ISSN 2347-1921

Nevanlinna Theory for the Uniqueness of Difference Polynomials and Meromorphic Functions by sharing one Small Function

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ABSTRACT

The purpose of this paper is to extend the usual Nevanlinna theory to the periodic functions, difference operators and difference polynomials of meromorphic functions concerning their uniqueness after sharing one small function and satisfying certain conditions on the number of zeros and poles of the functions.

Indexing terms/Keywords:

Shared value; Uniqueness; Shift, Difference operator; Difference polynomials.

Academic Discipline And Sub-Disciplines

Mathematics; Complex Analysis.

SUBJECT CLASSIFICATION (2010) :

P rimary30D35, Secondary30D30, 39A05.

Council for Innovative Research

Peer Review Research Publishing System

Journal: Journal of Advances in Mathematics

Vol 9, No 1

editor@cirjam.org

www.cirjam.com, www.cirworld.com

INTRODUCTION

A function f(z) is called meromorphic if it is analytic in the complex plane except at poles. Throughout this paper, we assume the reader is familiar with the standard notations and fundamental results of Nevanlinna theory of Meromorphic functions such as the Characteristic function T(r, f), proximity function m(r, f), counting function N(r, f) and so on. (see e.g.[2], [4]). In addition, S(r, f) denotes any quantity that satisfies the condition that S(r, f)= o(T(r, f)) as r→ ∞ outside a possible exceptional set of finite logarithimic measure. For a meromorphic function f(z), we use S(f) to denote the family of all meromorphic functions a(z) that satisfy T(r,a)=S(r,f). Such functions are called **small functions** with respect to f(z). Let c be a non-zero complex costant then for a meromorphic function f(z) , we define its shift by f(z+c) and its difference operator by

$$\Delta_c f(z) = f(z+c) - f(z)$$

$$\nabla_c f(z) = f(z) - f(z-c),$$

$$\Delta_{mc} f(z) = f(z+mc) - f(z)$$

where m is a positive integer,

$$\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z))$$

n$\in \mathbb{N}$, **n**$\geq$ **2**

$$= \sum_{k=0}^{n} \frac{n!}{k!.(n-k)!} f(z+\overline{n-k}.c) .$$

In particular,

$$\Delta_c^n f(z) = \Delta^n f(z)$$

for c=1.

We define **Shift Monomial** as

$$M[f] = \prod_{i=1}^{k} f^{n_i}(z+c_i),$$

where $c_i \in \mathbb{C}, n_i \in \mathbb{N}, i = 1,2,...,k$.

Then $d_M = n_1 + n_2 + ... + n_k$,

Is the degree of the Shift Monomial M[f].

Definition 1

Let $M_1[f]$, $M_2[f]$, …, $M_n[f]$ denote the distinct monomials in f, and a_1, a_2, … , a_n be small meromorphic functions including complex numbers then

$$D[f] = \sum_{j=1}^{n} a_j M_j[f] \quad \dots (1)$$

will be called a **Difference Polynomial** in f of degree

$$d = \underset{j=1}{\overset{n}{Max}}.d_{M_j} .$$

Linear Difference Polynomial is defined as the Difference Polynomial of degree one i.e.

$$L[z, f] = \sum_{i=1}^{n} a_i f(z+c_i).$$

A difference operator is a special case of Linear Differential Polynomial in f in which sum of the co- efficients is zero.

For a given non- constant meromorphic function f(z), we recall the definition of the order of f(z) as

$$\rho[f] = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r}.$$

Let f(z) and g(z) be two meromorphic functions and let a(z) be a small function with respect to f(z) and g(z). We say that f(z) and g(z) share a(z) IM, provided that f(z) – a(z) and g(z) – a(z) have the same zeros (ignoring multiplicities), and we say that f(z) and g(z) share a(z) CM, provided that f(z) – a(z) and g(z) – a(z) have the same zeros (counting multiplicities).

Uniqueness Theory of Meromorphic functions is an important part of Nevanlinna Theory. Recently, number of papers have focused on the Nevanlinna Theory with respect to difference operators see e.g. [1 – 3], [5 – 10]. Then many authors started to investigate the uniqueness of meromorphic functions sharing values with their shifts or difference operators.

The following results concerning the uniqueness of difference operators have already been proved:

Theorem A [6]:

Let f(z) be a non- constant entire function of finite order. If f(z) and f(z+c) share two finite values CM, then

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f(z) ≡ f(z+c).

Theorem B [1]:

Let f(z) be a non- constant entire function of finite order and let a(z) (≠ 0) ϵ S (f) be a periodic entire function with period c . If three functions f(z), $\Delta_c f(z), \Delta_c^2 f(z)$ share a(z) CM, then $\Delta_c^2 f(z) = \Delta_c f(z)$.

Theorem C [3]:

Let f(z) be a non- constant transcendental meromorphic function of finite order, such that N(r,f) + N(r, 0, f) = S(r, f), and let a(z) (≠ 0, ∞) ϵ S (f). For m a positive integer and c a non- zero complex constant, if f(z) and f(z+mc) – f(z) = $\Delta_{mc} f$ share a(z) CM then $\Delta_{mc} f = f$.

Theorem D [3]:

Let f(z) be a non- constant meromorphic function of finite order, such that N(r,f) + N(r, 0, f) = S(r, f), and let a(z) (≠ 0, ∞) ϵ S (f). For m a positive integer and c a non- zero complex constant, if f(z) and

$\sum_{k=1}^{n} a_k \Delta_c^k f(z)$ share a(z) then $\sum_{k=1}^{n} a_k \Delta_c^k f(z) = f(z)$.

Theorem E [8]:

Let f(z) be a meromorphic function of hyper order less than one. Let L[z, f] be a linear differential polynomial in f and let a, b be two distinct small meromorphic functions. Let f(z) and L[z, f] share a, b, ∞ CM and one of the following cases hold:

i). L[z, a] – a = L[z, b] – b ≡ 0

ii). L[z, a] – a ≡0 or L[z, b] – b ≡0 and N(r, f) < λ T(r, f), λ ϵ (0, 1)

iii). ρ [f] $\notin \mathbb{N} \cup \infty$

then f(z) = L[z, f] .

Theorem F [10]:

Let f(z) be a non- constant meromorphic function of finite order. Let c be a complex constant and a be a non- zero small periodic function with period c . If $f(z)^n$, $f(z+c)^n$ share a and ∞ CM, n ≥ 4, then f(z) = w f(z+ c)

for a constant w and integer n such that $w^n = 1$.

QUESTION

It is natural to ask about the pattern in uniqueness of a non- constant meromorphic function with a general difference polynomial (not a small meromorphic function) in f as defined in (1), when they share small meromorphic function IM? It is pertinent to mention that General Difference polynomial includes linear difference polynomials as well as difference operators and shifts. In this connection, we have the following results:

MAIN RESULTS

Theorem 1:

Let f(z) be a non- constant meromorphic function of finite order such that

$$\overline{N}(r,f)+N(r,0,f)=\lambda T(r,f),\lambda\in(0,\frac{1}{2}).$$

Let D[f] be a non- constant General Difference Polynomial of degree d, defined as in definition 1 such that T(r, D) ≠ S(r, f). If D[f] and f^d share a non- zero small function a(z) (≠ ∞) IM then D[f] = f^d .

Remark 1:

If d = 1, then D[f] becomes linear difference polynomial and difference operators and shifts are particular cases of linear difference polynomials.

If the condition on N(r, f) is dropped, then we have the following results:

Theorem 2:

Let f(z) be a non- constant meromorphic function of finite order. Let D[f] be a non- constant General Difference Polynomial of degree d such that T(r, D) ≠ S(r, f). If D[f] and f^d share two distinct non- zero small functions a(z), b(z) (≠ ∞) IM and

$$N(r,0,f)=\lambda T(r,f),\lambda\in(0,\frac{1}{2}),$$

then

$$\mathrm{D[f]}=\mathrm{f}^{d}$$

Theorem 3:

Let f(z) be a non- constant entire function of finite order. Let D[f] be a non- constant General Difference Polynomial of degree d such that T(r, D) ≠ S(r, f). If D[f] and f^d share one non- zero small functions a(z) (≠ ∞) IM and

$$N(r,0,f)=\lambda T(r,f),\lambda\in(0,\frac{1}{2}),$$

then

$$\mathrm{D[f]}=\mathrm{f}^{d}$$

EXAMPLES

Ex. 1:

Let

$$f(z)=1+\tan^{2}(z)=\sec^{2}z\,,$$

then we take

$$D[f]=f(z\pm\frac{\pi}{2})=1+\cot^{2}z=\cos ec^{2}z\,.$$

Here f(z) is non- constant meromorphic function of finite order and D[f] is of degree one. Also we observe that N(r, 0, f) = S(r, f). It can be easily seen that D[f], f^d share 2 (one finite non- zero value IM) and

D[f] ≠ f^d .

Thus in Theorem 2, the number of shared values can not be further reduced which implies that two is best possible.

Ex. 2:

Let

$$f(z)=\sin^{2}z,$$

then

$$D[f] = f(z \pm \frac{\pi}{2}) = \cos^2 z .$$

Here f(z) is a non- constant entire function of finite order and D[f] is of degree one. It can be seen that D[f], f^d share 1/2 (one finite non- zero value IM) and

D[f] ≠ f^d .

The reason being that N(r, 0, f) ≠ λ T(r, f), λ ϵ (0, 1/2).

Hence the condition that N(r, 0, f) = λ T(r, f), λ ϵ (0, 1/2) is essential.

Ex. 3:

Let

$$f(z) = e^{\sin z} .$$

Then

$$D[f] = f(z \pm \pi) = e^{-\sin z} .$$

Here f(z) is a non- constant entire function of infinite order and D[f] is of degree one. The condition that

N(r, 0, f) = λ T(r, f), λ ϵ (0, 1/2) is satisfied. . It can be seen that D[f], f^d share 1, -1 and

D[f] ≠ f^d .

The reason being that the condition for f to be of finite order is essential.

Ex. 4:

Let

$$f(z) = e^{z \log 2} ,$$

and

D[f] = f(z+1) – f(z).

Here f(z) is a non- constant entire function of finite order and D[f] is of degree one. Also N(r, 0, f) = S(r, f) and f(z) and D[f] share a IM and we have D[f] = f^d .

For the proofs of the results we need the following lemma:

LEMMA[12]:

Let f(z) be a non- constant meromorphic function of finite order such that

$$f^n P[z, f] = Q[z, f] ,$$

where are difference polynomial $P[z, f], Q[z, f]$ are difference polynomials in f. If the degree of $Q[z, f]$ as a polynomial in f and its shifts is at most n, then

$$m(r, P[z, f]) = S(r, f) ,$$

Where the exceptional set associated with S(r, f) is of at most finite logarithimic measure.

PROOFS OF THE MAIN RESULTS

Proof of Theorem 1:

Suppose on the contrary, then

D[f] ≠ f^d .

Using Nevanlinna's Second Fundamental Theorem, Lemma1 and the given condition, we get

ISSN 2347-1921

$$dT(r,f)=T(r,f^d)+O(1)$$
$$\leq \overline{N}(r,f^d)+\overline{N}(r,0,f^d)+\overline{N}(r,a(z),f^d)+S(r,f)$$
$$\leq \overline{N}(r,\frac{D[\mathrm{f}]}{f^d},1)+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$\leq T(r,\frac{D[\mathrm{f}]}{f^d})+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$=N(r,\frac{D[\mathrm{f}]}{f^d})+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$\leq N(r,\frac{1}{f^d})+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$=2\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$\prec dT(r,f)+\mathrm{S(r,f)},$$

which is a contradiction, therefore

$$D[f]\equiv f^d.$$

Proof of Theorem 2:

Suppose on the contrary, then

D[f] ≠ f^d .

Using Nevanlinna's Second Fundamental Theorem, Lemma1 and the given condition, we get

$$dT(r,f)=T(r,f^d)+O(1)$$
$$\leq \overline{N}(r,\mathrm{a}(z),f^d)+\overline{N}(r,0,f^d)+\overline{N}(r,\mathrm{b}(z),f^d)+\mathrm{S(r,f)}$$
$$\leq \overline{N}(r,\frac{D[\mathrm{f}]}{f^d},1)+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$\leq T(r,\frac{D[\mathrm{f}]}{f^d})+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$=N(r,\frac{D[\mathrm{f}]}{f^d})+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$\leq N(r,\frac{1}{f^d})+\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$=2\lambda dT(r,f)+\mathrm{S(r,f)},\lambda\in(0,\frac{1}{2})$$
$$\prec dT(r,f)+\mathrm{S(r,f)},$$

which is a contradiction, therefore

$$D[f]\equiv f^d.$$

Proof of Theorem 3:

The proof is on the same lines as in Theorem 1 and Theorem 2.

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