ISSN 2347-1921

Optimization of insurance broker's investment, consumption and the probability of survival with constant rate of return under exponential utility function

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ABSTRACT

In this study, we take the risk reserve of an insurance broker to follow Brownian motion with drift and tackle an optimal portfolio selection problem of the company. The investment case considered was insurance broker that trades two assets: the money market account (bond) growing at a rate r and a risky stock with an investment behavior in the presence of a stochastic cash flow or a risk process, continuously in the economy. Our focus was on obtaining investment strategies that are optimal in the sense of optimizing the returns of the company. We established among others that, the optimized investment in the assets and the optimal value function are dependent on horizon and the wealth. It is recommended that the broker should take into consideration this horizon dependency when making policy decisions.

Keywords:

Probability of survival; Optimization; Investment of returns; Insurance Broker; consumption; exponential utility.

Academic Discipline And Sub-Disciplines:

Mathematical Finance, Risky Theory.

Mathematics Subject Classification: 91B30, 62P20.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics
Vol 7, No. 1
editor@cirworld.com
www.cirworld.com, member.cirworld.com

1. INTRODUCTION

In this work, we find an optimal investment strategy, consumption and probability of survival for an insurance broker that maximizes his utility using power utility function where the risk-free asset has a constant rate of return. This investment strategy will decide how the company should invest between the risk-free bond and a risky stock subject to its obligation to pay policy holders when claims occur.

Optimal portfolio selection problem is of practical importance in finance and insurance Mathematics. Earlier work in this area can be traced back to Markowitz's mean variance model Markowitz (1959) which Samuelson (1969) extended to a dynamic set up. Samuelson used a dynamic stochastic programming approach and obtained the optimal decision for consumption investment model. Stochastic optimal control approach was used by Merton (1971) in continuous-time financing and under specific assumptions about the asset returns and the investor preferences to obtain a closed form solution to optimal portfolio strategy problem.

These days, insurance brokers invest both in the money market and stocks. Due to the high risks involved in the stock market, investment strategies and risk management are becoming more important.

Hipp and Plum (2000) determined the strategy of investment which minimizes the probability of ruin modeling the price of the stocks by geometric Brownian motion using the Hamilton-Jacobi-Bellman equation. Under the same hypothesis as Hipp and Plum, Gaier et. al., (2003), obtained an exponential bound with a rate that improves the classical Lundberg parameter and found an optimal trading strategy which involved investing in the stock a constant amount of money independent of the reserve.

In this work, we used the conjectured that maximizing the exponential utility of terminal wealth is intrinsically related to minimizing the probability of ruin. Browne (1995) verified the conjecture in a model without interest rate where the stock followed a geometric Brownian motion and the risk process of an insurance company being Brownian motion with drift.

We considered the wealth process of the reserve of the insurance broker, modeling the risk process as Brownian motion with drift when there are investments in both risk free and risky (stock) assets in which the risky asset followed a geometric Brownian motion.

The optimal investment problem consists of finding the portfolio strategy that maximizes the expected value of the utility function at some predetermined moment in future.

Browne (1999) considered a portfolio problem in continuous time where the objective of the investor or money manager was to exceed the performance of a given stochastic benchmark, as is often the case of institutional money management. The benchmark in this work was a stochastic process that needed not be perfectly correlated with the investment opportunities and the market is in a sense incomplete.

Unlike in our model, Castillo and Parrocha (2008) considered an insurance business with a fixed amount available for investment in a portfolio consisting of one non-risky asset and one risky asset. They presented the Hamilton-Jacobi-Bellman (HJB) equation and demonstrated its use in finding the optimal investment strategy based on some given criteria. The objective of the resulting control problem was to determine the investment strategy that minimized infinite ruin probability. The existence of a solution to the resulting HJB equation was then shown by verification theorem. A numerical algorithm is also given for analysis.

Hipp and Plum (2000) modeled the risk process of an insurance company as a compound Poison process unlike ours where the risk process is modeled by Brownian motion with drift. In their paper, they applied stochastic control to answer the following question: if an insurer has the possibility of investing part of his surplus into a risky asset, what is the optimal strategy to minimize the probability of ruin. They observed that the probability of ruin of the risk process can be minimized by a suitable choice of an investment strategy by a capital market index.

The optimal strategy was computed using the Bellman equation. They also proved the existence of a smooth solution and a verification theorem, and give explicit solutions in some cases with exponential claim distributions as well as numerical results in a case with Pareto claim size distribution. It was observed that the optimal amount invested cannot be bounded for that last case.

Promislow and Young (2005) extended the work of Browne (1995) and Schmidli (2001), in which they minimized the probability of ruin of an insurer facing a claim process modeled by a Brownian motion with drift. They consider two controls to minimize the probability of ruin;

1. Investing in a risky asset (constrained and the non-constrained cases)

2. Purchasing quota-share reinsurance.

They obtained an analytic expression for the minimum probability of ruin and their corresponding optimal controls. They also demonstrate their results with numerical results.

Bayraktar and Young (2008) worked on a problem involving individual consumers and especially beneficiaries of endowment funds who generally employ strategies such that consumption never decreases (ratcheted) or at least they try to do this. They assumed that an agent's rate of consumption is ratcheted; that is it forms a non-decreasing process. They assumed that the agent invests in a financial market with one risk less asset and one risky asset with the latter's price following geometric Brownian motion as in Black Scholes model. Given the rate of consumption of the agent, they act as financial advisers and find optimal investment strategies for the agent who wishes to minimize his/her probability of running out of money either before dying or before the organization holding the endowment fails due to causes other than the ruin of the fund itself. They solved this minimization problem using stochastic optimal control techniques.

Liu and Yang (2004) studied optimal investment strategies of an insurance company. They assume that an insurance company receives premiums at a constant rate, the total claims are modeled by a compound Poison process, and the insurance company can invest in the money market (bonds) and in a risky asset such as stocks. Their model generalizes that in the Hipp and Plum (2000) by including a risky free asset. The investment behavior in this paper was investigated numerically for various claim size distributions. The optimal policy and the solution of the associated Hamilton Jacobi Bellman equation are then computed under each assumed distribution. The effects of the changes in the various factors such as the stock volatility, on optimal investment strategies and survival probability are investigated. They further generalize to cases in which borrowing constraints or reinsurance is present.

Blane and Jeng-Eng (2008) considered an optimal asset allocation problem of an insurance company. In their model, an insurance company is represented by a compound Poison risk process which is perturbed by diffusion and has investments. The investments are in both risky and risk-free types of assets similar to stocks/real estates and bonds respectively. The insurance company can borrow at a constant interest rate in the event of a negative surplus. Numerical analysis appears to show that an optimal asset allocation range can be estimated for certain parameters and can be compared with using insurance data. Using a conservative method to minimize the probability of ruin, they were able to show that a reasonable optimal asset allocation range for a typical insurance is about 4.5 to 8 percent invested in risky stock/real estate assets. An inequality and the exact solution are obtained for the pure diffusion equation. In addition the asymptotic form of the ruin probability is shown to be a power function.

Oksendal and Sulem (2002) investigated a market with one risk-free and one risky asset in which the dynamics of the risky asset are governed by a geometric Brownian motion. They considered an investor who consumes from a bank account and has the opportunity at any time to transfer funds between two assets, and assumes that these transfers involve a fixed transaction cost which was independent of the size of the transaction plus its cost proportional to the size of transaction.

Their objective was to maximize the cumulative expected utility of consumption over planning horizon and they formulated the problem as a combined stochastic control/ impulse control.

Qian and Lin (2009) considered an insurance company whose surplus (reserve) is modeled by a jump diffusion risk process. The insurance company can invest part of its surplus in n risky assets and purchase a proportional reinsurance for claims.

Their main goal is to find an optimal investment and proportional reinsurance policy which minimizes ruin probability. They apply stochastic control theory to solve the problem. They obtained closed form expression for the minimum probability, optimal investment and proportional reinsurance policy. They found out that the minimum ruin probability satisfies the Lundberg equality. They also investigated the diffusion volatility parameter, the market price of risk and the correlation coefficient on the minimal ruin probability, optimal investment and proportional reinsurance policy through numerical calculations.

Azcue and Muler, (2009) considered that the reserve of an insurance company follows a Cramer-Lundberg process. They considered that the management of an insurance company had the possibility of investing part of the reserve in a risky asset. They considered that the risky asset was a stock as it is with most of the rest of the studies whose price process was a geometric Brownian motion.

Their main aim was to find a dynamic choice of investment policy which would minimize the ruin probability of the insurance company. They imposed that the proportion of the reserve invested in the risky asset was to be smaller than a given positive bound a for instance the case a = 1 meant that the company could not borrow money to buy stocks.

They characterized the optimal value function as the classical solution of the associated Hamilton-Jacobi-Bellman equation which was a non linear second order integro-differential equation. Numerical solutions were obtained for comparison with the results of the unconstrained cases that were studied earlier by Hipp and Plum (2000). Their study revealed that the optimal strategies of the constrained and unconstrained problems do not coincide.

Kostadinova (2007) considered a stochastic model for the wealth of an insurance company which has the possibility to invest into a risky asset and a risk-less asset under constant mix strategy. This total claim amount being modeled by compound Poison process and the price of the risky asset considered to follow a general Levy process. They investigated the resulting integrated risk process and the corresponding discounted net loss process. This opened up a way to measure the risk of a negative outcome of the integrated risk process in a stationary way.

They provided an approximation of the optimal investment strategy, that maximizes the expected wealth of the insurance company under the risk constraint on the Value-at-Risk.

Grandits and Gaier (2002) studied the infinite ruin probability problem in the classical Cramer-Lundberg model, where the company was allowed to invest their money in a stock whose price followed geometric Brownian motion. Starting from an integro-differential equation for the maximal survival probability, they analyzed the case of claim sizes which have distribution functions F with regularly varying tails. Their results showed that if $1 - F$ is regularly varying with index

$\rho < -1$, then the ruin probability is also regularly varying with same index $\rho < -1$ and this was under an assumption of zero interest rates.

Bai and Liu (2007) considered a classical risk process model and allowed investment into a risk-free asset as well as proportional reinsurance. The optimal proportional reinsurance strategy was found to minimize the probability of ruin of an insurance company. The problem was treated under two cases: The first case was trivial, the corresponding probability of ruin and the optimal proportional reinsurance strategy were obtained directly. The second case, firstly the existence of the solution to the Hamilton-Jacobi-Bellman (HJB) equation was provided. Then the minimal probability of ruin and the optimal proportional reinsurance strategy were obtained by a verification theorem.

Liu, Bai and Yiu (2012) considered a constrained investment problem with the objective of minimizing the ruin probability. In their paper, they formulated the cash reserve and investment model for the insurance company and analyzed the Value-at-Risk (VaR) in a short time horizon. For risk regulation, they imposed it as a risk constraint dynamically. Then the problem was therefore to minimize the probability of ruin together with the imposed risk constraint. By solving the corresponding Hamilton-Jacobi-Bellman equations, they derived analytic expressions for the optimal value function and the corresponding optimal strategies. Looking at the Value-at-Risk alone, they were able to show that it was possible to reduce the overall risk by an increased exposure to the risky assets with the stochastic of the fundamental insurance business. Studying the optimal strategies, they found out that a different investment strategy would be in place depending on the Sharpe ratio of the risky asset.

Luo (2008) considered an optimal dynamic control problem for an insurance company with opportunities of proportional reinsurance and investment. The company can purchase proportional reinsurance to reduce the risk level and invest its surplus in a financial market that a Black-Schole risky asset and a risk-free asset.

Unlike in our model, when investing in the risk-free asset, three practical borrowing constraints are studied individually :(B1) the borrowing rate is higher than the lending (saving) rate, (B2) the dollar amount borrowed is no more than $k > 0$, where k is a fixed limit, and finally (B3) the proportion of the borrowed amount to the surplus level is no more than k > 0. Under each of the constraints, the objective is to minimize the probability of ruin. Classical stochastic control theory is applied to solve the problem. Specifically, the minimal ruin probability functions are obtained in closed form by solving Hamilton-Jacobi-Bellman (HJB) equations, and other associated optimal reinsurance investment policies are found by verification theorem.

From the discussions above, we note that the risk process of an insurance company in most of the papers is modeled by the Cramer Lundberg model while investment is done with either two assets or with a single asset and reinsurance.

This work aims at optimization of investment returns of the insurance company, build a foundation for further research and also contribute to improved insurance business.

2. The Problem Formulation.

2.1 Weak Convergence of Risk process to Brownian motion.

To approximate the risk process of an insurance broker by Brownian motion, consider a sequence of risk process $R_n(t)$ defined as follows;

$$R_n(t) = u_n + c_n t - \sum_{i=1}^{N(t)} Y_k^{(n)} \qquad (1)$$

where u_n is the initial risk reserve of the insurance company, c_n is the gross risk premium per unit time paid by the policy holders and the sequence $\{Y_k^{(n)} : k \in N\}$,describes the consecutive claim sizes. Assume also that $E\left(Y_k^{(n)}\right) = \mu_n$ and $var\left(Y_k^{(n)}\right) = \sigma_n^{\ 2}$.

The point process $N = \{N(t) : t \geq 0\}$ counts claims appearing up to time t , that is

$$N(t) = \max\{k : \sum_{i=1}^{k} T_i \leq t\} \qquad (2)$$

where $\{T_k : k \in N\}$ is an identically independent sequence of non negative random variables describing the times between the arriving claims with $E(\{T_k) = \frac{1}{\lambda} \geq 0$.

If T_k are exponentially distributed then $N(t)$ is a Poison process with intensity λ.

2.2: The Model

Adapting the formulation of Osu and Ihedioha (2012), the insurance broker trades two assets continuously in the economy. The first asset is the money market account (bond) growing at a r which dynamics is given as;

$$dB_t = rdt. \quad (3)$$

The second asset is a risky stock which price S_t at any time t has the price processs that follows the geometric Brownian motion:

$$dS_t = S_t dZ_t, \quad (4)$$

where Z_t is a Brownian motion with drift μ and diffusion parameter σ, that is, $dZ_t = \mu dt + \sigma dB_t^{(2)}$, where μ and σ are constants and $B_t^{(2)}: t \geq 0$ is a standard Brownian motion.

In classical risk theory, the true net claim process say $\{R_t\}$ is usually modeled as;

$$R_t = u + ct - \sum_{i=1}^{N(t)} Y_i. \quad (5)$$

where u is the initial risk reserve, c is the premium income rate per unit time, N_t is the number of claims up to time t usually modeled as a stationary renewal process with rate λ and Y_i is the size of the i^{th} claim with $\{Y_i: i \geq 1\}$ assumed to be an identically independent sequence as shown in the previous section, $\alpha = c - \mu\lambda$ and $\beta^2 = \sigma^2\lambda$ and these can also be written as $\alpha = c\lambda E(Y_1)$ and $\beta^2 = \lambda E(Y_1{}^2)$ So the parameter α can be understood as the relative safety loading of the claims process.

We investigate the investment behavior in the presence of a stochastic cash flow denote by $R_t: t \geq 0$ which describes a Brownian motion with drift α and diffusion parameter σ that is R_t satisfies the stochastic differential equation;

$$dR_t = \alpha dt + \beta\, dB_t^{(1)} \quad (6)$$

where α and β are constants (with $\beta \geq 0$).

We also allow the two Brownian motions to be correlated and we denote their correlation coefficient by ρ that is $E(B_t^{(1)} B_t^{(2)}) = \rho t$. We will not consider the uninteresting case of ρ^2, in which case there would be only one source of randomness in the model.

The company is allowed to invest its surplus in the risky stock and we will denote the total amount of money invested in the risky stock at time t under an investment policy π as π_t where $\{\pi_t\}$ is a suitable admissible adapted control process, that is, π_t is a non anticipative function and satisfies for any T,

$$\int_0^T \pi_t{}^2 dt < \infty, \quad (7)$$

almost surely.

We assume that W_t is the total wealth of an insurance company. We also assume that the insurance company allocates its wealth as follows: Let π_t be the total amount of the company's wealth that is invested in risky assets and remaining balance $(W_t - \pi_t)$ be invested in a risk-less asset (bond/market).

We note that π_t may become negative, which is to be interpreted as short selling a stock. The amount invested in the bond, $W_t - \pi_t$ may also be negative, and this amounts to borrowing at the interest rate r. Assuming $B_t^{(1)}$ and $B_t^{(2)}$ are correlated standard Brownian motions, with correlation coefficient ρ then for any policy π, the total wealth process of the insurance broker evolves according to the stochastic differential equation as;

$$dW_t^{\pi} = [wr + \pi_t(\mu - r) + \alpha]dt + \pi_t \sigma dB_t^{(2)} + \beta dB_t^{(1)}. \quad (8)$$

with the quadratic variation of the wealth process given as;

$$d < W >_t = ({\pi_t}^2\sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)dt. \quad (9)$$

If $\rho^2 \neq 1$, this model is incomplete in a very strong sense in that the random cash flow or the random endowment R_t, can not be traded on the security market, and therefore the risk to the investor cannot be eliminated under any circumstance.

We put no constraints on the control π_t except for the particular case where the possibility of borrowing is not allowed, we allow $\pi < 0$ as well as $\pi_t > W_t^\pi$. In the first instance the company is shorting stock while in the second instance the company borrows money to invest long in the stock.

We assume that the broker can always borrow money as long as he has a positive net worth, that is, $W_t^\pi > 0$ and borrowing is not allowed once it's bankrupt and then the possibility of ruin is of real concern.

Suppose now that the broker is interested in maximizing the utility of his wealth say at time T. The utility function is $U(w)$ and satisfies $U' > 0$ and $U'' < 0$.

Let $V(t,w) = sup_\pi E(U(W_t^\pi)|W_t^\pi = w$ and $\pi_t^* : \{0 \leq t \leq T\}$ denote the optimal investment policy and the exponential utility function given by:

$$U(w) = \lambda e - \frac{\gamma}{\theta} e^{-\theta w}, \quad (10)$$

where $>$ and $\theta > 0$ then the coefficient of absolute risk aversion equal to a constant:

$$-\frac{U''(w)}{U'(w)} = \theta. \quad (11)$$

The insurance broker's problem, therefore, can be written as:

$$V(t,w) = sup_\pi E(U(W_t^\pi)|W_t^\pi = w, \quad (12)$$

subject to:

$$dW_t^\pi = [wr + \pi_t(\mu - r) + \alpha]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)}. \quad (13)$$

2.3 The problem

Under the assumptions that the insurance broker makes intermediate consumption decision on the admissible consumption space, C, which satisfies $\int_0^t |c_s| ds < \infty, \forall t \in [0,T]$, and through the risk-free asset's account, the insurance company's problem becomes;

$$J(W,t;T) = sup_\pi E\left[\int_0^T e^{-\rho\tau}\left(\lambda - \frac{\gamma}{\theta}e^{-\theta c_\tau}\right)d\tau + e^{-\rho T}\left(\lambda - \frac{\gamma}{\theta}e^{-\theta w_T}\right)\right] \quad (14)$$

subject to:

$$dW_t^\pi = [wr + \pi_t(\mu - r) + \alpha - c_t]dt + \pi_t\sigma dB_t^{(2)} + \beta dB_t^{(1)}$$

The value function should also satisfy the terminal condition:

$$J(W,T;T) = \lambda - \frac{\gamma}{\theta}e^{-\theta W_T}. \quad (15)$$

The first term of the value function, J represents discounted utility from consumption flows, while the second term captures the idea that terminal wealth gives utility to the participant from time T upwards.

Optimal investment strategies are chosen so as to maximize the final wealth at a deterministic time T.

Define the value function at time T as;

$$J(W,t;T) = sup_\pi E(U(W_t^\pi)|W_t^\pi = w, \quad (16)$$

subject to:

$$dW_t^{\pi} = [wr + \pi_t(\mu - r) + \alpha - c_t]dt + \pi_t \sigma dB_t^{(2)} + \beta dB_t^{(1)}$$

3.1 The optimal consumption, investment and value function.

The HJB equation becomes;

$$\lambda - \frac{\gamma}{\theta} e^{-\theta c} + J_t + [\mu\pi_t + (w - \pi_t)r + \alpha - c_t]J_W + \frac{[(\mu\pi_t)^2 + \beta^2 + 2\sigma\beta\rho\pi_t]}{2} J_{WW} - \vartheta J = 0. \quad (17)$$

The optimal consumption here is obtained using the first order condition, thus;

$$c^* = \ln\left(\frac{\gamma}{J_W}\right)^{\frac{1}{\theta}}. \quad (18)$$

Considering the nature of the objective function, the restriction and the terminal condition, let; $G(w,t;\ T) = \left[\lambda - \frac{\gamma}{\theta} e^{-\theta w}\right] f(t;T)$, where $f(t;T)$ is a function of time, be the new value function, then with;

$$G_t = f'\left[\lambda - \frac{\gamma}{\theta} e^{-\theta w}\right];\ G_w = \gamma e^{-\theta w} f;\ G_{ww} = -\theta\gamma e^{-\theta w} f$$

The HJB equation becomes;

$$-\theta\gamma e^{-\theta w} \frac{[(\mu\pi_t)^2 + \beta^2 + 2\sigma\beta\rho\pi_t]}{2} f - \vartheta\left[\lambda - \frac{\gamma}{\theta} e^{-\theta w}\right] f = 0. \quad (19)$$

Differentiating with respect to π_t and simplifying yields;

$$\pi_t^* = \left[\frac{[\mu - r]}{\theta\sigma^2} - \frac{\beta\rho}{\sigma}\right], \quad (20)$$

the optimal investment in the risky asset that is independent of the wealth at hand.

Lemma:

The optimal value function is given by;

$$G(w,t;\ T) = D'\left[\lambda - \frac{\gamma}{\theta} e^{-\theta w}\right] e^{-\left[\frac{b(T-t)}{a}\right]} \int_t^T e^{-[b(T-\tau)]}\, d\tau$$

Proof:

On simplifying equation (17); we get;

$$af' + bf = d \quad (21)$$

where,

$a = \left[\lambda - \frac{\gamma}{\theta} e^{-\theta w}\right];\ b = [\mu\pi_t + (w - \pi_t)r + \alpha - c_t]\gamma e^{-\theta w} - \theta\gamma e^{-\theta w} \frac{[(\mu\pi_t)^2 + \beta^2 + 2\sigma\beta\rho\pi_t]}{2} - \vartheta\left[\lambda - \frac{\gamma}{\theta} e^{-\theta w}\right]$ and $d = \lambda - \frac{\gamma}{\theta} e^{-\theta c}$

,a first order linear differential equation with integrating factor;

$$R(t) = e^{\frac{b(T-t)}{a}} \quad (22)$$

Therefore, the solution to the linear differential equation (21) is;

$$f = D' e^{-\left(\frac{b(T-t)}{a}\right)} \int_t^T e^{\frac{b(T-\tau)}{a}}\, d\tau, \quad (23)$$

for which when $t \to T$ $f \to 1$ and,

$$G(w,t;\ T) = \left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right] f(t;T)$$

$$= \left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right] D' e^{-\left(\frac{b(T-t)}{a}\right)} \int_t^T e^{\frac{b(T-\tau)}{a}} d\tau.$$

So, the optimal value function is then given as;

$$G(w,t;\ T) = D'\left[\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right] e^{-\left[\frac{b(T-t)}{a}\right]} \int_t^T e^{-\left[\frac{b}{a}(T-\tau)\right]}.$$

3.2 Maximization of the probability of survival of the insurance company.

For the case of maximizing the insurance company's probability of survival under exponential utility function, we formalize the problem by letting $V^*(w)$ be the maximal probability of beating the benchmark when starting from the state w. That is $W_o = w$ and u and v be given constants such that $l < w < m$ and $V^*(w) = sup_\pi P_w(\tau_u^\pi < \tau_v^\pi)$.

The HJB equation reduces to;

$$\lambda - \frac{\gamma}{\theta}e^{-\theta c} + [\mu\pi_t + (w - \pi_t)r + \alpha - c_t]V_W + \frac{[(\mu\pi_t)^2 + \beta^2 + 2\sigma\beta\rho\pi_t]}{2} V_{WW} - \vartheta V = 0.$$

$V(w)$ is now independent of time and subject to boundary conditions; $V(l) = 0 \ and \ V(m) = 1$ for $u < w < v$.

Let there exit a classical solution V that satisfies the condition that $V_{WW} < 0$, given by;

$$J(w) = \varphi\left(\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right) \tag{24}$$

where φ is a constant, then,

$$J_W = \gamma\varphi e^{-\theta w} \ and \ J_{WW} = -\gamma\theta\varphi e^{-\theta w}. \tag{25}$$

Substituting these in the HJB equation above, we obtain;

$$\varphi\left(\lambda - \frac{\gamma}{\theta}e^{-\theta c}\right) + [\mu\pi_t + (w - \pi_t)r + \alpha - c_t]\gamma\varphi e^{-\theta w} - \gamma\theta\varphi e^{-\theta w}\frac{[(\mu\pi_t)^2 + \beta^2 + 2\sigma\beta\rho\pi_t]}{2} \vartheta\varphi(\lambda - \frac{\gamma}{\theta}e^{-\theta w}) = 0$$

. (26)

To obtain the optimal value of π_t, we differentiate equation with respect to π_t the HJB equation above and simplify thus;

$$\pi_t^* = \left[\frac{(\mu - r)}{h\sigma^2} - \frac{\beta\rho}{\sigma}\right], \tag{27}$$

where $h = \varphi\theta$.

Also, to obtain the optimal consumption, we apply the first order condition with respect to c to get;

$$c_t^* = w - \frac{\ln\varphi}{\theta} \tag{28}$$

Substituting (27) into the HJB equation, we get;

$$\left(\lambda - \frac{\gamma}{\theta}e^{-\theta c}\right) + \left[wr + \left[\frac{(\mu - r)}{h\sigma^2} - \frac{\beta\rho}{\sigma}\right](\mu - r) + \alpha - c_t\right]\gamma\varphi e^{-\theta w} + \frac{-\gamma\theta}{2}\left[\left[\frac{(\mu - r)}{h\sigma^2} - \frac{\beta\rho}{\sigma}\right]^2 \sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)\right]\varphi e^{-\theta w} = 0$$

(29)

Lemma :

The optimal value function is;

$$H(w) = \frac{e^{\theta(w-v)}\left(e^{-\theta w} - e^{-\theta u}\right)}{\left(e^{-\theta v} - e^{-\theta u}\right)} \quad (30)$$

Proof:

Let

$$\left.\begin{aligned} a &= \left[wr + \left[\frac{(\mu - r)}{h\sigma^2} - \frac{\beta\rho}{\sigma}\right](\mu - r) + \alpha - c_t\right]\gamma \\ b &= \frac{-\gamma\theta}{2}\left[\left[\frac{(\mu-r)}{h\sigma^2} - \frac{\beta\rho}{\sigma}\right]^2 \sigma^2 + \beta^2 + 2\sigma\rho\beta\pi_t)\right] and -z = \left(\lambda - \frac{\gamma}{\theta}e^{-\theta c}\right)\end{aligned}\right\}, \quad (31)$$

then (29) reduces to the equation;

$$(b+a)e^{-\theta w}\varphi = z, \quad (32)$$

yielding

$$\varphi = \frac{z}{(b+a)e^{-\theta w}} \;. \quad (33)$$

Therefore,

$$J(w,t;T) = \frac{z}{(b+a)e^{-\theta w}}\left(\lambda - \frac{\gamma}{\theta}e^{-\theta w}\right), \quad (34)$$

and applying the boundary conditions, we get;

$$\lambda = \frac{\gamma}{\theta}e^{-\theta u} \quad (35a)$$

$$\frac{z}{(b+a)} = \frac{e^{-\theta v}}{\frac{\gamma}{\theta}\left(e^{-\theta u} - e^{-\theta v}\right)}. \quad (35b)$$

Applying (35a) and (35b) in (34) we obtain the optimal value function;

$$J^*(w) = \frac{e^{\theta(w-v)}\left(e^{-\theta w} - e^{-\theta u}\right)}{\left(e^{-\theta v} - e^{-\theta u}\right)}$$

as expected.

This value satisfies the boundary conditions, $V(u) = 0 \; and \; V(v) = 1$ for $u < w < v$.

4. CONCLUSION

The problem of optimizing investment returns and consumption of an insurance broker with constant rate of return was dealt with for exponential utility function. The basis for this study was the fact that risk reserve process of brokers can weakly converge to a Brownian motion process. The main emphasis has been on how the utility function affects the broker's portfolio selection given investment and consumption choices. It was observed that the optimal consumption has a logarithmic expression and the optimized investment in the risky asset independent of wealth at hand.

5. RECOMMENDATION

The findings of this work show that the value functions are horizon dependent, so the managers of the assets of the insurance company should take into consideration this horizon dependency when making policy decisions.

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