ISSN 2347-1921

The Real Matrices forms of the Bicomplex Numbers and Homothetic Exponential motions

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ABSTRACT

In this paper, a bicomplex number is described in four- dimensional space and its a variety of algebraic properties is presented. In addition, Pauli-spin matrix elements corresponding to base the real matrices forms of the bicomplex numbers are obtained and its the algebraic properties are given. Like i and j in two different spaces are defined terms of Euler's formula. In the last section velocities become higher order by giving an exponential homothetic motion for the bicomplex numbers. And then, Due to the way in which the matter is presented, the paper gives some formula and facts about exponential homothetic motions which are not generally known.

KEYWORDS: Homothetic exponential motion; bicomplex number; Pauli-spin matrix; regular motion

2000 MATHEMATICS SUBJECT CLASSIFICATION: 15A90, 53A05, 53A17.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 8, No. 1
editor@cirjam.com
www.cirworld.org, www.jam.cirworld.com

ISSN 2347-1921

INTRODUCTION

In 1892, in search for special algebras, Corrado Segre (1860-1924) published a paper (see [1]) in which explained an algebra whose elements are called bicomplex numbers. In recent years many papers have been written on the extension of the formalism of quantum mechanics. These generalizations have been done mainly over quaternions or over the Cayley algebra (octonions), see for instance (see [2, 3, 4]). Theory of bicomplex numbers and bicomplex functions has found many applications,(see [5,6]). Bicomplex numbers is a commutative ring with unity which contains the field of complex numbers and the commutative ring of hyperbolic numbers. In 2006, Dominic Rochon and S. Tremblay, puplished a paper based on bicomplex quantum mechanics: II. The Hilbert Spaces (see [7, 8]). The bicomplex (hyperbolic) numbers are given in this paper from a number of different points of view of Hilberty Spaces for quantum mechanics.

In this study, the commutative algebra of bicomplex numbers variable is considered. This algebra of the 4-th rank has the properties of division, conjugation, taking the root and factorization of Euler's formula as that of complex numbers [see [9]). Later, Given the algebraic properties of Pauli-Spin matrices. In [10,11] Hamilton motion has been defined and investigated in four dimensional Euclidean space E^4. In the final section, It is shown that this study can be done for bicomplex number, which is a homothetic exponential motion and this homothetic exponential motion satisfies all of the properties.

BICOMPLEX NUMBERS

Bicomplex numbers is defined (see [1,8]) , as a complex number depending on four units +1, i, j, k where

$$i^2 = j^2 = -1 \text{ and } k^2 = 1$$

$$ij = ji = k \;\; ; ik = ki = -j \;\; ; jk = kj = -i.$$

Where k has the properties of a hyperbolic unit.Thus, a bicomplex number ξ is defined as

$$\xi = \omega_0 + i\omega_1 + j\omega_2 + k\omega_3$$

where ω_0, ω_1, ω_2, ω_3, are reel number components of ξ. Note that ´´$\mathbb{C}_0 \cong$IR´´. All points of the set of bicomplex numbers $\mathbb{C}_2$ is given by

$$\mathbb{C}_2 = \{ \xi : \xi = \omega_0 + i\omega_1 + j\omega_2 + k\omega_3 \;\; ; \; \omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{C}_0 \}$$

It is also convenient to write the set of bicomplex numbers as,

$$\mathbb{C}_2 = \{ z_1 + j\, z_2 \;\; | \;\; z_1, z_2 \in \mathbb{C}_1 \}.$$

Furthermore, rule of multiplication and addition of two complex numbers have the following algebraic properties.

$$(z_1 + j. z_2).(z_3 + j. z_4) = (z_1 + z_3 - z_2 z_4) + j (z_1 + z_4 + z_2 z_3)$$

$$(z_1 + j. z_2) + (z_3 + j. z_4) = (z_1 + z_3) + j (z_2 + z_4)$$

$$z_1 + j. z_2 = z_3 + j. z_4 \Leftrightarrow z_1 = z_3 \text{ and } z_2 = z_4.$$

The system $\{\mathbb{C}_2, \oplus, R, +, ., \circ, \otimes\}$ is a commutative algebra. This algebra is called bicomplex numbers algebra and are denoted by $\mathbb{C}_2$, Note that one of the basis of this algebra is {1, i, j, k} and the dimension is 4.

Definition (Multiplication Operator) We define,

$$\otimes : T \times T \to T$$

$$(u,w) \to u \otimes w = w \otimes u$$

The multiplication rule is given as follows;

$$\begin{aligned} u \otimes w &= w \otimes u \\ &= (u_0 w_0 - u_1 w_1 - u_2 w_2 + u_3 w_3) + i(u_0 w_1 + u_1 w_0 - u_2 w_3 - u_3 w_2) \\ &\quad + j(u_0 w_2 - u_1 w_3 + u_2 w_0 - u_3 w_1) + k(u_0 w_3 + u_1 w_2 + u_2 w_1 + u_3 w_0) \end{aligned}$$

Definition (*The Concept of Conjugacy for Bicomplex Numbers*) if $\xi = (\omega_0 + i\omega_1) + j(\omega_2 + i\omega_3)$ is bicomplex number conjugates $\xi^*(i)$, $\xi^*(j)$, $\xi^*(k)$ are given as follows;

$$\mathbf{1.}\; \xi * (i) = \omega_0 - i\omega_1 + j\omega_2 - ij\omega_3$$

$$\mathbf{2.}\; \xi * (j) = \omega_0 + i\omega_1 - j\omega_2 - ij\omega_3$$

$$\mathbf{3.}\; \xi * (k) = \omega_0 - i\omega_1 - j\omega_2 + ij\omega_3$$

Definition (*Norms of Bicomplex Numbers*) The norm of bicomplex of ξ is defined by the norms of the components i, j, k respectively, In particular,

$$\|\xi(i)\|, \|\xi(j)\|, \|\xi(k)\| \text{ and } \operatorname{Im}\xi(i), \operatorname{Im}\xi(j), \operatorname{Im}\xi(k) = 0,$$

where,

$$\|\xi(i)\| = \sqrt{(\omega_0{}^2 + \omega_1{}^2 - \omega_2{}^2 - \omega_3{}^2)}$$

$$\|\xi(j)\| = \sqrt{(\omega_0{}^2 - \omega_1{}^2 + \omega_2{}^2 - \omega_3{}^2)}$$

$$\xi(k)\| = \sqrt{(\omega_0{}^2 + \omega_1{}^2 + \omega_2{}^2 + \omega_3{}^2)}$$

Definition *(Idempotent Element)* It is also important to know that every bicomplex number (z_1+jz_2) has the following unique idempotent representation:

$$z_1 + j\, z_2 = (z_1 - iz_2)\, e_1 + (z_1 + iz_2)\, e_2$$

where

$$e_1 = (1+k)/2\ , e_2 = (1-k)/2\ ;\ e_1 + e_2 = 1, e_1.e_2 = 0.$$

REAL MATRICES FORM OF BICOMPLEX NUMBERS

We can obtain the matrix in the 4-dimension for bicomplex number with Pauli-Spin matrices

(see [9]) . They are defined as

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_2 = \begin{bmatrix} 0 & -j \\ j & 0 \end{bmatrix} \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Consequently, we have

$$\Omega_0 = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \Omega_1 = \begin{bmatrix} -j\sigma_2 & 0 \\ 0 & -j\sigma_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Omega_2 = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad \Omega_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Expression of a bicomplex number in the form of 4x4 matrix can be written as

$$\xi = \omega_0\, \Omega_0 + \omega_1\, \Omega_1 + \omega_2\, \Omega_2 + \omega_3\, \Omega_3.$$

Hence, we end up with

$$A^{-}(\xi) = A^{+}(\xi) = \begin{bmatrix} \omega_0 & -\omega_1 & -\omega_2 & \omega_3 \\ \omega_1 & \omega_0 & -\omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & \omega_0 & -\omega_1 \\ \omega_3 & \omega_2 & \omega_1 & \omega_0 \end{bmatrix}$$

Note that, $A^{-}(\xi)$ and $A^{+}(\xi)$ are similar to Hamilton operators. (for Hamilton operators see [5,6]). Here only one matrix is obtain. Let E be the set of matrices $A(\xi)$. Then E is a commutative algebra with respect to matrix addition and product.

Lemma: E and C_2 are isomorphic algebras.

Proof: Let us defined $\beta : C_2 \rightarrow E$, by $\xi \rightarrow \beta(\xi) = A^{+}(\xi)$. Since $\beta(\xi \otimes \alpha) = \beta(\xi)\, \beta(\alpha) = A(\xi)A(\alpha)$ and $\beta(\xi+\alpha) = \beta(\xi) + \beta(\alpha)$ and $\beta(\mu\, \xi) = \mu\,(\beta(\xi)) = \mu\, A^{+}(\xi)$, then β is a algebra isomorphism.

Following relations hold for Ω_0, Ω_1, Ω_2 and Ω_3 :

$$\Omega_1{}^2 = \Omega_2{}^2 = -\Omega_0\ ;\ \Omega_1\Omega_2 = \Omega_2\Omega_1 = \Omega_3\ ;\ \Omega_2\Omega_3 = \Omega_3\Omega_2 = -\Omega_1.$$

THE EULER FORMULAS

If $\xi = \omega_0 + i\omega_1 + j\omega_2 + k\omega_3$ is a bicomplex number and considering i and j as a product of spaces we have

$$\{(a+ib).(c+jd) = \omega_0 + i\omega_1 + j\omega_2 + k\omega_3\}$$

where ω_0=ac, ω_1=bc, ω_2=ad, ω_3=bd. Here, Re $\xi=\omega_0+j\,\omega_2$ and Im $\xi=\omega_1+j\omega_3$. This gives rise to

ISSN 2347-1921

$$\|\xi\| = \sqrt{a^2 c^2 + b^2 c^2 + a^2 d^2 + b^2 d^2}$$

$$\sqrt[n]{(a+ib)\,(c+jd)} = \sqrt[n]{\|\xi\|}\,.exp[(iarctan(b/a)+jarctan(d/c)+2\pi(s\, i+ r\, j))\ \backslash n],$$

where r,s =0,1,...,n-1 are natural numbers. Another interesting case is hypercomplex represented by only two components:

$$P = a+kd = exp\ (k\ \phi) = cosh\phi + k\ sinh\phi.$$

For $|d / a| \neq 1$, there is the following representation:

$$P = a+kd = \sqrt{|a^2 - d^2|}\ .exp\ (k.arctanh\ (d/a)).$$

Not that arctan(d/a) is real for $|(d / a)| < 1$. Otherwise it is complex in either i-, or in the j- spaces. Analogously, for $| d/a | > 1$ there exists an additional representation in the i, j-space

$$P = k(d+ka) = k\ \sqrt{|a^2 - d^2|}\ .\ exp\ (\ k.\ arctanh\ (a/d))$$

$$= \sqrt{|a^2 - d^2|}.exp\ (\ (i+j)\ (\pi/2) + k\ arctanh\ (a/d))$$

In last case, when $\|P\| \neq 0$, we can give the Euler's formula in the following way:

$$P \cong a + ib + jc + kd = exp\ (\omega_0 + i\omega_1 + j\omega_2 + k\omega_3) \cong exp\ (\xi)$$

where a relationship between P and ξ may be found from system: $\omega_0 = ln\ \|P\|$

$$x = sin\omega_1\ cos\omega_2\ cosh\omega_3 - cos\omega_1\ sin\omega_2\ sinh\omega_3$$

$$y = cos\omega_1\ sin\omega_2\ cosh\omega_3 - sin\omega_1\ cos\omega_2\ sinh\omega_3$$

$$v = sin\omega_1\ sin\omega_2\ cosh\omega_3.$$

where $x = b / \|P\|$, $y=c / \|P\|$ and $v = d / \|P\|$ are normalized components.

Example: Let (a, b, c, d) $=(1,1,1,\sqrt{3})$ be real numbers and then ξ have angles ϕ_i and ϕ_j.

HOMOTHETIC EXPONENTIAL MOTIONS

Definition: *Let* $f(t)=e^{tA}$ *and* $B(t)=h(t)f(t)$ *be the orthogonal matrix. Where* $h\ (\mathbf{t}) \neq$ *constant,* $\mathbf{t} \in R$.

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} X_0 \\ 1 \end{bmatrix} = \begin{bmatrix} hA & C \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} X_0 \\ 1 \end{bmatrix}$$

Which is called homothetic exponential motion in C_2 . The homothetic scale h the elements of *f* are continuously differentiable function of a real parameters *t* [10,11].

Where X and X_0 correspond to position vectors the same point with respect to the rectangular coordinate systems of the moving space R_0 and the fixed space R, At first time $t=t_o$ we consider coordinate systems of R_o and R coincident. Then homothetic exponential motions of R_o with respect to R will be denoted by $R_o \backslash R$. $\forall t$,

Let $f\,' = Af$, and suppose $C \neq 0$ then

$$B' = dB / dt = h'f + hf' = (h' + hA)f.$$

where $h=h(t)$ is a scalar matrix, its inverse and transpose are

$$h^{-1} = (1/h).I\ \ ,\ h^T = h$$

respectively. Since f is an orthogonal matrix, the inverse of B is

(1)... $B^{-1} = h^{-1} f^T$, $f^{-1} = f^T$; " $Im\ (\ f(t)\ ) = 0 \cong \omega_0\omega_3 - \omega_1\omega_2 = 0$ ".

Theorem: The matrix defined by the equation (1) is a orthogonal matrix.

Proof: Since $AA^T = A^T A = I_4$,´´ $\omega_0\omega_3 - \omega_1\omega_2 = 0$´´ and $\det A = 1$.

Theorem: Homothetic exponential motion given by equation (1) is regular for all n and it is independent of *h*.

Proof: $B' = h' f + h f' = (\ h' + hA)\ f = h\ f(\ A + \lambda\ I)$, where if we define $\lambda(t) = -(h'(t) / h(t))$, then last equation is

(2) .. $B' = h\ f(A - \lambda\ I)$.

From equation (2), we find that $\det B' = \det(\ hf\ )\ \det(\ A - \lambda\ I)$. As $\det B' = 0$, that is, B' is singular ,we get h=0 or det(A-λI)=0. Here h≠0. Otherwise,the motion will be pure translation.B' is always regular.

POLE POINTS OF EXPONENTIAL MOTIONS

Definition: To find the pole points, we have to solve the equation

$$B'X_0+C'.$$

Any solution of the equation in (4,1) is a pole point of the motion at that instantin R_0. The equation in (4,1) has only one solution

$$X_0= - (B')^{-1} C$$

at every t-instant [7].

Theorem: $f'(t)$ is a derivation of $f(t)$ .then, the pole points corresponding t_0 each t- instant in R_0 is the rotation by B' of the speed vector C' of the translation vector at the momoent.

Proof: Since B' is orthogonal , then the matrix $(B')^T$ is orthogonal. Thus it makes a rotation.

Theorem: If f is a orthogonal nxn matrix, the n^{th} -order derivatives of B are given by

$$B^{(n)} = [\sum_{j=0}^{n} \binom{n}{j} h^{(n-j)} A^j]f.$$

Proof: The proof of this theoram can be defined by induction. For n=1,

$$B' = \left[\sum_{j=0}^{1} \binom{1}{j} h^{(1-j)} A^j\right] f.$$

Thus we have shown that it is true for (n-1) and

$$B^{(n-1)} = \left[\sum_{j=0}^{n-1} \binom{n-1}{j} h^{(n-1-j)} A^j\right] f.$$

It can now be shown that it is true for n. Thus,,for n:

$$B^{(n)} = \left[\sum_{j=0}^{n} \binom{n}{j} h^{(n-j)} A^j\right] f.$$

Theorem: In spaces of n-dimension the high order velocities of homothetic exponential motions are given by

$$X^{(n)} = \sum_{j=0}^{n} \left[\sum_{i=0}^{n-j} \binom{n-j-i}{i} h^{(n-j)} A^i\right] f . C^{(n-1)} + C^{(n)}.$$

Example: $\xi = \frac{1}{\sqrt{2}}(\sin\eta, \cos\eta, \sin\eta, \cos\eta).$

Example: $\xi = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

Let $f(t)=e^{tA}$ be function and then $f'(t), f''(t), f^{(3)}(t),.., f^{(n)}(t)$ is available and provides related conditions.

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