ISSN 2347-1921

New Generalized Definitions of Rough Membership Relations and Functions from Topological Point of View

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Abstract

In this paper, we shall integrate some ideas in terms of concepts in topology. In fact, we introduce two different views to define generalized membership relations and functions as mathematical tools to classify the sets and help for measuring exactness and roughness of sets. Moreover, we define several types of fuzzy sets. Comparisons between the induced operations were discussed. Finally, many results, examples and counter examples to indicate connections are investigated.

AMS Subject Classifications: 54A05, 54C10.

Keywords: j-Neighborhood Space; Topology; Near Operators; Membership Relations; Rough Set; Membership Functions and Fuzzy Set.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 8, No 3

editor@cirjam.org

www.cirjam.com, www.cirworld.com

1 Introduction.

In order to extract useful information hidden in voluminous data, many methods in addition to classical logic have been proposed. These include fuzzy set theory [16], rough set theory [35, 36], computing with words [17-19] and computational theory for linguistic dynamic systems [12]. Rough set theory, proposed by Pawlak [35], is a new mathematical approach to deal with imprecision, vagueness and uncertainty in data analysis and information system. Rough set theory has many applications in several fields (see [2-13, 19-27]). The classical rough set theory is based on equivalence relations. However, the requirement of equivalence relations as the indiscernibility relation is too restrictive for many applications. In light of this, many authors introduced some extensions (generalizations) on Pawlak's original concept (see [1-12, 19-27 and 29-34]). But most of them could not apply the properties of original rough set theory and thus they put some conditions and restrictions.

In our work [20], we have introduced frame work to generalize Pawlak's original concept. In fact, we have introduced the generalized neighborhood space $j-\boldsymbol{NS}$ as a generalization to neighborhood space. Moreover, in our approaches $j-\boldsymbol{NS}$, we have introduced different approximations that satisfy all properties of original rough set theory without any conditions or restrictions. In addition, we have introduced an important result as a new method to generate general topology from any neighborhood space and then from any binary relation. This technique opens the way for more topological applications in rough context and help in formalizing many applications from real-life data. Accordingly, our work [21] introduced some of the important topological applications named "*Near concepts*" as easy tools to classify the sets and help for measuring near exactness and near roughness of sets.

In the present paper, we introduce some new notions in $j-\boldsymbol{NS}$ such as "*j-rough membership relations, j-rough membership functions and j-fuzzy sets*". In addition, we apply near concepts on the above notions to define different tools for modification the original operations. Many results, examples and counter examples are provided to illustrate the properties and the connections of the introduced approaches.

Moreover, the introduced j-rough membership functions are more accurate than other rough member function such as Lin [28]; Lemma 4.2 & 4.3, prove this result. For first time, we use the new topological application named "*j-near concepts*" to define new different tools namely "*j-near rough membership relations and j-near rough membership functions*" to classify the sets and help for measuring near exactness and near roughness of sets. Considering the *j-near rough membership functions*, we introduce new different fuzzy sets in $j-\boldsymbol{NS}$. The introduced techniques are very interesting since it is give new connection between four important theories namely "rough set theory, fuzzy set theory and the general topology".

2 Preliminaries.

In this section, we introduce the fundamental concepts that were used through this paper.

Definition 2.1"*Topological Space*"[18]

A *topological space* is the pair (U,τ) consisting of a set U and family τ of subsets of U satisfying the following conditions:

(T1) $\emptyset \in \tau$ and $U \in \tau$.

(T2) τ is closed under finite intersection.

(T3) τ is closed under arbitrary union.

The pair (U,τ) is called "*space*", the elements of U are called *"points"* of the space, the subsets of U that belonging to τ are called "*open"* sets in the space and the complement of the subsets of U belonging to τ are called *"closed"* sets in the space; the family τ of open subsets of U is also called a *"topology"* for U.

Definition 2.2 "*Pawlak Approximation Space*"[36, 37]

Let U be a finite set, the universe of discourse, and R be an equivalence relation on U, called an indiscernibility relation. The pair $\mathcal{A}=(U,R)$ is called Pawlak approximation space. The relation R will generate a partition $U/R=\{[x]_R : x\in U\}$ on U, where$[x]_R$ is the equivalence class with respect to R containing *x*.

For any, $X\subseteq U$ the upper approximation $\overline{Apr}(X)$ and the lower approximation $\underline{Apr}(X)$ of a subset X are defined respectively as follow [36, 37]:

$$\overline{Apr}(X) = \cap\{Y\subseteq U/R : Y\cap X\neq\emptyset\} \text{ and } \underline{Apr}(X) = \cup\{Y\subseteq U/R : Y\subseteq X\}.$$

Let $\emptyset$ be the empty set, X^c is the complement of X in U, we have the following properties of the Pawlak's rough sets [36, 37]:

(L1) $\underline{Apr}(X) = \left[\overline{Apr}(X^c)\right]^c$.

(L2) $\underline{Apr}(U) = U$.

(U1) $\overline{Apr}(X) = \left[\underline{Apr}(X^c)\right]^c$.

(U2) $\overline{Apr}(U) = U$.

(L3) $\underline{Apr}(X \cap Y) = \underline{Apr}(X) \cap \underline{Apr}(Y)$.

(L4) $\underline{Apr}(X \cup Y) \supseteq \underline{Apr}(X) \cup \underline{Apr}(Y)$.

(L5) If $X \subseteq Y$, then $\underline{Apr}(X) \subseteq \underline{Apr}(Y)$.

(L6) $\underline{Apr}(\emptyset) = \emptyset$.

(L7) $\underline{Apr}(X) \subseteq X$.

(L8) $\underline{Apr}\ (\underline{Apr}(X)) = \underline{Apr}(X)$.

(L9) $\underline{Apr}\ (\overline{Apr}(X)) = \underline{Apr}(X)$.

(U3) $\overline{Apr}(X \cup Y) = \overline{Apr}(X) \cup \overline{Apr}(Y)$.

(U4) $\overline{Apr}(X \cap Y) \subseteq \overline{Apr}(X) \cap \overline{Apr}(Y)$.

(U5) If $X \subseteq Y$, then $\overline{Apr}(X) \subseteq \overline{Apr}(Y)$.

(U6) $\overline{Apr}(\emptyset) = \emptyset$.

(U7) $X \subseteq \overline{Apr}(X)$.

(U8) $\overline{Apr}(\overline{Apr}(X)) = \overline{Apr}(X)$.

(U9) $\overline{Apr}(\underline{Apr}(X)) = \overline{Apr}(X)$.

Definition 2.3[37] "Pawlak Membership function"

Rough sets can be also defined employing, instead of approximations, rough membership function as follow: $\mu_X^R: U \longrightarrow [0,1]$, *where*

$$\mu_X^R(x) = \frac{|[x]_R \cap X|}{|[x]_R|}$$, and $|X|$ denotes the cardinality of X.

Lin [28] have defined new rough membership function in the case of R is a general binary relation as the following definition illustrates.

Definition 2.4[28] "Lin Membership function"

Rough sets can be also defined employing, instead of approximations, rough membership function as follow: $\mu_X^R: U \longrightarrow [0,1], where$

$$\mu_X^R(x) = \frac{|xR \cap X|}{|xR|}$$

, and xR indicates to the after set of element $x \in U$.

3 Generalized Neighborhood Space and Near Concepts in Rough Sets.

In this section, we introduce the main ideas about the new j-neighborhood space (briefly $j - NS$) which represents a generalized type of neighborhood spaces; that was given in [20]. Moreover, we introduce a comprehensive survey about the near concepts in $j - \boldsymbol{NS}$ that were introduced in [21]. Different pairs of dual approximation operators were investigated and their properties being discussed. Comparisons between different operators were discussed. Many results, examples and counter examples were provided.

Definition 3.1 Let R be an arbitrary binary relation on a non-empty finite set U. The j-neighborhood of $x \in U$ $\left(N_j(x)\right), \forall\, j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, can be defined as follows:

(i) r-neighborhood: $N_r(x) = \{y \in U \mid xRy\}$,

(ii) l-neighborhood: $N_l(x) = \{y \in U \mid yRx\}$,

(iii) $\langle r\rangle$-neighborhood: $N_{\langle r\rangle}(x) = \bigcap_{x \in N_r(y)} N_r(y)$,

(iv) $\langle l\rangle$-neighborhood : $N_{\langle l\rangle}(x) = \bigcap_{x \in N_l(y)} N_l(y)$,

(v) i-neighborhood: $N_i(x) = N_r(x) \cap N_l(x)$,

(vi) u-neighborhood: $N_u(x) = N_r(x) \cup N_l(x)$,

(vii) $\langle i\rangle$-neighborhood: $N_{\langle i\rangle}(x) = N_{\langle r\rangle}(x) \cap N_{\langle l\rangle}(x)$,

(viii) $\langle u\rangle$-neighborhood: $N_{\langle u\rangle}(x) = N_{\langle r\rangle}(x) \cup N_{\langle l\rangle}(x)$.

Definition 3.2 Let R be an arbitrary binary relation on a non-empty finite set U and $\xi_j: U \longrightarrow P(U)$ be a mapping which assigns for each x in U its j-neighborhood in $P(U)$. The triple $\left(U, R, \xi_j\right)$ is called j-neighborhood space, in briefly $j - \boldsymbol{NS}$.

The following theorem is interesting since by using it we can generate eight different topologies.

Theorem 3.1 If $\left(U, R, \xi_j\right)$ is $j - \boldsymbol{NS}$, then the collection

$$\tau_j = \{A \subseteq U \mid \forall\, p \in A,\ N_j(p) \subseteq A\},$$

$\forall\, j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, is a topology on U.

Proof

$(T1)$ Clearly, U and $\emptyset$ belong to τ_j.

$(T2)$ Let $\{A_i \mid i \in I\}$ be a family of elements in τ_j and $p \in \bigcup_i A_i$. Then there exists $i_0 \in I$ such that $p \in A_{i_0}$. Thus $N_j(p) \subseteq A_{i_0}$ this implies $N_j(p) \subseteq \bigcup_i A_i$ and so $\bigcup_i A_i \in \tau_j$.

$(T3)$ Let $A_1, A_2 \in \tau_j$ and $p \in A_1 \cap A_2$. Then $p \in A_1$ and $p \in A_2$ which implies $N_j(p) \subseteq A_1$ and $N_j(p) \subseteq A_2$. Thus $N_j(p) \subseteq A_1 \cap A_2$ and then $A_1 \cap A_2 \in \tau_j$.

Accordingly τ_j is a topology on U. ■

Example 3.1 Let $U = \{a, b, c, d\}$ and

$R = \{(a,a),(a,d),(b,a),(b,c),(c,c),(c,d),(d,a)\}$. Thus we get

$N_r(a) = \{a,d\}, N_l(a) = \{a,b,d\}, N_i(a) = \{a,d\}$ and $N_u(a) = \{a,b,d\}$.

$N_r(b) = \{a,c\}, N_l(b) = \emptyset, N_i(b) = \emptyset$ and $N_u(b) = \{a,c\}$.

$N_r(c) = \{c,d\}, N_l(c) = \{b,c\}, N_i(c) = \{c\}$ and $N_u(c) = \{b,c,d\}$.

$$N_r(d) = \{a\}, N_l(d) = \{a,c\}, N_i(d) = \{a\} \; and \; N_u(d) = \{a,c\}.$$

$N_{\langle r\rangle}(a) = \{a\}, N_{\langle l\rangle}(a) = \{a\}, N_{\langle i\rangle}(a) = \{a\}$ and $N_{\langle u\rangle}(a) = \{a\}$.

$N_{\langle r\rangle}(b) = \emptyset, N_{\langle l\rangle}(b) = \{b\}, N_{\langle i\rangle}(b) = \emptyset$ and $N_{\langle u\rangle}(b) = \{b\}$.

$N_{\langle r\rangle}(c) = \{c\}, N_{\langle l\rangle}(c) = \{c\}, N_{\langle i\rangle}(c) = \{c\}$ and $N_{\langle u\rangle}(c) = \{c\}$.

$N_{\langle r\rangle}(d) = \{d\}, N_{\langle l\rangle}(d) = \{a,b,d\}, N_{\langle i\rangle}(d) = \{d\}$ $andN_{\langle u\rangle}(d) = \{a,b,d\}$.

Thus we get

$\tau_r = \{U, \emptyset, \{a,d\}, \{a,c,d\}\}$, $\tau_l = \{U, \emptyset, \{b\}, \{b,c\}\}$, $\tau_u = \{U, \emptyset\}$,

$\tau_i = \{U, \emptyset, \{b\}, \{c\}, \{a,d\}, \{b,c\}, \{a,b,d\}, \{a,c,d\}\}$, $\tau_{\langle r\rangle} = \tau_{\langle i\rangle} = \wp(U)$,

$\tau_{\langle l\rangle} = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\} = \tau_{\langle u\rangle}$.

Remark 3.1 From the results that were given in [20], the implications between different topologies τ_j are given in the following diagram (where $\longrightarrow$ means $\subseteq$).

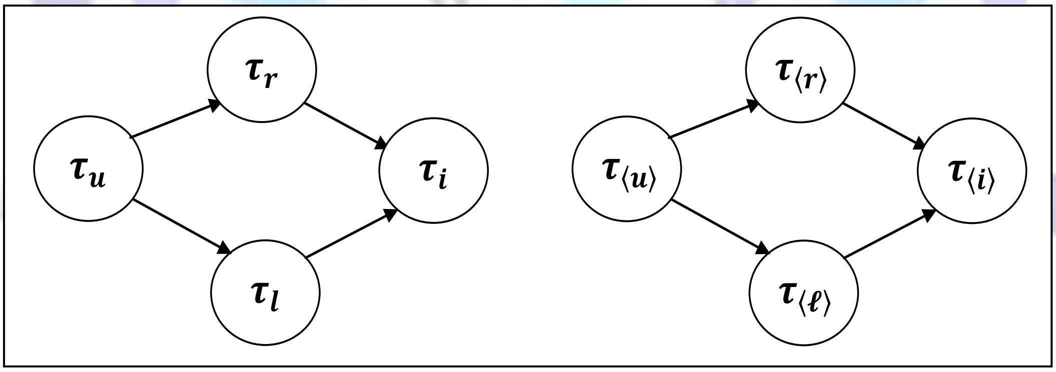

Diagram 3.1

By using the above topologies, we introduce eight methods for approximation rough sets using the interior and closure of the topologies τ_j, $\forall j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$.

Definition 3.4 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$. The subset $A \subseteq U$ is said to be j-open set if $A \in \tau_j$, the complement of j-open set is called j-closed set. The family Γ_j of all j-closed sets of a j-neighborhood space is defined by

$$\Gamma_j = \{ F \subseteq U \mid F^c \in \tau_j \}.$$

Definition 3.5 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. The j-lower and j-upper approximations of A are defined respectively by

$$\underline{R}_j(A) = \cup \{G \in \tau_j : G \subseteq A\} = j\text{-interior of } A,$$

$$\overline{R}_j(A) = \cap\{H \in \Gamma_j : A \subseteq H\} = j\text{-closure of } A.$$

Definition 3.6 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$ and $A \subseteq U$. The j-boundary, j-positive and j-negative regions of A are defined respectively by

$$B_j(A) = \overline{R}_j(A) - \underline{R}_j(A),$$

$$POS_j(A) = \underline{R}_j(A), \text{and}$$

$$NEG_j(A) = U - \overline{R}_j(A).$$

Definition 3.7 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$. Then subset A is called j-definable (exact) set if $\underline{R}_j(A) = \overline{R}_j(A) = A$. Otherwise, it is called j-rough.

Definition 3.8 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$. The j-accuracy of the approximations of $A \subseteq U$ is defined by

$$\delta_j(A) = \frac{|\underline{R}_j(A)|}{|\overline{R}_j(A)|}, where\ |\overline{R}_j(A)| \neq 0.$$

Remarks 3.2 It is clear that $0 \leq \delta_j(A) \leq 1$ and A is j-exact if $\mathcal{B}_j(A) = \emptyset$ and $\delta_j(A) = 1$. Otherwise, A is j-rough.

Remark 3.3 According to the above results, we can conclude that the using of τ_i in constructing the approximations of sets is accurate than τ_r, τ_l and τ_u. Also, the using of $\tau_{\langle i \rangle}$ in constructing the approximations of sets is accurate than $\tau_{\langle r \rangle}, \tau_{\langle l \rangle}$ and $\tau_{\langle u \rangle}$. Moreover, the topologies τ_i and $\tau_{\langle i \rangle}$ are not necessarily comparable and consequently so are $\alpha_i(A)$ and $\alpha_{\langle i \rangle}(A)$.

Some properties of the approximation operators $\underline{R}_j(A)$ and $\overline{R}_j(A)$ are imposed in the following proposition.

Proposition 3.1 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$ and $A, B \subseteq U$. Then

(1) $\underline{R}_j(A) \subseteq A \subseteq \overline{R}_j(A)$.

(2) $\underline{R}_j(U) = \overline{R}_j(U) = U$,

$\underline{R}_j(\emptyset) = \overline{R}_j(\emptyset) = \emptyset$.

(3) $\overline{R}_j(A \cup B) = \overline{R}_j(A) \cup \overline{R}_j(B)$.

(4) $\underline{R}_j(A \cap B) = \underline{R}_j(A) \cap \underline{R}_j(B)$.

(5) $If\ A \subseteq B\ then\ \underline{R}_j(A) \subseteq \underline{R}_j(B)$.

(6) $If\ A \subseteq B\ then\ \overline{R}_j(A) \subseteq \overline{R}_j(B)$.

(7) $\underline{R}_j(A \cup B) \supseteq \underline{R}_j(A) \cup \underline{R}_j(B)$.

(8) $\overline{R}_j(A \cap B) \subseteq \overline{R}_j(A) \cap \overline{R}_j(B)$.

(9) $\underline{R}_j(A) = [\overline{R}_j(A^c)]^c$, $where\ A^c\ is\ the\ complement\ of\ A$.

(10) $\overline{R}_j(A) = [\underline{R}_j(A^c)]^c$

(11) $\underline{R}_j(\underline{R}_j(A)) = \underline{R}_j(A)$

(12) $\overline{R}_j(\overline{R}_j(A)) = \overline{R}_j(A)$.

Proof By using properties of interior and closure, the proof is obvious. ■

Remark 3.4 The above proposition can be considered as one of the differences between our approaches and other generalizations such as [12, 18, 21, 25, and 27]. Although they used general binary relation but they added some conditions to satisfy the properties of Pawlak approximation operators. Our approaches satisfied most of the properties of Pawlak approximations. So, we can say that our approaches are the actual generalizations of Pawlak approximation space [36] and the other generalizations in [1, 4, 7, 9, 14, 16, 17, 27, 28 and 30-37].

The following table shows the comparisons between our approaches and some of other generalizations which used general relation.

ISSN 2347-1921

Properties of Pawlak approximations	Yao [35] and others [1, 4, 7, 9, 14, and 30]	$j - NS$
(L1)	*	*
(L2)	*	*
(L3)	*	*
(L4)	*	*
(L5)	*	*
(L6)		*
(L7)		*
(L8)		*
(L9)		*
(U1)	*	*
(U2)		*
(U3)	*	*
(U4)	*	*
(U5)	*	*
(U6)		*
(U7)		*
(U8)		*
(U9)		*

Table 3.1

The following example illustrates the comparison between our approaches and Yao's method [34].

Example 3.2 Let (U, R, ξ_j) is a $j - \boldsymbol{NS}$ where $U = \{a, b, c, d\}$ and

$$R = \{(a, c), (b, b), (c, a), (d, a)\}.$$

Then we compute the approximations of all subsets of U according to Yao method as follows:

Yao [35] defines the approximations of any subset $X \subseteq U$ as follow:

$$\underline{apr}(X) = \{x \in U: xR \subseteq X\} \text{ and } \overline{apr}(X) = \{x \in U: xR \cap X \neq \emptyset\}.$$

The following table gives the comparison between Yao approach and our approaches $j - \boldsymbol{NS}$, in case of $j = \langle r \rangle, \langle l \rangle$, and the other cases similarly.

ISSN 2347-1921

$\wp(U)$	Yao's approach		$j-\boldsymbol{NS}$			
	$\underline{apr}(A)$	$\overline{apr}(A)$	$\underline{R}_{\langle r\rangle}(A)$	$\overline{R}_{\langle r\rangle}(A)$	$\underline{R}_{\langle l\rangle}(A)$	$\overline{R}_{\langle l\rangle}(A)$
$\{a\}$	$\{c,d\}$	$\{c,d\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{c\}$	$\{a\}$	$\{a\}$	$\{c\}$	$\{c\}$	$\emptyset$	$\{c,d\}$
$\{d\}$	$\emptyset$	$\emptyset$	$\{d\}$	$\{d\}$	$\emptyset$	$\{c,d\}$
$\{a,b\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c\}$	$\{a,c\}$	$\{a\}$	$\{a,c,d\}$
$\{a,d\}$	$\{c,d\}$	$\{c,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$	$\{a,c,d\}$
$\{b,c\}$	$\{a,b\}$	$\{a,b\}$	$\{b,c\}$	$\{b,c\}$	$\{b\}$	$\{b,c,d\}$
$\{b,d\}$	$\{b\}$	$\{b\}$	$\{b,d\}$	$\{b,d\}$	$\{b\}$	$\{b,c,d\}$
$\{c,d\}$	$\{a\}$	$\{a\}$	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$
$\{a,b,c\}$	U	U	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b\}$	U
$\{a,b,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b\}$	U
$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$
$\{b,c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$
U	U	U	U	U	U	U
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 3.2: Exact Sets and Rough Set.

From the above table, we can notice that:

(i) $\underline{apr}(X) \nsubseteq X \nsubseteq \overline{apr}(X)$ e.g.$\{a\}$ and $\{d\}$. But in our approaches

$$\underline{R}_j(X) \subseteq X \subseteq \overline{R}_j(X)\text{, for any } X \subseteq U.$$

(ii) There are many subsets in U are rough in Yao's approach (except the shaded sets), but in our approaches $j-\boldsymbol{NS}$ there are many subsets are j-exacts such as the sets that shaded in the above table. Also, if there is exact set in Yao's approach, then it is exact in our approaches (but the converse is not true in general). Moreover, the boundary region was reduced and became smaller than Yao approach.

Remark 3.5 In Yao's approach $\underline{apr}(\emptyset) \neq \emptyset$ and $\overline{apr}(U) \neq U$ in general, as the following example illustrates.

Example 3.3 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$ where $U = \{a,b,c,d\}$ and $R = \{(a,a),(b,b),(c,c),(c,d)\}$. Then we get $aR = \{a\}, bR = \{b\}, cR = \{c,d\}$ and $dR = \emptyset$.

Accordingly, we have $\underline{apr}(\emptyset) = \{d\} \neq \emptyset$ and $\overline{apr}(U) = \{a,b,c\} \neq U$

In what follows, we introduce one of the important topological concepts named "j-near open sets". By using it, we define new forty approximations as mathematical tools to modify the j-approximations in the $j-\boldsymbol{NS}$. Properties of the introduced approximation operators are investigated, and their connections are examined.

Definition 3.9 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$. Then, for each $j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$, the subset $A \subseteq U$ is called:

(i) j-Regular-open (briefly R_j^*-open) if $A = \boldsymbol{int_j}(\boldsymbol{cl_j}(A))$.

(ii) j-Pre-open (briefly P_j-open) if $A \subseteq \boldsymbol{int_j}(\boldsymbol{cl_j}(A))$.

(iii) j-Semi-open (briefly S_j-open) if $A \subseteq \boldsymbol{cl_j}(\boldsymbol{int_j}(A))$.

(iv) γ_j-open if $A \subseteq \boldsymbol{int_j}(\boldsymbol{cl_j}(A)) \cup \boldsymbol{cl_j}(\boldsymbol{int_j}(A))$.

(v) α_j -open if $A \subseteq \boldsymbol{int_j}[\boldsymbol{cl_j}(\boldsymbol{int_j}(A))]$.

(vi) β_j -open (semi-pre-open) if $A \subseteq \boldsymbol{cl_j}[\boldsymbol{int_j}(\boldsymbol{cl_j}(A))]$.

Remarks 3.6

(i) The above sets are called j-near open sets and the families of j-near open sets of U denoted by $K_j O(U)$, for each $K = R^*, P, S, \gamma, \alpha$ and β.

(ii) The complements of the $j - near$ open sets are called j-near closed sets and the families of j-near closed sets of U denoted by $K_j C(U)$, for each

$K = R^*, P, S, \gamma, \alpha$ and β.

(iii) According to [21], $\alpha_j O(U)$ represent a topology on U, and then the j-near interior (resp. the j-near closure) represent the j-interior (resp. the j-closure).

Remark 3.7 According to the results in [21], the implications between the topologies τ_j and the above families of j-near open sets (resp. j-near closed sets) are given in the following diagram (where $\longrightarrow$ means $\subseteq$).

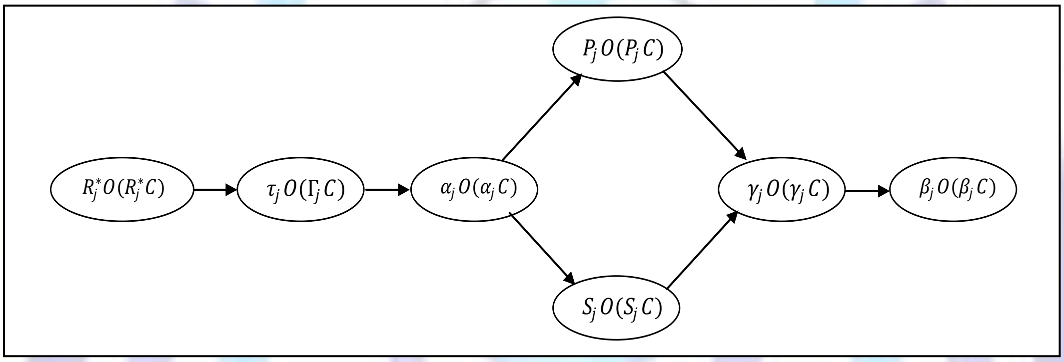

Diagram 3.2

By using the j-near open set, we can introduce new methods for approximation rough sets using the j-near interior and the j-near closure for each topology of τ_j as the following definitions illustrate.

Definition 3.10 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then, for each $j \in \{r, l, \langle r \rangle, \langle l \rangle, u, i, \langle u \rangle, \langle i \rangle\}$ and $k \in \{R^*, p, s, \gamma, \alpha, \beta\}$, the j-near lower and j-near upper approximations of A are defined respectively by

$$\underline{R}_j^k(A) = \cup \{G \in k_j O(U): G \subseteq A\} = j\text{-near interior of } A,$$

$$\overline{R}_j^k(A) = \cap \{H \in k_j C(U): A \subseteq H\} = j\text{-near closure of } A.$$

Definition 3.11 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then, for each

$j \in \{r, l, \langle r \rangle, \langle l \rangle, u, i, \langle u \rangle, \langle i \rangle\}$ and $k \in \{R^*, p, s, \gamma, \alpha, \beta\}$, the j-near boundary, j-near positive and j-near negative regions of A are defined respectively by

$$B_j^k(A) = \overline{R}_j^k(A) - \underline{R}_j^k(A),$$

$$POS_j^k(A) = \underline{R}_j^k(A) \text{ and}$$

$$NEG_j^k(A) = U - \overline{R}_j^k(A).$$

Definition 3.12 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then, for each

$j \in \{r, l, \langle r \rangle, \langle l \rangle, u, i, \langle u \rangle, \langle i \rangle\}$ and $k \in \{R^*, p, s, \gamma, \alpha, \beta\}$, the j-near accuracy of the j-near approximations of $A \subseteq U$ is defined by

$$\delta_j^k(A) = \frac{|\underline{R}_j^k(A)|}{|\overline{R}_j^k(A)|}, where\ |\overline{R}_j^k(A)| \neq 0.$$

It is clear that $0 \leq \delta_j^k(A) \leq 1$.

The following propositions give the fundamental properties of the j-near approximations.

Proposition 3.2 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$ and $A \subseteq U$. Then, for each

$j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$ and $k = r^*, p, s, \gamma, \alpha, \beta$

(1) $\underline{R}_j^k(A) \subseteq A$.

(2) $\underline{R}_j^k(U) = \overline{R}_j^k(U) = U$

(3) If $A \subseteq B$ then $\underline{R}_j^k(A) \subseteq \underline{R}_j^k(B)$.

(4) $\underline{R}_j^k(A \cap B) \subseteq \underline{R}_j^k(A) \cap \underline{R}_j^k(B)$.

(5) $\underline{R}_j^k(A \cup B) \supseteq \underline{R}_j^k(A) \cup \underline{R}_j^k(B)$.

(6) $\underline{R}_j^k(A) = [\overline{R}_j^k(A^c)]^c$, where A^c is the complement of A.

(7) $\underline{R}_j^k(\underline{R}_j^k(A)) = \underline{R}_j^k(A)$.

(8) $A \subseteq \overline{R}_j^k(A)$.

(9) $\underline{R}_j^k(\emptyset) = \overline{R}_j^k(\emptyset) = \emptyset$.

(10) If $A \subseteq B$ then $\overline{R}_j^k(A) \subseteq \overline{R}_j^k(B)$.

(11) $\overline{R}_j^k(A \cap B) \subseteq \overline{R}_j^k(A) \cap \overline{R}_j^k(B)$.

(12) $\overline{R}_j^k(A \cup B) \supseteq \overline{R}_j^k(A) \cup \overline{R}_j^k(B)$.

(13) $\overline{R}_j^k(A) = [\underline{R}_j^k(A^c)]^c$, where A^c is the complement of A.

(14) $\overline{R}_j^k(\overline{R}_j^k(A)) = \overline{R}_j^k(A)$.

Remark 3.8 Since the topologies τ_j are larger than the families of all regular open sets of U, $R_j^*O(U)$, (that is, $R_j^*O(U)$ represents a special case of the topologies τ_j) then we will not using it in our approaches.

The j-near approximations are very interesting in rough context since the use of the j-near structures can help for further developments in the theoretical and applications of rough sets. Moreover, the j-near approximations can help in the discovery of hidden information in data collected from real-life applications, since the boundary regions will decreased or cancelled by increasing the lower and decreasing the upper approximations, as the following results illustrate.

Proposition 3.3 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$ and $A \subseteq U$. Then, for each

$j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$and $k \in \{R^*, p, s, \gamma, \alpha, \beta\}$ such that $k \neq R^*$:

$$\underline{R}_j(A) \subseteq \underline{R}_j^k(A) \subseteq A \subseteq \overline{R}_j^k(A) \subseteq \overline{R}_j(A)$$

Proof Since the families of j-near open sets $k_jO(U)$ (resp. j-near closed sets $k_jC(U)$) are larger than the topologies τ_j (resp. the families of j-closed sets Γ_j). Then we have

$$\underline{R}_j(A) = \cup\{G \in \tau_j : G \subseteq A\} \subseteq \cup\{G \in k_jO(U) : G \subseteq A\} = \underline{R}_j^k(A).$$

By similar way, we can prove $\overline{R}_j^k(A) \subseteq \overline{R}_j(A)$.■

Corollary 3.1 Let (U, R, ξ_j) be $j-\boldsymbol{NS}$ and $A \subseteq U$. Then, for each

$j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$and $k \in \{R^*, p, s, \gamma, \alpha, \beta\}$ such that $k \neq R^*$:

(1) $B_j^k(A) \subseteq B_j(A)$. (2) $\delta_j(A) \leq \delta_j^k(A)$.

The aim of the following example is to show that, the above results and to illustrate the importance of using j-near concepts in rough context.

Example 3.4 Let the $j-\boldsymbol{NS}$ (U, R, ξ_j), where $U = \{a, b, c, d\}$ and

$R = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, a), (c, b), (c, c), (c, d), (d, d)\}$.

Then, we get $N_r(a) = \{a, b\}, N_r(b) = \{a, b\}, N_r(c) = U, N_r(d) = \{d\}$ this implies

$\tau_r = \{U, \emptyset, \{d\}, \{a, b\}, \{a, b, d\}\}$ and $\Gamma_r = \{U, \emptyset, \{c\}, \{c, d\}, \{a, b, c\}\}$.

We shall compute the j-near approximations for $j = r$ and $k = p, \gamma, \beta$ and the other cases similarly as follow:

$P_rO(U) = \{U, \emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$,

$P_r C(U) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$,

$\gamma_r O(U) = \{U, \emptyset, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$,

$\gamma_r C(U) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}$,

$\beta_r O(U) = \{U, \emptyset, \{a\}, \{b\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\},$
$\{a,c,d\}, \{b,c,d\}\}$ and

$\beta_r C(U) = \{U, \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,c,d\},$
$\{b,c,d\}\}$.

The following table introduces comparisons between the j-approximations and the j-near approximations of the all subsets of U as follow:

$\wp(U)$	τ_r		P_r		γ_r		β_r	
	$\underline{R}_r(A)$	$\overline{R}_r(A)$	$\underline{R}_r^p(A)$	$\overline{R}_r^p(A)$	$\underline{R}_r^\gamma(A)$	$\overline{R}_r^\gamma(A)$	$\underline{R}_r^\beta(A)$	$\overline{R}_r^\beta(A)$
$\{a\}$	$\emptyset$	$\{a,b,c\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
$\{b\}$	$\emptyset$	$\{a,b,c\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
$\{c\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\{c\}$	$\emptyset$	$\{c\}$
$\{d\}$	$\{d\}$	$\{c,d\}$	$\{d\}$	$\{c,d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\emptyset$	$\{a,b,c\}$	$\{a\}$	$\{a,c\}$	$\{a\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$
$\{a,d\}$	$\{d\}$	U	$\{a,d\}$	$\{a,c,d\}$	$\{a,d\}$	$\{a,c,d\}$	$\{a,d\}$	$\{a,d\}$
$\{b,c\}$	$\emptyset$	$\{a,b,c\}$	$\{b\}$	$\{b,c\}$	$\{b\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
$\{b,d\}$	$\{d\}$	U	$\{b,d\}$	$\{b,c,d\}$	$\{b,d\}$	$\{b,c,d\}$	$\{b,d\}$	$\{b,d\}$
$\{c,d\}$	$\emptyset$	$\{c,d\}$	$\{d\}$	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$
$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b,d\}$	$\{a,b,d\}$	U	$\{a,b,d\}$	U	$\{a,b,d\}$	U	$\{a,b,d\}$	U
$\{a,c,d\}$	$\{d\}$	U	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$
$\{b,c,d\}$	$\{d\}$	U	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$
U	U	U	U	U	U	U	U	U
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 3.3: ☐ Exact Set

From the above table, we can notice that:

(i) Applying the j-near approximations is very interesting for removing the vagueness of rough sets, and this would helps to extract and discovery of hidden information in data collected from real-life applications.

(ii) The best j-near approach is β_j (since β_j is more accurate than the other types of j-near open sets).

(iii) There are many rough sets in τ_r, but it is j-near exact such as the shaded sets.

4 j-Rough Membership Relations, j-Rough Membership Functions and j-Fuzzy Sets.

The present section is provided to introduce new definitions of "rough membership relations, rough membership functions and fuzzy sets" in $j - \boldsymbol{NS}$. Moreover, we introduce some differences between our approaches and some others approaches such as Lin [28]. In addition, we give some solutions to accurate the approximations and exactness of rough sets. In the last of the section we give some connections between rough set theory, fuzzy set theory and topology.

Definition 4.1 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then we say that:

(i) x is " j-surely" belongs to A, written $x \underline{\in}_j A$, if $x \in \underline{R}_j(A)$.

(ii) x is " j-possibly" belongs to X, written $x\ \overline{\in}_j\ A$, if $x \in \overline{R}_j(A)$.

These two membership relations are called " j-strong" and " j-weak" membership relations respectively, $\forall\ j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$.

Lemma 4.1 Let the triple (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then the following statements are true in general:

(i) If $x\ \underline{\in}_j\ A$ implies to $x \in A$.

(ii) If $x \in A$ implies to $x\ \overline{\in}_j\ A$.

Proof Straight forward. ■

Remark 4.1 The converse of the above lemma is not true in general, as the following example illustrates:

Example 4.1 Consider the triple (U, R, ξ_j) be $j - \boldsymbol{NS}$, where $U = \{a, b, c, d\}$ and $= \{(a, a), (b, b), (c, c), (c, b), (c, d), (d, a)\}$. Then we get

$$\tau_r = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}\text{and}$$

$$\Gamma_r = \{U, \emptyset, \{c\}, \{b, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}.$$

We show the above remark in case of $j = r$ and the other cases similarly. Suppose that $A = \{a, b, c\}$, then we get

$\underline{R}_r(A) = \{a, b\}$ and $\overline{R}_r(A) = U$.

Clearly $c \in A$ but $c\ \underline{\notin}_j\ A$ and $d\ \overline{\in}_r\ A$ but $d \notin A$.

Proposition 4.1 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A, B \subseteq U$. Then by using the properties of the j-approximations we can prove the following properties:

(i) If $A \subseteq B$, then ($x\ \underline{\in}_j\ A \Longrightarrow x\ \underline{\in}_j\ B$ and $x\ \overline{\in}_j\ A \Longrightarrow x\ \overline{\in}_j\ B$).

(ii) $x\ \overline{\in}_j\ (A \cup B) \Longleftrightarrow x\ \overline{\in}_j\ A$ or $x\ \overline{\in}_j\ B$.

(iii) $x\ \overline{\in}_j\ (A \cup B) \Longleftrightarrow x\ \overline{\in}_j\ A$ and $x\ \overline{\in}_j\ B$.

(iv) If $x\ \underline{\in}_j\ A$ or $x\ \underline{\in}_j\ B$, then $x\ \underline{\in}_j\ (A \cup B)$.

(v) If $x\ \underline{\in}_j\ (A \cup B)$, then $x\ \underline{\in}_j\ A$ and $x\ \underline{\in}_j\ B$.

(vi) $x\ \underline{\in}_j\ A^c \Longleftrightarrow \text{non}\ x\ \overline{\in}_j\ A$.

(vii) $x\ \overline{\in}_j\ A^c \Longleftrightarrow \text{non}\ x\ \underline{\in}_j\ A$.

Remarks 4.2 We can redefine the j-approximations by using $\underline{\in}_j$ and $\overline{\in}_j$ as follows, for any $A, B \subseteq U$:

$$\underline{R}_j(A) = \{x \in U | x\ \underline{\in}_j\ A\}\text{and } \overline{R}_j(A) = \{x \in U | x\ \overline{\in}_j\ A\}.$$

The following proposition is very interesting since it is give the relations between different types of j-rough membership relations $\underline{\in}_j$ and $\overline{\in}_j$. Accordingly, we will illustrate the importance of using these different types of membership relations.

Proposition 4.2 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then

(i) If $x\ \underline{\in}_i\ A \Longrightarrow x\ \underline{\in}_r\ A \Longrightarrow x\ \underline{\in}_u\ A$.

(ii) If $x\ \underline{\in}_i\ A \Longrightarrow x\ \underline{\in}_l\ A \Longrightarrow x\ \underline{\in}_u\ A$.

(iii) If $x\ \overline{\in}_u\ A \Longrightarrow x\ \overline{\in}_r\ A \Longrightarrow x\ \overline{\in}_i\ A$.

(iv) If $x\ \overline{\in}_u\ A \Longrightarrow x\ \overline{\in}_l\ A \Longrightarrow x\ \overline{\in}_i\ A$.

(v) If $x\ \underline{\in}_{\langle i\rangle}\ A \Longrightarrow x\ \underline{\in}_{\langle r\rangle}\ A \Longrightarrow x\ \underline{\in}_{\langle u\rangle}\ A$.

(vi) If $x\ \underline{\in}_{\langle i\rangle}\ A \Longrightarrow x\ \underline{\in}_{\langle l\rangle}\ A \Longrightarrow x\ \underline{\in}_{\langle u\rangle}\ A$.

(vii) If $x\ \overline{\in}_{\langle u\rangle}\ A \Longrightarrow x\ \overline{\in}_{\langle r\rangle}\ A \Longrightarrow x\ \overline{\in}_{\langle i\rangle}\ A$.

$(viii)$ If $x\ \overline{\in}_{\langle u\rangle}\ A \Longrightarrow x\ \overline{\in}_{\langle l\rangle}\ A \Longrightarrow x\ \overline{\in}_{\langle i\rangle}\ A$.

Proof We will prove first statement and the others similarly:

(i) If $x \underline{\in}_i A \Longrightarrow x \in \underline{R}_i(A) \Longrightarrow x \in \underline{R}_r(A) \Longrightarrow x \underline{\in}_r A$.

Also, if $x \underline{\in}_r A \Longrightarrow x \in \underline{R}_r(A) \Longrightarrow x \in \underline{R}_u(A) \Longrightarrow x \underline{\in}_u A$. ■

Remark 4.3 The converse of the above proposition is not true in general as the following example illustrates.

Example 4.2 Let the triple (U, R, ξ_j) be $-\boldsymbol{NS}$, where $U = \{a, b, c, d\}$ and

$R = \{(a, a), (a, b), (b, c), (b, d), (c, a), (d, a)\}$. Then we get

$\tau_{\langle r\rangle} = \{U, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$and $\Gamma_{\langle r\rangle} = \{U, \emptyset, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}\}$.

$\tau_{\langle l\rangle} = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$and $\Gamma_{\langle l\rangle} = \{U, \emptyset, \{b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

$\tau_{\langle i\rangle} = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\} = \Gamma_{\langle i\rangle}$.

$\tau_{\langle u\rangle} = \{U, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}\}$and $\Gamma_{\langle u\rangle} = \{U, \emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$.

Suppose that $A = \{b, c, d\}$. Thus we get

$\underline{R}_{\langle u\rangle}(A) = \emptyset$, $\underline{R}_{\langle r\rangle}(A) = \{c, d\}$, $\underline{R}_{\langle l\rangle}(A) = \{b\}and\underline{R}_{\langle i\rangle}(A) = \{b, c, d\}$.

Accordingly, $c \underline{\in}_{\langle r\rangle} A$ and $b \underline{\in}_{\langle l\rangle} A$ but $b \underline{\notin}_{\langle u\rangle} A$ and $c \underline{\notin}_{\langle u\rangle} A$.

Also $b \underline{\in}_{\langle i\rangle} A$ and $c \underline{\in}_i A$ but $b \underline{\notin}_r A$ and $c \underline{\notin}_l A$.

By similar way, we can illustrate the others cases.

Definition 4.2 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then $\forall\, j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}$ and $x \in U$ we define the j-rough membership functions of $j - \boldsymbol{NS}$ as follows:

The j- rough membership functions on Ufor subset A are $\mu_A^j: U \to [0,1]$, where

$$\mu_A^j(x) = \frac{|\{\cap N_j(x)\} \cap A|}{|\cap N_j(x)|}$$

and $|A|$denotes the cardinality of A.

The rough j-membership function expresses conditional probability that x belongs to A given R and can be interpreted as a degree that x belongs to A in view of information about x expressed by R. Moreover, in case of infinite universe, the above membership function μ_A^j can be use for spaces having locally finite minimal neighborhoods for each point.

Remark 4.4 The rough j-membership functions can be used to define j-approximations of a set A, as shown below:

$$\underline{R}_j(A) = \{x \in U |\ \mu_A^j(x) = 1\} \text{ and}$$

$$\overline{R}_j(A) = \{x \in U |\ \mu_A^j(x) > 0\}.$$

The following results give the fundamental properties of the above j- rough membership functions.

Proposition 4.3 Let(U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A, B \subseteq U$. Then

(i) $\mu_A^j(x) = 1 \Leftrightarrow x \underline{\in}_j A$.

(ii) $\mu_A^j(x) = 0 \Leftrightarrow x \in U - \overline{R}_j(A)$.

(iii) $0 < \mu_A^j(x) < 1 \Leftrightarrow x \in \mathcal{B}_j(A)$.

(iv) $\mu_{U-A}^j(x) = 1 - \mu_A^j(x)$ for any $x \in U$.

(v) $\mu_{A\cup B}^j(x) \geq \max(\mu_A^j(x), \mu_B^j(x)) for\ any\ x \in U$.

(vi) $\mu_{A\cap B}^j(x) \leq \min(\mu_A^j(x), \mu_B^j(x)) for\ any\ x \in U$.

Proof We will prove (i), and the others similarly.

$$x \underline{\in}_j A \Leftrightarrow x \in \underline{R}_j(A) \Leftrightarrow x \in A, N_j(x) \subseteq A \Leftrightarrow \mu_A^j(x) = 1. ■$$

Remark 4.5 The rough j-membership functions divides the universe U by using the j-boundary, j-positive and j-negative regions of $A \subseteq U$, respectively as follow:

ISSN 2347-1921

$$\mathcal{B}_j(A) = \{x \in U |\ 0 < \mu_A^j(x) < 1\},$$

$$POS_j(A) = \{x \in U |\ \mu_A^j(x) = 1\} \text{ and}$$

$$NEG_j(A) = \{x \in U |\ \mu_A^j(x) = 0\}.$$

Lemma 4.2 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then for every $x \in U$

(i) $\mu_A^u(x) = 1 \implies \mu_A^r(x) = 1 \implies \mu_A^i(x) = 1$.

(ii) $\mu_A^u(x) = 1 \implies \mu_A^l(x) = 1 \implies \mu_A^i(x) = 1$.

(iii) $\mu_A^{\langle u \rangle}(x) = 1 \implies \mu_A^{\langle r \rangle}(x) = 1 \implies \mu_A^{\langle i \rangle}(x) = 1$.

(iv) $\mu_A^{\langle u \rangle}(x) = 1 \implies \mu_A^{\langle l \rangle}(x) = 1 \implies \mu_A^{\langle i \rangle}(x) = 1$.

Proof

(i) If $\mu_A^u(x) = 1 \implies x \in_u A \implies x \in_r A \implies \mu_A^r(x) = 1$.

Also, if $\mu_A^r(x) = 1 \implies x \in_r A \implies x \in_i A \implies \mu_A^i(x) = 1$.

(ii) , (iii) and (iv) Similarly as (i).■

Lemma 4.3 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then for every $x \in U$

(i) $\mu_A^u(x) = 0 \implies \mu_A^r(x) = 0 \implies \mu_A^i(x) = 0$.

(ii) $\mu_A^u(x) = 0 \implies \mu_A^l(x) = 0 \implies \mu_A^i(x) = 0$.

(iii) $\mu_A^{\langle u \rangle}(x) = 0 \implies \mu_A^{\langle r \rangle}(x) = 0 \implies \mu_A^{\langle i \rangle}(x) = 0$.

(iv) $\mu_A^{\langle u \rangle}(x) = 0 \implies \mu_A^{\langle l \rangle}(x) = 0 \implies \mu_A^{\langle i \rangle}(x) = 0$.

Proof

(i) If $\mu_A^u(x) = 1 \implies N_u(x) \cap A = \emptyset \implies N_r(x) \cap A = \emptyset$

$\implies \mu_A^r(x) = 0$.

Also, if $\mu_A^r(x) = 0 \implies N_r(x) \cap A = \emptyset \implies N_i(x) \cap A = \emptyset$

$\implies \mu_A^i(x) = 0$.

(ii) , (iii) and (iv) Similarly as (i).■

Remarks 4.6

(i) According to the above results and by using Proposition 4.2, we can prove that μ_A^i is more accurate than the others types, this means that:

(1)If $x \in A \implies \mu_A^u(x) \le \mu_A^r(x) \le \mu_A^i(x)$ and

if $x \in A \implies \mu_A^u(x) \le \mu_A^l(x) \le \mu_A^i(x)$.

(2)If $x \notin A \implies \mu_A^i(x) \le \mu_A^r(x) \le \mu_A^u(x)$ and

if $x \notin A \implies \mu_A^i(x) \le \mu_A^l(x) \le \mu_A^u(x)$.

(3)If $x \in A \implies \mu_A^{\langle u \rangle}(x) \le \mu_A^{\langle r \rangle}(x) \le \mu_A^{\langle i \rangle}(x)$ and

if $x \in A \implies \mu_A^{\langle u \rangle}(x) \le \mu_A^{\langle l \rangle}(x) \le \mu_A^{\langle i \rangle}(x)$.

(4)If $x \notin A \implies \mu_A^{\langle i \rangle}(x) \le \mu_A^{\langle r \rangle}(x) \le \mu_A^{\langle u \rangle}(x)$ and

if $x \notin A \implies \mu_A^{\langle i \rangle}(x) \le \mu_A^{\langle l \rangle}(x) \le \mu_A^{\langle u \rangle}(x)$.

(ii) The converse of the above lemmas is not true in general.

The following example illustrates Remarks 4.6.

Example 4.3 According to Example 4.2, consider the subset $A = \{b, c, d\}$. Then we get

$\mu_A^{\langle r\rangle}(a) = \frac{|\{a\}\cap A|}{|\{a\}|} = 0.$ $\mu_A^{\langle l\rangle}(a) = \frac{|\{a\}\cap A|}{|\{a\}|} = 0.$

$\mu_A^{\langle r\rangle}(b) = \frac{|\{a,b\}\cap A|}{|\{a,b\}|} = \frac{1}{2}.$ $\mu_A^{\langle l\rangle}(b) = \frac{|\{b\}\cap A|}{|\{b\}|} = 1.$

$\mu_A^{\langle r\rangle}(c) = \frac{|\{c,d\}\cap A|}{|\{c,d|} = 1.$ $\mu_A^{\langle l\rangle}(c) = \frac{|\{a,c,d\}\cap A|}{|\{a,c,d\}|} = \frac{2}{3}.$

$\mu_A^{\langle r\rangle}(d) = \frac{|\{c,d\}\cap A|}{|\{c,d\}|} = 1.$ $\mu_A^{\langle l\rangle}(d) = \frac{|\{a,c,d\}\cap A|}{|\{a,c,d\}|} = \frac{2}{3}.$

$\mu_A^{\langle i\rangle}(a) = \frac{|\{a\}\cap A|}{|\{a\}|} = 0.$ $\mu_A^{\langle u\rangle}(a) = \frac{|\{a\}\cap A|}{|\{a\}|} = 0.$

$\mu_A^{\langle i\rangle}(b) = \frac{|\{b\}\cap A|}{|\{b\}|} = 1.$ $\mu_A^{\langle u\rangle}(b) = \frac{|\{a,b\}\cap A|}{|\{a,b\}|} = \frac{1}{2}.$

$\mu_A^{\langle i\rangle}(c) = \frac{|\{c,d\}\cap A|}{|\{c,d\}|} = 1.$ $\mu_A^{\langle u\rangle}(c) = \frac{|\{a,c,d\}\cap A|}{|\{a,c,d\}|} = \frac{2}{3}.$

$\mu_A^{\langle i\rangle}(d) = \frac{|\{c,d\}\cap A|}{|\{c,d\}|} = 1.$ $\mu_A^{\langle u\rangle}(d) = \frac{|\{a,c,d\}\cap A|}{|\{a,c,d\}|} = \frac{2}{3}.$

Remark 4.7 Lin [28] have defined rough membership function for any binary relation, this membership function coincide with our membership function μ_A^jin case of $j = r$(r-rough membership function μ_A^r) only. So, our approaches represent generalization for Lin approach. Moreover, our membership functions are accurate more than Lin membership function.

One of the key issues in all fuzzy sets is how to determine fuzzy membership functions. The membership function fully defines the fuzzy set, which represent the basic tool in fuzzy theory. A membership functions provides a measure of the degree of similarity of element to fuzzy set. The following definition uses the j-rough membership functions μ_A^j to define four different types of fuzzy sets in $j - \boldsymbol{NS}$.

Definition 4.3 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then we define j-fuzzy sets in U is a set of ordered pairs:

$\tilde{A}_j = \{(x, \mu_A^j(x)) | x \in U\}.$

Example 4.4 According to Example 4.2, consider the subset $A = \{b, c, d\}$. Then we get

$$\tilde{A}_{\langle r\rangle} = \{(a,0),(b,\tfrac{1}{2}),(c,1),(d,1)\},\ \tilde{A}_{\langle l\rangle} = \{(a,0),(b,1),(c,\tfrac{2}{3}),(d,\tfrac{2}{3})\},$$

$$\tilde{A}_{\langle u\rangle} = \{(a,0),(b,\tfrac{1}{2}),(c,\tfrac{2}{3}),(d,\tfrac{2}{3})\},\ \text{and}\ \tilde{A}_{\langle i\rangle} = \{(a,0),(b,1),(c,1),(d,1)\}.$$

5 j -Near Rough Membership Relations, j -Near Rough Membership Functions and j-Fuzzy Sets in $j - \boldsymbol{NS}$.

By considering j-near concepts, we introduce the new concepts j-near rough membership relations (resp. j-near rough membership functions) to modify and generalize the j-membership relations (resp. j-membership functions) in $j - \boldsymbol{NS}$. The near rough membership functions are considered as easy tools to classify the sets and help for measuring near exactness and near roughness of sets. The existence of near rough membership functions made us to introduce the concept of near fuzzy sets.

Definition 5.1 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then

$\forall\, j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}, k \in \{p, s, \gamma, \alpha, \beta\}$, we say that:

(i) x is " j-near surely"(briefly k_j-surely) belongs to A, written $x \underline{\in}_j^k A$, if $x \in \underline{R}_j^k(A)$.

(ii) x is " j-near possibly"(briefly k_j- possibly) belongs to A, written $x \overline{\in}_j^k A$, if $x \in \overline{R}_j^k(A)$.

These two membership relations are called " j-near strong" and " j-near weak" membership relations respectively.

Lemma 5.1 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then

$\forall\, j \in \{r, l, \langle r\rangle, \langle l\rangle, u, i, \langle u\rangle, \langle i\rangle\}, k \in \{p, s, \gamma, \alpha, \beta\}$, the following statements are true in general:

(i) If $x \underline{\in}_j^k A$ implies to $x \in A$.

(ii) If $x \in A$ implies to $x \overline{\in}_j^k A$.

Proof Straight forward. ■

The converse of the above lemma is not true in general, as the following example illustrates:

Example 5.1 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$, where $U = \{a, b, c, d\}$ and $R = \{(a,a),(b,b),(b,a),$

$(c,a),(c,d),(d,a),(d,c),(d,d)\}$.Thus we get

$$N_{\langle r\rangle}(a)=\{a\}\,,\ N_{\langle r\rangle}(b)=\{a,b\}\,,\ N_{\langle r\rangle}(c)=\{a,c,d\},\ \ N_{\langle r\rangle}(d)=\{d\}.$$

We will show the above remark in case of ($j=\langle r\rangle$ and $k=p$) and the other cases similarly.

$$P_{\langle r\rangle}O(U)=\{U,\emptyset,\{a\},\{d\},\{a,b\},\{a,d\},\{a,b,d\},\{a,c,d\}\} \text{ and}$$

$$P_{\langle r\rangle}C(U)=\{U,\emptyset,\{b\},\{c\},\{b,c\},\{c,d\},\{a,b,c\},\{b,c,d\}\}$$

Suppose that $A=\{b,d\}$, then we get

$\underline{R}^{p}_{\langle r\rangle}(A)=\{d\}$ and $\overline{R}^{p}_{\langle r\rangle}(A)=\{b,c,d\}$. Clearly $b\in A$, but $b\ \underline{\notin}^{p}_{\langle r\rangle}\ A$ and $c\ \overline{\in}^{p}_{\langle r\rangle}\ A$ but $c\notin A$.

Remarks 5.1 We can redefine the j-near approximations by using $\underline{\in}^{k}_{j}$ and $\overline{\in}^{k}_{j}$ as follows:

For any $A,B\subseteq U$

$$\underline{R}^{k}_{j}(A)=\{x\in U|x\ \underline{\in}^{k}_{j}\ A\}\text{and }\overline{R}^{k}_{j}(A)=\{x\in U|x\ \overline{\in}^{k}_{j}\ A\}.$$

The following proposition is very interesting since it is give the relations between the j- rough membership relations and j-near rough membership relations. Accordingly, we will illustrate the importance of using these different types of j-near rough membership relations.

Proposition 5.1 Let (U,R,ξ_j) be $j-\boldsymbol{NS}$ and $A\subseteq U$. Then$\forall\ j\in\{r,l,\langle r\rangle,\langle l\rangle,u,i,\langle u\rangle,\langle i\rangle\}$,

$k\in\{p,s,\gamma,\alpha,\beta\}$, the following statements are true in general:

(i) If $x\ \underline{\in}_{j}\ A\Longrightarrow x\ \underline{\in}^{k}_{j}\ A$. (ii) If $x\ \overline{\in}^{k}_{j}\ A\ \Longrightarrow x\ \overline{\in}_{j}\ A$.

Proof We will prove first statement and the other similarly:

(i) If $x\ \underline{\in}_{j}\ A\Longrightarrow x\in\underline{R}_{j}(A)\Longrightarrow x\in\underline{R}^{k}_{j}(A)\Longrightarrow x\ \underline{\in}^{k}_{j}\ A$. ■

Remark 5.2 The converse of the above proposition is not true in general as the following example illustrates.

Example 5.2 Consider Example 5.1, where $U=\{a,b,c,d\}$ and

$R=\{(a,a),(b,b),(b,a),(c,a),(c,d),(d,a),(d,c),(d,d)\}$. Thus we get

$N_{\langle r\rangle}(a)=\{a\}\,,\ N_{\langle r\rangle}(b)=\{a,b\}\,,\ N_{\langle r\rangle}(c)=\{a,c,d\},\ \ N_{\langle r\rangle}(d)=\{d\}$.

We will show the above remark in case of ($j=\langle r\rangle$ and $k=s$) and the other cases similarly.

$$S_{\langle r\rangle}O(U)=\{U,\emptyset,\{a\},\{d\},\{a,b\},\{a,d\},\{a,c\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\} \text{ and}$$

$$S_{\langle r\rangle}C(U)=\{U,\emptyset,\{b\},\{c\},\{d\},\{a,b\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{b,c,d\}\}$$

Suppose that $A=\{a,c\}$ and $B=\{a,b\}$, then we get $\underline{R}_{\langle r\rangle}(A)=\{a\}$ and $\underline{R}^{s}_{\langle r\rangle}(A)=\{a,c\}$. Clearly $c\ \underline{\in}^{s}_{\langle r\rangle}\ A$, but $c\ \underline{\notin}_{\langle r\rangle}\ A$ although $c\in A$.

Also $\overline{R}_{r}(B)=\{a,b,c\}$ and $\overline{R}^{s}_{r}(B)=\{a,b\}$. Clearly $c\ \overline{\notin}^{s}_{\langle r\rangle}\ B$, but $c\ \overline{\in}_{\langle r\rangle}\ B$

although $c\notin B$.

Definition 5.2 Let (U,R,ξ_j) be $j-\boldsymbol{NS}$ and $A\subseteq U$. Then we define the j-near rough membership functions for $j-\mathrm{NS}$ as follows:

For each $j\in\{r,l,\langle r\rangle,\langle l\rangle,u,i,\langle u\rangle,\langle i\rangle\},k\in\{p,s,\gamma,\alpha,\beta\}$ and $x\in U$:

The j-near rough membership functions on U for subset A are $\mu^{k_j}_{A}:U\rightarrow[0,1]$ where

$$\mu^{k_j}_{A}(x)=\begin{cases}1 & if\ 1\in\Psi^{k_j}_{A}(x).\\ \boldsymbol{min}\left(\Psi^{k_j}_{A}(x)\right) & Otherwise.\end{cases}$$

And $\Psi^{k_j}_{A}(x)=\left\{\frac{|k_j(x)\cap A|}{|k_j(x)|}\middle|x\in k_j(x)\right\}$ such that $k_j(x)$ is a j-near open set in U.

Remark 5.3 The j-near rough membership functions can be used to define j-near approximations as shown below:

$$\underline{R}_j^k(A) = \{x \in U \mid \mu_A^{k_j}(x) = 1\} \text{and}$$

$$\overline{R}_j^k(A) = \{x \in U \mid \mu_A^{k_j}(x) > 0\}.$$

The following results give the fundamental properties of the j-near rough membership functions.

Proposition 5.2 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A, B \subseteq U$. Then,$\forall\, j \in \{r, l, \langle r\rangle, \langle l\rangle, u,$ $, \langle u\rangle, \langle i\rangle\}, k \in \{p, s, \gamma, \alpha, \beta\}$ and $x \in U$:

(i) $\mu_A^{k_j}(x) = 1 \Leftrightarrow x \underline{\in}_j^k A.$

(ii) $\mu_A^{k_j}(x) = 0 \Leftrightarrow x \in U - \overline{\mathcal{R}}_j^k(A).$

(iii) $0 < \mu_A^{k_j}(x) < 1 \Leftrightarrow x \in \mathcal{B}_j^k(A).$

(iv) $\mu_{U-A}^{k_j}(x) = 1 - \mu_A^{k_j}(x)\, for\ any\ x \in U.$

(v) $\mu_{A\cup B}^{k_j}(x) \geq \max(\mu_A^{k_j}(x), \mu_B^{k_j}(x)) for\ any\ x \in U.$

(vi) $\mu_{A\cap B}^{k_j}(x) \leq \min(\mu_A^{k_j}(x), \mu_B^{k_j}(x)) for\ any\ x \in U.$

Proof We will prove (i), and the others similarly.

First, $x \underline{\in}_j^k A \Leftrightarrow x \in \underline{R}_j^k(A)$. Since $\underline{R}_j^k(A)$ is j-near open set contained in A, then

$\frac{|\underline{R}_j^k(A)\cap A|}{|\underline{R}_j^k(A)|} = \frac{|\underline{R}_j^k(A)|}{|\underline{R}_j^k(A)|} = 1$. Thus $1 \in \Psi_A^{k_j}(x)$and accordingly $\mu_A^{k_j}(x) = 1.$ ■

Remark 5.4 The j-rough membership functions can be divide the universe U by using the j-near boundary, j-near positive and j-near negative regions of $A \subseteq U$, respectively as follow:

$$\mathcal{B}_j^k(A) = \{x \in U \mid 0 < \mu_A^{k_j}(x) < 1\},$$

$$POS_j^k(A) = \{x \in U \mid \mu_A^{k_j}(x) = 1\} \text{ and}$$

$$NEG_j^k(A) = \{x \in U \mid \mu_A^{k_j}(x) = 0\}.$$

The following result is very interesting since it is gives the relation between the j-rough membership functions and j-near rough membership functions. Moreover, it is illustrates the importance of j-near rough membership functions.

Lemma 5.2 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A, B \subseteq U$. Then $\forall\, j \in \{r, l, \langle r\rangle, \langle l\rangle, u,\ i, \langle u\rangle, \langle i\rangle\}, k \in \{p, s, \gamma, \alpha, \beta\}$, the following is true in general:

(i) $\mu_A^j(x) = 1 \implies \mu_A^{k_j}(x) = 1, \forall x \in U.$

(ii) $\mu_A^j(x) = 0 \implies \mu_A^{k_j}(x) = 0, \forall x \in U.$

Proof

(i) If $\mu_A^j(x) = 1 \implies x \underline{\in}_j A \implies x \underline{\in}_j^k A \implies \mu_A^{k_j}(x) = 1, \forall x \in U.$

(ii) If $\mu_A^j(x) = 0 \implies x \in U - \overline{R}_j(A) \implies x \in U - \overline{R}_j^k(A)$

$\implies \mu_A^{k_j}(x) = 0, \forall x \in U.$ ■

Remarks 5.5

(i) According to the above result and by using Proposition 5.1, we can prove that $\mu_A^{k_j}$ is

(ii) More accurate than μ_A^j, this means that:

(1)If $x \in A \implies \mu_A^j(x) \leq \mu_A^{k_j}(x).$

(2)If $x \notin A \implies \mu_A^{k_j}(x) \leq \mu_A^j(x).$

(iii) The converse of Lemma 5.2 is not true in general.

The following example illustrates Remarks 5.5.

Example 5.3 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$, where $U = \{a, b, c, d\}$ and

$R = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b), (c, c), (c, d), (d, d)\}$.

We will show the above result in case of $j = r$ and $k = s$ the other cases similarly as follow:

The family of r-semi open sets is:

$$S_r O(U) = \{U, \emptyset, \{a\}, \{d\}, \{a, b\}, \{a, d\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\} \text{and}$$

$$S_r C(U) = \{U, \emptyset, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}\}.$$

Now consider the subset $A = \{a, c\}$, then the r-rough membership functions of $A, x \in U$ are

$\mu_A^r(a) = \frac{|\{a,b\} \cap A|}{|\{a,b\}|} = \frac{1}{2}$. $\mu_A^r(c) = \frac{|U \cap A|}{|U|} = \frac{1}{2}$.

$\mu_A^r(b) = \frac{|\{a,b\} \cap A|}{|\{a,b\}|} = \frac{1}{2}$. $\mu_A^r(d) = \frac{|\{d\} \cap A|}{|\{d\}|} = 0$.

But the r-semi rough membership functions of $A, x \in U$ are

$$\Psi_A^{S_r}(a) = \left\{\frac{|\{a\} \cap A|}{|\{a\}|} = 1, \frac{|\{a,b\} \cap A|}{|\{a,b\}|} = \frac{1}{2}, \dots\right\} \Rightarrow \mu_A^{S_r}(a) = 1.$$

$$\Psi_A^{S_r}(b) = \left\{\frac{|\{a,b\} \cap A|}{|\{a,b\}|} = \frac{1}{2}, \frac{|\{a,b,c\} \cap A|}{|\{a,b,c\}|} = \frac{2}{3}, \frac{|\{a,b,d\} \cap A|}{|\{a,b,d\}|} = \frac{1}{3}\right\} \Rightarrow \mu_A^{S_r}(b) = \frac{1}{3}.$$

$$\Psi_A^{S_r}(c) = \left\{\frac{|\{a,c\} \cap A|}{|\{a,c\}|} = 1, \frac{|\{c,d\} \cap A|}{|\{c,d\}|} = \frac{1}{2}, \dots\right\} \Rightarrow \mu_A^{S_r}(c) = 1.$$

$$\Psi_A^{S_r}(d) = \left\{\frac{|\{d\} \cap A|}{|\{d\}|} = 0, \frac{|\{a,d\} \cap A|}{|\{a,d\}|} = \frac{1}{2}, \dots\right\} \Rightarrow \mu_A^{S_r}(a) = 0.$$

The j-near rough membership functions $\mu_A^{k_j}$ allow us to define forty different types of fuzzy sets in $j - \mathrm{NS}$ as the following definition illustrates.

Definition 5.3 Let (U, R, ξ_j) be $j - \boldsymbol{NS}$ and $A \subseteq U$. Then $\forall\, j \in \{r, l, \langle r \rangle, \langle l \rangle, u, i, \langle u \rangle, \langle i \rangle\}$ and $k \in \{p, s, \gamma, \alpha, \beta\}$, the j-near fuzzy set in U is a set of ordered pairs:

$$\tilde{A}_j^k = \{(x, \mu_A^{k_j}(x)) | x \in U\}$$

Example 5.4 According to Example 5.3, the r-semi fuzzy set of a subset $A = \{a, c\}$ is

$\tilde{A}_r^s = \{(a, 1), \left(b, \frac{1}{3}\right), (c, 1), (d, 0)\}$. But the r-fuzzy set of a subset $A = \{a, c\}$ is

$\tilde{A}_r = \{\left(a, \frac{1}{2}\right), \left(b, \frac{1}{2}\right), \left(c, \frac{1}{2}\right), (d, 0)\}$.

Conclusion

In this paper, we have integrated some ideas in terms of concepts in topology. Topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics, but also in many real life applications. We believe that topological structure will be an important base for modification of knowledge extraction and processing.

We have introduced some the important topological applications named “Near concepts” as easy tools to classify the sets and help for measuring near exactness and near roughness of sets. Near rough membership functions allowed us to introduce different types of near fuzzy sets. Accordingly, we introduced a useful connection between four important theories namely “rough set theory, fuzzy set theory and the general topology” that will be useful in applications.

Finally, we introduced an important application to illustrate the importance of using near concepts. So, we can say that the introduced structures are useful in the applications and thus these techniques open the way for more topological applications in rough context and help in formalizing many applications from real-life data.

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