_n) = (gx_1, gx_2, \dots, gx_n)$.

The following lemma is crucial for the proof of our main theorem

Lemma2 [18] Let (X, d) be a MS and let $\{x_n\}$ be a sequence in X such that $lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. If $\{x_n\}$ is not a Cauchy sequence in (X, d), then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tends to ϵ^+ when $k \to \infty$:

$d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{n_k-1}), d(x_{m_k+1}, x_{n_k}), d(x_{m_k+1}, x_{n_k-1})$.

Theorem 4 Let (X, d, $\preceq$) be a partially ordered metric space and $f, g: X \to X$ be mappings satisfying the following

(i) f is g-non-decreasing with respect to $\preceq$ and $f(X) \subseteq g(X)$

(ii) g is continuous and the commutes with f,

(iii) There exist $x_0 \in X$ such that $gx_0 \preceq fx_0$,

(iv) $d(fx, fy) \le \varphi(d(gx, gy))$, for all $x, y \in X$ for which $gx \preceq gy$ or $gx \succeq gy$.

Also, suppose that either

(a) f is continuous or

(b) X has the sequential monotone property.

Then f and g have a coincidence point.

Proof : If $gx_0 = fx_0$ then x_0 is a coincidence point of f and g. Therefore, $gx_0 \prec fx_0$. Since

$f(X) \subseteq g(X)$ we obtain sequence $y_n = fx_n = gx_{n+1}$ for all $n = 0,1,2, \dots$ where $x_n \in X$ and by induction we get that $y_n \preceq y_{n+1}$. If $y_n = y_{n+1}$ for some $n \in N$ then x_n is a coincidence point of f and g.

Therefore, suppose that $y_n \ne y_{n+1}$ for each n. Now, we shall prove the following :

(1) $d(y_n, y_{n+1}) \to 0$ as $n \to \infty$;

(2) $\{y_n\}$ is a Cauchy Sequence.

By putting $x = x_n, y = x_{n+1}$ in (iv) we get

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) \le \varphi(d(gx_n, gx_{n+1})) = \varphi(d(y_{n-1}, y_n)) < d(y_{n-1}, y_n)$$

This gives $\{y_n\}$ is decreasing and consequently there exists 'd' such that

$lim_{n\to\infty} d(y_n, y_{n+1}) = d \ge 0$. If $d > 0$, we get from previous relation

$d = lim_{n\to\infty} d(y_n, y_{n+1}) \le lim_{n\to\infty} \varphi(d(y_{n-1}, y_n)) < d$,

which is a contradiction and hence $lim_{n\to\infty} d(y_n, y_{n+1}) = d = 0$.

Further using Lemma 2, we shall prove that $\{y_n\}$ is a Cauchy sequence. Suppose that is not true. Then by Lemma 2, there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to ϵ^+ when $k \to \infty$.

$d(x_{m_k}, x_{n_k})$, $d(x_{m_k}, x_{n_{k-1}})$, $d(x_{m_{k+1}}, x_{n_k})$, $d(x_{n_{k-1}}, x_{m_{k+1}})$

Putting $x = x_{m_{k+1}}, y = x_{n_k}$ in (iv)

$$d(y_{m_{k+1}}, y_{n_k}) = d(fx_{m_{k+1}}, fx_{n_k}) \le \varphi\left(d(gx_{m_{k+1}}, gx_{n_k})\right) = \varphi\left(d(y_{m_k}, y_{n_{k-1}})\right) < d(y_{m_k}, y_{n_{k-1}})$$

Letting $k \to \infty$, we get $\epsilon \le \varphi(\epsilon) < \epsilon$, which is a contradiction with $\epsilon > 0$.Hence we have proved that $\{y_n\}$ is a Cauchy sequence in (X,d). Since (X, d) is complete there exists $z \in X$ such that $y_n \to z$. Then

$d(fx_n, z) \to 0$ and $d(gx_n, z) \to 0$ as $n \to \infty$. As g is continuous so $ggx_n, gfx_n \to gz$ and (f, g) is commuting so we have

$$lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

Now, consider the case (a) when f is continuous, using triangle inequality we get

$d(fz, gz) \le d(fz, fgx_n) + d(fgx_n, gz) \le d(fz, fgx_n) + d(fgx_n, gfx_n) + d(gfx_n, gz) \to 0.$
This gives $gz = fz$.

If case (b) holds. Since f is g-non-decreasing and $gx_n \to z$ so $gx_n \le z$. Also, $gfx_n \to gz$ as g is continuous.

$$\begin{aligned} d(gz, fz) &\le d(gz, gfx_n) + d(gfx_n, fz) \\ &\le lim_{n\to\infty} d(gz, gfx_n) + lim_{n\to\infty} d(gfx_n, fz) \\ &= lim_{n\to\infty} d(gz, gfx_n) + lim_{n\to\infty} d(fgx_n, fz) \\ &\le lim_{n\to\infty} d(gz, gfx_n) + lim_{n\to\infty} \varphi\big(d(ggx_n, gz)\big) \to 0 \end{aligned}$$

Thus $gz = fz$ and hence the theorem follows completely.

For all i, the mapping σ_i is a permutation of $\{1,2,\dots,n\}$ and so for all i and all $A, B \in X^n$,

$$\sum_{i=1}^{n} d\big(a_{\sigma_i(j)}, b_{\sigma_i(j)}\big) = \sum_{i=1}^{n} d\big(a_j, b_j\big) = D_n(A, B)$$

Theorem 3 Theorem 2 follows from Theorems 3 and 4.

Proof: Consider the product space $Y = X^n$ provided with the metric D_n and the partial order $\subseteq$ on Y. Then $(Y, D_n, \subseteq)$ is a complete ordered MS. Since F has mixed monotone property, item 1 of theorem 1 shows that $F_\gamma: Y \to Y$ is non-decreasing $w.r.t. \subseteq$. By item 2 of theorem 1. If F is continuous, then F_γ is also continuous. If $x_0 = (x_0^1, x_0^2, \dots, x_0^n) \in Y$, then condition (5) is equivalent to $x_0 \subseteq F_{\gamma x_0}$. By Proposition 1.2, given $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n)$ such that $X \subseteq Y$, the points

$(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ and $(y_{\sigma(1)}, y_{\sigma(2)}, \dots, y_{\sigma(n)})$ are comparable by $\subseteq$. Therefore, (4) can be applied to these points, and it follows that

$$D_n\left(F_\gamma(X), F_\gamma(Y)\right) = D_n\left(F_\gamma(x_1, x_2, \dots, x_n), F_\gamma(y_1, y_2, \dots, y_n)\right)$$

$$= D_n\left(\begin{pmatrix} F\big(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \dots, x_{\sigma_1(n)}\big), \\ F\big(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \dots, x_{\sigma_2(n)}\big), \\ \vdots \\ F\big(x_{\sigma_n(1)}, x_{\sigma_n(2)}, \dots, x_{\sigma_n(n)}\big) \end{pmatrix}, \begin{pmatrix} F\big(y_{\sigma_1(1)}, y_{\sigma_1(2)}, \dots, y_{\sigma_1(n)}\big), \\ F\big(y_{\sigma_2(1)}, y_{\sigma_2(2)}, \dots, y_{\sigma_2(n)}\big), \\ \vdots \\ F\big(y_{\sigma_n(1)}, y_{\sigma_n(2)}, \dots, y_{\sigma_n(n)}\big) \end{pmatrix}\right)$$

$$= \sum_{i=1}^{n} \left[\left(F\big(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}\big)\right) - \left(F\big(y_{\sigma_i(1)}, y_{\sigma_i(2)}, \dots, y_{\sigma_i(n)}\big)\right)\right]$$

$$\le \sum_{i=1}^{n} \varphi\left(\frac{1}{n}\sum_{j=1}^{n} d\left(g(\sigma_i x_j), g(\sigma_i y_j)\right)\right) = \varphi\left(D_n\left(g_\gamma(X), g_\gamma(Y)\right)\right)$$

Theorems 1 and 2 imply that F_γ and g_γ have a coincidence point, which is a γ-coincidence point of F and g by item 3 of Theorem 1.

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