ISSN 2347-1921

Matrices of inversions for permutations: Recognition and Applications

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Abstract

This work provides a criterion for a binary strictly upper triangle matrices to be a matrix of inversions for a permutation. It admits an invariant matrices for permutations to being well recognizable. Then it provides a complete algorithmic classification of elements in the symmetric group S_n. Also it gives an algorithm for generating and writing a permutation in a unique canonical form, as a word of transpositions.

Council for Innovative Research
Peer Review Research Publishing System
Journal: Journal of Advances in Mathematics
Vol 9, No 1
editor@cirjam.org
www.cirjam.com, www.cirworld.com

ISSN 2347-1921

1 Introduction:

Symmetric groups are important to many studies in mathematics such as group theory, representation theory, combinatorics and invariant theory [1]. Also they are powerful in classifying chemicals and spectral properties of molecules [2], [3], as well as quantum mechanics [4].

For a permutation $\pi:\{1,2,...,n\}\rightarrow\{1,2,...,n\}$, let us denote by π_i for $\pi(i)$ and $\pi=(\pi_1\pi_2...\pi_n)$ in S_n. An inversion of a permutation π is a pair (i,j) with $i<j$ and $\pi_i>\pi_j$, the inversion number of π is the total number of its inversions, i.e. $Inv(\pi)=\left|\{(i,j):i<j,\pi_i>\pi_j,\}\right|$ [5]. The notion called matrix of inversions for a permutation $\pi=(\pi_1\pi_2...\pi_n)$ in S_n is introduced in [6]. That any permutation π in S_n has a unique matrix $M_\pi=(m_{ij})_{n\times n}$ of its inversions, where $m_{ij}=1$ if $i<j$ and $\pi_i>\pi_j$, otherwise $m_{ij}=0$. Then any matrix of inversions is a binary strictly upper triangular matrix. As an example, consider the permutation $\pi=(6713254)$ in S_7 which has inversions $\{(1,3),(1,4),(1,5),(1,6),(1,7),(2,3),(2,4),(2,5),(2,6),(2,7),(4,5),(6,7)\}$, then

$$M_\pi=\begin{bmatrix}0&0&1&1&1&1&1\\0&0&1&1&1&1&1\\0&0&0&0&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\end{bmatrix}$$

We can, directly, extract the matrix M_π of π by considering the permutation as π a function $\begin{pmatrix}1&2&.&.&.&n\\\pi_1&\pi_2&.&.&.&\pi_n\end{pmatrix}$, then look to each restriction $\begin{pmatrix}i&j\\\pi_i&\pi_j\end{pmatrix}$ of π to $\{i,j\}$ for all $i<j$, and find m_{ij}. For the permutation $\pi=(6713254)$, the restrictions $\begin{pmatrix}3&6\\1&5\end{pmatrix},\begin{pmatrix}4&5\\3&2\end{pmatrix}$ give $m_{36}=0, m_{45}=1$, respectively.

A binary operation on $M_n(F)=\{M_\pi:\pi\in S_n, m_{ij}\in F=\{0,1\}\}$ is defined in [6], as $M_\alpha+M_\beta=M_\alpha+_{mod2}\alpha^{-1}(M_\beta)$ for each α,β in S_n, and if $M_\alpha=(m_{ij})$, then,

$$\beta(M_\alpha)=\beta(m_{ij})=\left\{\begin{array}{lll}0 & , & i\geq j\\ m_{\beta(i)\beta(j)} & , & i<j,\ \beta(i)<\beta(j)\\ m_{\beta(j)\beta(i)} & , & i<j,\beta(i)>\beta(j)\end{array}\right\}$$

The set $M_n(F)$ with the above operation is a group which is isomorphic to S_n. Where $F=\{0,1\}$ is the field with the addition $+_{mod2}$, while the associative operations $+$, . on $F=\{0,1\}$ are defined as: $0+0=1+1=0$, $1+0=0+1=1$, $0\,.\,0=1\,.\,0=0\,.\,1=0$, $1\,.\,1=1$.

In section two we provide a criterion for a binary strictly upper triangle matrices to be a matrix of inversions for a permutation. It admits an invariant matrix for permutations to being well recognizable. Then it provides a complete algorithmic classification of elements in the symmetric group S_n. In section three we give an algorithm for generating and writing a unique canonical word for a permutation as a word of transpositions. We hope that this work will be useful for the representation of braid groups of Hecke algebra.

2 Recognition of matrices of inversions for permutations

Binary matrices are of interest in combinatorics, information theory, cryptology, and graph theory [7], [8]. For each natural number n, there are 2^{n^2} binary $n\times n$ matrices. But not every such matrix is a matrix of inversions for a permutation in

S_n for some natural number n . e.g. for $n=2$ we have sixteen binary matrices, where there are only two matrices of inversions , and for $n=3$ we have $2^{3^2}=512$ binary matrices, where there are only six matrices of inversions.

As above, every permutation $\pi \in S_n$ has a unique matrix of inversions $M_\pi = (m_{ij})$ which is $n \times n$ binary strictly upper triangular matrix. In fact, each binary strictly upper triangular $n \times n$ matrix has $n(n-1)/2$ entries in the upper triangle, then there are $2^{n(n-1)/2}$ of such these matrices, but not every such matrix is a matrix of inversions of a permutation. For $n=3$ there are eight binary strictly upper triangular matrices, but we have only 6 matrices of inversions . The following example gives a binary strictly upper triangular matrix which does not a matrix of inversions of any permutation .

Example 1 *Consider the binary strictly upper triangular* 4×4 *matrix*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

it can not be a matrix of inversions of any permutation in S_4. To see that take $\pi = (\pi_1\pi_2\pi_3\pi_4)$ in S_4, the first row in A implies that $\pi_1 < \pi_2, \pi_1 > \pi_3, \pi_1 > \pi_4$, while the second row gives $\pi_2 < \pi_3, \pi_2 > \pi_4$, then $\pi_1 < \pi_2 < \pi_3$ which contradicts $\pi_1 > \pi_3$, hence A does not a matrix of inversions.

Now we are going to give a necessary and sufficient condition for a binary strictly upper triangle matrix to be a matrix of inversions of a permutation, then we can recognize matrix of inversions. We are going to establish the $(n+1)\times(n+1)$ matrices of inversions from $n \times n$ matrices of inversions. Given an $n \times n$ matrix of inversions A_π , we can enlarge it by $(n+1)\times 1$ column and $1 \times (n+1)$ row vectors.

Definition 2 *For each* n *in* $\mathbb{N}$ *and for a permutation* $\pi = (\pi_1\pi_2...\pi_n)$ *in* S_n, *define the lifting operator* $T: S_n \to S_{n+1}$, $T(\pi) = (1\,\pi_1+1\,\pi_2+1...\pi_n+1)$. *Also for* $r = 1,2,...,n+1$, *define the* (r,n) *inserting permutation* $I_n^r(\pi)$ *of* π *in* S_{n+1}, *by* $I_n^r: S_n \to S_{n+1}$, $I_n^r(\pi) = T(\pi) \circ C_r$, *where*

$$C_r = \begin{Bmatrix} id. & , & r=1 \\ (r\ 1\ 2\ 3...r-1\ r+1...n\ n+1) & , & 2 \le r \le n \\ (n+1\ 1\ 2\ 3...n\ ) & , & r=n+1 \end{Bmatrix}$$

In fact the lifting operator T is the inserting $(1,n)$, i.e. $T = I_n^1$. A geometric representation of the permutations $\pi, T(\pi), C_r$, and $I_n^r(\pi)$ is illustrated in Figure 1.

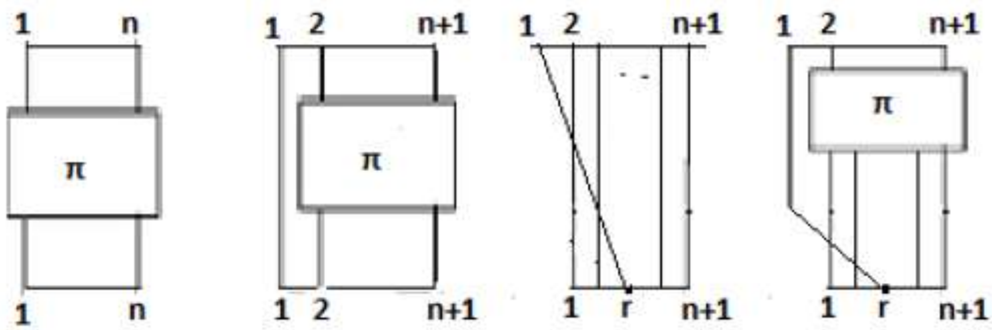

Figure 1: From left to right, a geometric representation of the permutations π in S_n and $T(\pi)$, C_r and $I_n^r(\pi)$ in S_{n+1}

ISSN 2347-1921

Lemma 3 *For a permutation* π *in* S_n, *the* (r,n) *inserting permutation* $\theta = I_n^r(\pi)$ *of* π *in* S_{n+1} *is*

$$\theta_i = \left\{ \begin{array}{lll} r & , & i=1 \\ \pi_{i-1} & , & \pi_i < r, 1 < r \leq n+1 \\ \pi_{i-1}+1 & , & \pi_i \geq r, 1 < r \leq n+1 \end{array} \right\}$$

Proof. From the definition above $\theta = I_n^r(\pi) = T(\pi) \circ C_r$, then $\theta_1 = (T(\pi)\circ C_r)_1 = C_r(T(\pi)_1) = C_r(1) = r$. Now for $i \in \{2,3,...,n\}$ and for π_i with $\pi_i < r$, then $\theta_i = \pi_{i-1}$, but for $\pi_i > r$, we have $\theta_i = \pi_{i-1}+1$. So that,

$$I_n^r(\pi) = T(\pi)\circ C_r = \theta = \left\{ \begin{array}{lll} \theta_i = r & , & i=1 \\ \theta_i = \pi_{i-1} & , & \pi_i < r, i \in \{2,3,...,n+1\} \\ \theta_i = \pi_{i-1}+1 & , & \pi_i \geq r, i \in \{2,3,...,n+1\} \end{array} \right\}$$

Proposition 4 *For a permutation* π *in* S_n *with matrix permutation inversion* $M_\pi = (m_{ij})$, *the* $r-th$ *embedding* $I_n^r(\pi) = T(\pi)\circ C_r$ *of* π *in* S_{n+1} *has matrix of inversions*

$$M_{I_n^r(\pi)} = M_{T(\pi)\circ C_r} = (n_{ij}) = \left\{ \begin{array}{lll} m_{i-1j-1} & , & 2 \leq i,j \leq n+1 \\ n_{j1} = 0 & , & 1 \leq j \leq n+1 \\ n_{1j} = 0 & , & \pi_j \geq r \\ n_{1j} = 1 & , & \pi_j < r \end{array} \right\}$$

Proof. Let π be a permutation in S_n with matrix of inversions $M_\pi = (m_{ij})$, then the $(1,n)$ inserting permutation $I_n^1(\pi)$ of π in S_{n+1} is $I_n^1(\pi) = (1\,\pi_1+1\,\pi_2+1 \ldots \pi_n+1)$, see second graph in Figure1. Then $(I_n^1(\pi))_1 = 1 < (I_n^1(\pi))_i$ for all $i > 1$, so that the entries in the first row of its matrix of inversions $M_{I_n^1(\pi)}$ will be zeros. Then

$$M_{I_n^1(\pi)} = \begin{bmatrix} 0 & 0 & . & 0 \\ 0 & & & \\ . & & M_\pi & \\ 0 & & & \end{bmatrix}$$

Now, for $\pi_{i-1} < r$, then $\theta_j = \pi_{i-1}, i \in \{2,3,...,n+1\}$ and for the pair $(1,i)$ where $1 < i$, we have $\theta_1 = r > \theta_i$, so we have inversion, hence $n_{1i} = 1$. Also, for $\pi_{i-1} \geq r$, then $\theta_i = \pi_{i-1}+1, i \in \{2,3,...,n+1\}$ and for the pair $(1,i)$ where $1 < j$, we have $\theta_1 = r < r+1 \leq \pi_{i-1}+1 = \theta_i$, so we have no inversion, hence $n_{1i} = 0$. Therefore the first row of $M_{I_n^1(\pi)}$ has r ones, where $\pi_{i-1} < r$, otherwise zeros

Example 5 *Consider the permutation* $\pi = (6713254)$ *in* S_7, *then for* $r=5$, *we have* $\pi_i < 5, i = 3,4,5,7$ *and* $\pi_i \geq 5, i = 1,2,6$. *Let* $I_8^5(\pi) = \theta = (\theta_1\theta_2 \ldots \theta_8)$, *then* $\theta_1 = r = 5$, $\theta_i = \pi_{i-1}$ *for* $i = 4,5,6,8$ *and* $\theta_i = \pi_{i-1}+1$ *for* $i = 2,3,7$. *Then the first row in the matrix* $M_{I_8^5(\pi)}$ *has exactly* $r-1=4$ *ones, and* $\theta = (57813264)$ *in* S_8.

Figure 2 illustrates a geometric representation of permutations $\pi = (6713254)$ *and* $I_8^5(\pi) = \theta = (57813264)$. *The associated matrices of inversions* M_π *and* $M_{I_8^5(\pi)}$ *are,*

ISSN 2347-1921

$$M_\pi = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, M_{I_8^5(\pi)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

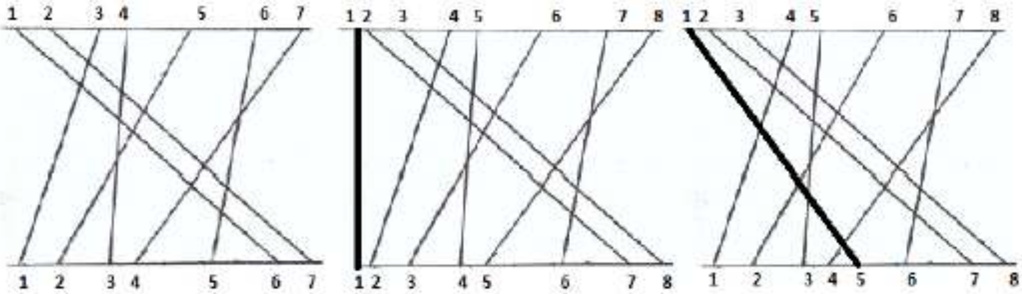

Figure 2: A geometric representation of the permutations $\pi = (6713254)$ in S_7, $T(\pi) = I_8^1(\pi) = (17824365)$ and $I_8^5(\pi) = (57813264)$ in S_8

Definition 6 *A submatrix of a matrix M is the matrix obtained from $M = (m_{ij})$ by deleting rows and columns but without permuting the remaining rows and columns. The submatrix obtained from a $n \times n$ matrix M by deleting the first $n-k$ rows and columns, $k = 1,2,\ldots,n-1$ is called the $k-th$ lower right submatrix, $LR(M)_k$, of the matrix M.*

Theorem 7 *A binary strictly upper triangle $n \times n$ matrix M is a matrix of inversions of a permutation π in S_n for some positive integer n if and only if for every $1 \le k \le n-1$ there exists $1 \le r \le k$ such that $LR(M)_k$ = $M_{I_n^r(\theta)}$ for some θ in S_{k-1}.*

Proof. For the necessity, let π be a permutation in S_n and $M_\pi = (m_{ij})$ be its matrix of inversions. Then take k such that $1 \le k \le n-1$, so by deleting the strands $(i\ \pi_i)$, from $i = 1$, then $i = 2$, up to $i = k$ from the permutation π, then we have a new permutation $\theta = (\theta_1\theta_2 \ldots \theta_{n-k})$ in S_{n-k} , where its matrix of inversions will be $LR(M_\pi)_k$, as in proposition above. For the converse, let M be a binary strictly upper triangle $n \times n$ matrix such that $LR(M)_k$ = $M_{I_n^r(\theta)}$ for some θ in S_{k-1} for every $1 \le k \le n-1$ and for some r with $1 \le r \le k$. If so we apply the process lemma above and use induction, which ends the proof.

Example 8 *For $\pi = (68237541) \in S_8$, starting from $m_{11} = 0$ where M_π is a strictly upper triangle matrix, then compare the value $\pi_1 = 6$ with the next values of $\pi_i, i = 2,3,\ldots,8$, then $m_{1i} = 0$ for $\pi_1 < \pi_i$ and $m_{1i} = 1$ for $\pi_1 > \pi_i$. The first row of the matrix M_π will be $(0,0,1,1,0,1,1,1)$. For the second row $m_{21} = m_{22} = 0$, then compare the value $\pi_2 = 8$ with the next values of $\pi_i, i = 3,4,\ldots,8$, then $m_{2i} = 1$ for all $i \ge 3$. Then the second row of the matrix M_π will be $(0,0,1,1,1,1,1,1)$. Following this process, we have M_π as,*

$$M_\pi = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$LR(M_\pi)_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with associated permutations $\theta = (4321)$ and $I_5^1(\theta) = (25431)$, and with matrix of inversions

$$M_{I_5^1(\theta)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3 Generating and writing a unique canonical word for a permutation

The most common way for writing a permutation in a unique form is by decomposing it into combinations of cycles [1]. Here we give an algorithm for generating and writing a permutation in a standard canonical form as a composition of transpositions.

Algorithm 9 *Generating and writing down a permutation from its matrix of inversions in a unique canonical form:*

1. For a permutation π in S_n, find its matrix of inversions M_π.
2. Each row will be producing a word as a product of transpositions.
3. The row that all its entries are zeros will contribute by the identity word, id..
4. If the number of ones in the entries of the i^{th} row is k, then the corresponding word will be $w_i = \tau_i \tau_{i+1} ... \tau_{i+k-1}$.
5. Then writes $\pi = w_n w_{n-1} ... w_1$.

Consider the permutation $\pi = (531642)$, then its matrix of inversions is

$$M_\pi = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $w_1 = \tau_1\tau_2\tau_3\tau_4, w_2 = \tau_2\tau_3, w_3 = id., w_4 = \tau_4\tau_5, w_5 = \tau_5, w_6 = id.,$ and the associated canonical word is

$\pi = w_6 w_5 ... w_1 = id..\tau_5.\tau_4\tau_5.id..\tau_2\tau_3.\tau_1\tau_2\tau_3\tau_4 = \tau_5.\tau_4\tau_5.\tau_2\tau_3.\tau_1\tau_2\tau_3\tau_4$. Notice that the arrangement of the words w_i is according to the tower of the associated lower right corner submatrices of the matrix M_π, where the associated $k-th$ lower right submatrices, $LR(M)_k$, $k = 0,1,...,5$, are

$LR(M_\pi)_5$	$LR(M_\pi)_4$	$LR(M_\pi)_3$	$LR(M_\pi)_2$	$LR(M_\pi)_1$	$LR(M_\pi)_0$
$\begin{bmatrix}0\end{bmatrix}$	$\begin{bmatrix}0&1\\0&0\end{bmatrix}$	$\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0&0\\0&0&1&1\\0&0&0&1\\0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0&0&1\\0&0&0&0&0\\0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&1&0&1&1\\0&0&1&0&0&1\\0&0&0&0&0&0\\0&0&0&0&1&1\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}$

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