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A Modified Newton-type Method with Order of Convergence Seven for Solving Nonlinear Equations

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ABSTRACT

In this paper, we mainly study the iterative method for nonlinear equations. We present and analyze a modified seventh-order convergent Newton-type method for solving nonlinear equations. The method is free from second derivatives. Some numerical results illustrate that the proposed method is more efficient and performs better than the classical Newton's method.

Keywords

Nonlinear equation; iterative method; order of convergence; Newton's method.

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Introduction

Nonlinear problem is an important direction of research in the field of numerical calculation. The solution of nonlinear equations is one of the most investigated topics in applied mathematics, and the problem of solving nonlinear equations by numerical methods has gained more importance than before, since many practical problems in practice can be transformed into nonlinear equations to solve.

In this paper, to improve the efficiency, we consider iterative method to find a simple root x^* of a nonlinear equation $f(x)=0$, where $f:\Gamma\subseteq R\to R$ for an open interval Γ is a scalar function and it is sufficiently differentiable in a neighborhood of x^*.

It is well known that the classical Newton's method (NM) is a basic and important method for solving non-linear equation [1] by the iterative scheme

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \tag{1}$$

which is quadratically convergent in the neighborhood of x^*.

In [2], Potra and Pták presented the following iterative method (PPM)

$$x_{n+1}=x_n-\frac{f(x_n)+f(y_n)}{f'(x_n)} \tag{2}$$

which is cubically convergent.

In recent years, much attention has been given to develop iterative methods for solving nonlinear equations and a vast literature has been produced [3-8].

Motivated and inspired by the on going activities in this direction, in this paper, based on PPM method (2) and Newton’s method (1), we present a seventh-order convergent iterative method. The method is free from second derivatives. Several numerical experiments are given to illustrate the efficiency and advantage of the algorithm.

The Modified Method and its Convergence

Let us consider the following iterative method.

Algorithm 1. For given x_0, we consider the following iteration scheme

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \tag{3}$$

$$z_n=x_n-\frac{f(x_n)+f(y_n)}{f'(x_n)}, \tag{4}$$

$$w_n=y_n-\frac{f(y_n)+f(z_n)}{f'(x_n)}. \tag{5}$$

$$x_{n+1}=w_n+\frac{f'(x_n)+f'(y_n)}{f'(x_n)-3f'(y_n)}\frac{f(w_n)}{f'(x_n)}. \tag{6}$$

Now, we are in the position to give the convergence of Algorithm 1.

Theorem 1. Assume that the function $f:\Gamma\subseteq R\to R$ has a single root $x^*\in\Gamma$, where Γ is an open interval. If $f(x)$ has first, second and third derivatives in the interval Γ, then Algorithm 1 is sixth-order convergent in a neighborhood of x^* and it satisfies error equation

$$e_{n+1}=8c_2^4(c_2^2-c_3)e_n^7+O(e_n^8), \tag{7}$$

where

$$e_n = x_n - x^*, \ c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)}, k = 1, 2, \cdots. \tag{8}$$

Proof. Let α be the simple root of $f(x)$,

$$c_k = \frac{f^{(k)}(x^*)}{k!\, f'(x^*)}, k = 1, 2, \cdots \text{ and } e_n = x_n - x^*.$$

Consider the iteration function $F(x)$ defined by

$$F(x) = w(x) - \frac{f'(x) + f'(y(x))}{f'(x) - 3f'(y(x))} \frac{f(w(x))}{f'(x)} \tag{9}$$

where

$$w(x) = y(x) - \frac{f(y(x)) + f(z(x))}{f'(x)}, z(x) = x - \frac{f(x) + f(y(x))}{f'(x)}, \ y(x) = x - \frac{f(x)}{f'(x)}. \tag{10}$$

By some computations using Maple we can obtain

$$\begin{aligned} &F(x^*) = x^*, F^{(i)}(x^*) = 0, i = 1, 2, 3, 4, 5, 6, \\ &F^{(7)}(x^*) = \frac{210 f^{(2)}(x^*)^4 (3 f^{(2)}(x^*)^2 - 2 f^{(3)}(x^*) f'(x^*))}{f'(x^*)^6}. \end{aligned} \tag{11}$$

Furthermore, from the Taylor expansion of $F(x_n)$ around x^*, we have

$$\begin{aligned} x_{n+1} = F(x_n) &= F(x^*) + F'(x^*)(x_n - x^*) + \frac{F^{(2)}(x^*)}{2!}(x_n - x^*)^2 + \frac{F^{(3)}(x^*)}{3!}(x_n - x^*)^3 \\ &+ \frac{F^{(4)}(x^*)}{4!}(x_n - x^*)^4 + \frac{F^{(5)}(x^*)}{5!}(x_n - x^*)^5 + \frac{F^{(6)}(x^*)}{6!}(x_n - x^*)^6 \\ &+ \frac{F^{(7)}(x^*)}{7!}(x_n - x^*)^7 + O((x_n - x^*)^8). \end{aligned} \tag{12}$$

Substituting (11) into (12) yields

$$x_{n+1} = x^* + e_{n+1} = x^* + 8c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8).$$

Therefore, we have

$$e_{n+1} = 8c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8)$$

which shows that the algorithm 1 is seventh-order convergent.

Numerical Results

Now, we employ Algorithm 1 presented in the paper to solve some nonlinear equations and compare it with NM and PPM. Displayed in Table 1 are the number of iterations (ITs) required such that $|f(x_n)| < 1.E-14$.

Table 1. Comparison of Algorithm 1, Newton's method and PPM

Functions	x_0	NM	PPM	Algorithm 1
f_1	1	5	4	3
	1.45	4	3	2
f_2	0.55	5	3	3
	3.1	7	4	3
f_3	2.5	6	4	3
	1.5	7	56	5
f_4	4	19	14	7
	3.5	12	8	5
f_5	7.9	15	11	5
	-0.6	4	4	3
f_6	1.9	5	4	4
	2.1	5	4	3

In table 1, we use the following functions.

$$f_1(x) = x^3 + 4x^2 - 10,\ \ x^* = 1.36523001341410.$$

$$f_2(x) = \cos x - x,\ \ x^* = 0.73908513321516.$$

$$f_3(x) = (x-1)^3 - 1,\ \ x^* = 2.$$

$$f_4(x) = e^{x^2+7x-30} - 1,\ \ x^* = 3.$$

$$f_5(x) = (x+2)e^x - 1,\ x^* = -0.44285440096708.$$

$$f_6(x) = \sin^2(x) - x^2 + 1,\ x^* = 1.40449164885154.$$

The computational results in Table 1 show that Algorithm 1 requires less ITs than NM and PPM. Therefore, the present seventh-order convergent method is of practical interest and can compete with NM.

Conclusion

With the wide development of sience and technology, the problem of solving nonlinear equations by numerical methods has gained more importance than before. In order to obtain efficient algorithm for the nonlinear equations which come from the practical problems, in this paper, we present and analyze a modified seventh-order convergent Newton-type iterative method for solving nonlinear equations. The method is free from second derivatives. Several numerical results illustrate the convergence behavior and computational efficiency of the proposed method. Computational results demonstrate that it is more efficient and performs better than the classical Newton's method.

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Liang Fang is an associate professor at Taishan University. He obtained his PhD from Shanghai Jiaotong University in June, 2010. His research interests are in the areas of cone optimizations and complementarity problems.

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