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Theory of Linear Hahn difference equations

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ABSTRACT

Hahn introduced the difference operator $D_{q,\omega} f(t) = (f(qt+\omega) - f(t))/(t(q-1)+\omega)$ in 1949, where $0 < q < 1$ and $\omega > 0$ are fixed real numbers. This operator extends the classical difference operator $\Delta_{\omega} f(t) = (f(t+\omega) - f(t))/\omega$ as well as Jackson $q-$ difference operator $D_q f(t) = (f(qt) - f(t))/(t(q-1))$. In this paper, our target is to give a rigorous study of the theory of linear Hahn difference equations of the form

$$a_0(t)D_{q,\omega}^{n}x(t) + a_1(t)D_{q,\omega}^{n-1}x(t) + ... + a_n(t)x(t) = 0.$$

We introduce its fundamental set of solutions when the coefficients are constant and the Wronskian associated with $D_{q,\omega}$. Hence, we obtain the corresponding Liouville's formula. Also, we derive solutions of the first and second order linear Hahn difference equations with non-constant coefficients. Finally, we present the analogues of the variation of parameter technique and the annihilator method for the non-homogeneous case.

Keywords: Hahn difference operator; Jackson $q-$ difference operator.

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Introduction and Preliminaries

Hahn introduced his difference operator which is defined by

$$D_{q,\omega}f(t)=\frac{f(qt+\omega)-f(t)}{t(q-1)+\omega},t\neq\theta$$

where $0<q<1$ and $\omega>0$ are fixed real numbers , $\theta=\omega/(1-q)$ [12, 13]. This operator unifies and generalizes two well-known difference operators. The first is Jackson $q-$ difference operator defined by

$$D_q f(t)=\frac{f(qt)-f(t)}{t(q-1)},t\neq 0$$

where q is fixed. Here f is supposed to be defined on a $q-$ geometric set $A\subset R$ for which $qt\in A$ whenever $t\in A$, see [2, 3, 5, 6, 9, 10, 15, 17, 18]. The second operator is the forward difference operator

$$\Delta_\omega f(t)=\frac{f(t+\omega)-f(t)}{\omega}$$

where $\omega>0$ is fixed, see [7, 8, 16, 19]. Hahn's operator was applied to construct families of orthogonal polynomials as well as to investigate some approximation problems, see [21, 22, 23]. Another direction of interest is to establish a calculus based on this operator. This was recently studied by M. H. Annaby, A. E. Hamza and K. A. Aldwoah in [4]. They proved a fundamental theorem of Hahn's calculus. An essential function which plays an important role in this calculus is $h(t)=qt+\omega$. This function is normally taken to be defined on an interval I which contains the number θ. One can see that the $k-$th order iteration of $h(t)$ is given by

$$h^k(t)=q^k t+\omega[k]_q,t\in I.$$

The sequence $h^k(t)$ is uniformly convergent to θ on I. Here $[k]_q$ is defined by

$$[k]_q=\frac{1-q^k}{1-q}.$$

Throughout this paper I is any interval of R containing θ and X is a Banach space.

Definition 1.1. Assume that $f:I\to X$ is a function and let $a,b\in I$. The $q,\omega-$ integral of f from a to b is defined by

$$\int_a^b f(t)d_{q,\omega}t=\int_\theta^b f(t)d_{q,\omega}t-\int_\theta^a f(t)d_{q,\omega}t$$

where

$$\int_\theta^x f(t)d_{q,\omega}t=(x(1-q)-\omega)\sum_{k=0}^{\infty}q^k f(h^k(x)),\quad x\in I,$$

provided that the series converges at $x=a$ and $x=b$.

Definition 1.2 [4]. For certain $z\in\mathbb{C}$, the q,ω-exponential functions $e_z(t)$ and $E_z(t)$ are defined by

$$e_z(t)=\sum_{k=0}^{\infty}\frac{(z(t(1-q)-\omega))^k}{(q;q)_k}=\frac{1}{\prod_{k=0}^{\infty}(1-zq^k(t(1-q)-\omega))},\tag{1.1}$$

and

$$E_z(t)=\sum_{k=0}^{\infty}\frac{q^{\frac{1}{2}k(k-1)}(z(t(1-q)-\omega))^k}{(q;q)_k}=\prod_{k=0}^{\infty}(1+zq^k(t(1-q)-\omega)). \quad (1.2)$$

To guarantee the convergence of the infinite product in (1.1) with $t\in C,$ we assume additionally that

$$|t-\theta|<\frac{1}{|z(1-q)|},$$

see [1, 20]. For a fixed $z\in C,$ (1.2) converges for all $t\in C,$ defining an entire function of order zero. For the proofs of the equalities in (1.1) and (1.2), see [11, Section 1.3] and [20]. Here the $q-$shifted factorial $(b;q)_n$ for a complex number b and $n\in \mathrm{N}_0$ is defined to be

$$(b;q)_n=\begin{cases}\prod_{j=1}^{n}(1-bq^{j-1}), if \quad n\in \mathrm{N},\\ 1, \qquad\qquad\qquad if \quad n=0.\end{cases}$$

Definition 1.3. For $z\in C,$ the q,ω-trigonometric functions $\cos_{q,\omega}(z;\cdot)$ and $\sin_{q,\omega}(z;\cdot)$ are defied on C by

$$\cos_{q,\omega}(z;t)=\frac{e_{iz}(t)+e_{-iz}(t)}{2},$$

$$\sin_{q,\omega}(z;t)=\frac{e_{iz}(t)-e_{-iz}(t)}{2i}$$

and the functions $Cos_{q,\omega}(z;\cdot)$ and $Sin_{q,\omega}(z,\cdot)$ in C by

$$Cos_{q,\omega}(z;t)=\frac{E_{iz}(t)+E_{-iz}(t)}{2},$$

$$Sin_{q,\omega}(z;t)=\frac{E_{iz}(t)-E_{-iz}(t)}{2i}.$$

The following lemma gives us the q,ω derivative of sum, product and quotients of q,ω differentiable functions.

Lemma 1.4. Let $f,g:I\to \mathrm{R}$ be q,ω-differentiable at $t\in I$. Then:

(i) $D_{q,\omega}(f+g)(t)=D_{q,\omega}f(t)+D_{q,\omega}g(t)$,

(ii) $D_{q,\omega}(fg)(t)=D_{q,\omega}(f(t))g(t)+f(h(t))D_{q,\omega}g(t)$,

(iii) For any constant $c\in X$, $D_{q,\omega}(cf)(t)=cD_{q,\omega}(f(t))$,

(iv) $D_{q,\omega}(f/g)(t)=(D_{q,\omega}(f(t))g(t)-f(t)D_{q,\omega}g(t))/(g(t)g(h(t)))$ provided that $g(t)g(h(t))\neq 0$.

We notice that (ii) and (iv) are true even if $f:I\to X$. Also, (i) is true if $f,g:I\to X$.

The following theorem is important and we will use it later on.

Theorem 1. 5 [4]. Assume $f:I\to R$ is continuous at θ. Then the following statements are true.

(i) $\{f((sq^k)+\omega[k]_q)\}_{k\in \mathrm{N}}$ converges uniformly to $f(\theta)$ on I,

(ii) $\sum_{k=0}^{\infty} q^k \left| f(sq^k + \omega[k]_q) \right|$ is uniformly convergent on I and consequently f is q,ω-integrable over I ., (iii) Define

$$F(x) := \int_{\theta}^{x} f(t) d_{q,\omega} t. \quad x \in I.$$

Then F is continuous at θ. Furthermore; $D_{q,\omega}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega} f(t) d_{q,\omega} t = f(b) - f(a) \quad forall\ a,b \in I.$$

Based on the results in [14], A. E. Hamza and S. M. Ahmed deduced new results concerning the calculus associated with Hahn difference operator like Mean Value Theorems, Gronwall's and Bernoulli's Inequalities. Also, they established existence and uniqueness theorems of solutions of Hahn difference equations. They gave the required conditions for the existence and uniqueness of solutions of the Cauchy problem

$$\left.\begin{array}{l} a_0(t)D_{q,\omega}^{n}x(t) + a_1(t)D_{q,\omega}^{n-1}x(t) + ... + a_n(t)x(t) = b(t), \\ D_{q,\omega}^{i-1}x(\theta) = y_i, \quad i = 1,...,n. \end{array}\right\} \tag{1.3}$$

These conditions can be stated in the following theorem.

Theorem 1. 6. Assume the functions $a_j(t) : I \to \mathbb{C}, 0 \le j \le n,$ and $b(t) : I \to X$ satisfy the following conditions:

(i) $a_j(t),\ j = 1,...,n$ and $b(t)$ are continuous at θ with $a_0(t) \neq 0 \quad \forall\, t \in I$,

(ii) $a_j(t)/a_0(t)$ is bounded on $I,\ j \in \{1,...,n\}$.

Then, for any elements $y_r \in X$, Equation (3) has a unique solution on a subinterval $J \subset I$ containing θ.

The following lemma will be needed in our study.

Lemma 1. 7. Let $(X, \mathbb{K})$ be a vector space, and let T be a linear operator on X . For any $\lambda \in \mathbb{K}$ if there exist $y_0, y_1, ..., y_{m-1}$ in X such that

$$\begin{array}{l} Ty_0 = \lambda y_0 \\ Ty_i = \lambda y_i + y_{i-1} \quad (1 \le i \le m-1), \end{array}$$

then $y_0, ..., y_{m-1}$ are linearly independent [5].

Let us briefly summarize the organization of this paper. In Section 2, we investigate a necessary and sufficient condition for the existence of a fundamental set for the homogeneous equation

$$a_0(t)D_{q,\omega}^{n}x(t) + a_1(t)D_{q,\omega}^{n-1}x(t) + ... + a_n(t)x(t) = 0. \tag{1.4}$$

In Section 3, we introduce q,ω-Wronskian and prove its properties. We show that it is an effective tool to determine whether set of solutions is a fundamental set or not. See Corollary 3.5. Hence, we obtain Liouville's formula for Hahn difference equations. In Sections 4 and 5, we derive solutions of the first and second order linear Hahn difference equations with non-constant coefficients. In Section 6, we are concerned with constructing a fundamental set of solutions for (1.4) when the coefficients $a_j (0 \le j \le n)$ are constant. In Section 7, we present the analogues of the variation of parameter technique and the annihilator method to solve the nonhomogeneous linear Hahn difference equation

$$a_0(t)D^n_{q,\omega}x(t)+a_1(t)D^{n-1}_{q,\omega}x(t)+...+a_n(t)x(t)=b(t). \tag{1.5}$$

Finally, in Section 8 we propose to a future work .

2. Homogeneous Linear Hahn difference equation

In this Section, the coefficients $a_j(t), 0\le j\le n$ are assumed to satisfy the conditions of Theorem 1.6. The following two lemmas can be checked easily.

Lemma 2.1. If $x_1(t)$ and $x_2(t)$ are two solutions of Equation (1.4), then $c_1x_1(t)+c_2x_2(t)$ is also a solution where c_1 and c_2 are constants.

The second lemma is an immediate consequence of Theorem 1.6.

Lemma 2.2. If $x(t)$ is a solution of Equation (1.4) in J such that $D^i_{q,\omega}x(\theta)=0, 0\le i\le n-1$, then $x(t)=0 \ \forall t\in J$.

Definition 2.3. A set of n solutions of Equation (1.4) is said to be a fundamental set of Equation (1.4) valid in J if it is linearly independent in J.

The following results are analogous to the case of linear differential equations. Their proofs are similar and will be omitted.

Theorem 2.4. Let $b_{ij}, 1\le i,j\le n$ be any real or complex numbers and, for each j, $\psi_j(t)$ is the unique solution of Equation (1.4) which satisfies the initial conditions

$$D^{i-1}_{q,\omega}\psi_j(\theta)=b_{ij}, i,j=1,...,n.$$

Then, $\{\psi_j(t)\}_{j=1}^n$ is a fundamental set of Equation (4) if and only if $\det(b_{ij})\neq 0$.

Theorem 2.5. Let $\psi(t)$ be any solution of Equation (1.4) and $\psi_j, 1\le j\le n)$ form a fundamental set for Equation (1.4) valid in J. Then, there are unique constants c_j such that

$$\psi(t)=c_1\psi_1(t)+...+c_n\psi_n(t) \quad \forall\ t\in J. \tag{2.1}$$

3 A Hahn-Wronskian

Definition 3.1. We define the q,ω-Wronskian of the functions $x_1,...,x_n$, with domain I, by

$$W_{q,\omega}(x_1,...,x_n)(t)=\begin{vmatrix} x_1(t) & \dots & x_n(t)\\ D_{q,\omega}x_1(t) & \dots. & D_{q,\omega}x_n(t)\\ \vdots & \ddots & \vdots\\ D^{n-1}_{q,\omega}x_1(t) & \dots & D^{n-1}_{q,\omega}x_n(t)\end{vmatrix}$$

provided that $x_1,...,x_n$ are q,ω-differentiable functions.

Throughout this paper, we write $W_{q,\omega}$ instead of $W_{q,\omega}(x_1,...,x_n)$ unless there is ambiguity.

Lemma 3. 2. Let $x_1(t),x_2(t),...,x_n(t)$ be functions defined on I. Then, for any $t\in I, t\neq\theta$,

$$D_{q,\omega}Wq,\omega(t)=\begin{vmatrix} x_1(h(t)) & ... & x_n(h(t))\\ D_{q,\omega}x_1(h(t)) & ... & D_{q,\omega}x_n(h(t))\\ \vdots & \ddots & \vdots\\ D^{n-2}_{q,\omega}x_1(h(t)) & ... & D^{n-2}_{q,\omega}x_n(h(t))\\ D^n_{q,\omega}x_1(t) & ... & D^n_{q,\omega}x_n(t)\end{vmatrix} \tag{3.1}$$

Proof: We prove by induction on n. The lemma is trivial when $n=1$. Then suppose that it is true for $n=k$. Our objective is to show that it holds for $n = k+1$. Now, we expand $W_{q,\omega}(x_1,\dots,x_{k+1})$ in terms of the first row to obtain

$$W_{q,\omega}(x_1,\dots,x_{k+1}) = \sum_{j=1}^{k+1}(-1)^{j+1}x_j(t)W_{q,\omega}^{(j)}(t),$$

where

$$W_{q,\omega}^{(j)} = \begin{cases} W_{q,\omega}(D_{q,\omega}x_2,\dots,D_{q,\omega}x_{k+1}), \; j=1 \\ W_{q,\omega}(D_{q,\omega}x_1,\dots,D_{q,\omega}x_{j-1},D_{q,\omega}x_{j+1},\dots,D_{q,\omega}x_{k+1}), 2 \le j \le k \\ W_{q,\omega}(D_{q,\omega}x_1,\dots,D_{q,\omega}x_k), \; j=k+1. \end{cases}$$

Consequently,

$$\begin{aligned} D_{q,\omega}W_{q,\omega}(x_1,\dots,x_{k+1})(t) &= \sum_{j=1}^{k+1}(-1)^{j+1}D_{q,\omega}x_j(t)W_{q,\omega}^{(j)}(t) \\ &+\sum_{j=1}^{k+1}(-1)^{j+1}x_j(h(t))D_{q,\omega}W_{q,\omega}^{(j)}(t). \end{aligned}$$

Simple calculations show that

$$\sum_{j=1}^{k+1}(-1)^{j+1}D_{q,\omega}x_j(t)W_{q,\omega}^{(j)}(t) = 0,$$

and

$$\sum_{j=1}^{k+1}(-1)^{j+1}x_j(h(t))D_{q,\omega}W_{q,\omega}^{(j)}(t) = \begin{vmatrix} x_1(h(t)) & x_2(h(t)) & \dots & x_{k+1}(h(t)) \\ D_{q,\omega}x_1(h(t)) & D_{q,\omega}x_2(h(t)) & \dots & D_{q,\omega}x_{k+1}(h(t)) \\ \vdots & \vdots & \ddots & \vdots \\ D_{q,\omega}^{k-1}x_1(h(t)) & D_{q,\omega}^{k-1}x_2(h(t)) & \dots & D_{q,\omega}^{k-1}x_{k+1}(h(t)) \\ D_{q,\omega}^{k+1}x_1(t) & D_{q,\omega}^{k+1}x_2(t) & \dots & D_{q,\omega}^{k+1}x_{k+1}(t) \end{vmatrix}.$$

Thus, we have

$$D_{q,\omega}W_{q,\omega}(x_1,\dots,x_{k+1})(t) = \begin{vmatrix} x_1(h(t)) & x_2(h(t)) & \dots & x_{k+1}(h(t)) \\ D_{q,\omega}x_1(h(t)) & D_{q,\omega}x_2(h(t)) & \dots & D_{q,\omega}x_{k+1}(h(t)) \\ \vdots & \vdots & \ddots & \vdots \\ D_{q,\omega}^{k-1}x_1(h(t)) & D_{q,\omega}^{k-1}x_2(h(t)) & \dots & D_{q,\omega}^{k-1}x_{k+1}(h(t)) \\ D_{q,\omega}^{k+1}x_1(t) & D_{q,\omega}^{k+1}x_2(t) & \dots & D_{q,\omega}^{k+1}x_{k+1}(t) \end{vmatrix}$$

as required.

In the rest of this section, J is a subinterval of I containing θ.

Theorem 3.3. If $x_1,\dots,x_n$ are solutions of Equation (1.4) in J, then their $q,\omega-$Wronskian satisfies the first order Hahn difference equation

$$D_{q,\omega}W_{q,\omega}(t) = -R(t)W_{q,\omega}(t) \quad \forall \; t \in J \setminus \{\theta\}, \tag{3.2}$$

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where $R(t)=\sum_{k=0}^{n-1}(t-h(t))^k a_{k+1}(t)/a_0(t)$.

Proof: First, we show by induction that the following relation

$$D_{q,\omega}W_{q,\omega}(x_1,\dots,x_n)=\sum_{k=1}^{n}(-1)^{k-1}(t-h(t))^{k-1}\begin{vmatrix} x_1(t) & \dots & x_n(t) \\ D_{q,\omega}x_1(t) & \dots & D_{q,\omega}x_n(t) \\ \vdots & \ddots & \vdots \\ D_{q,\omega}^{n-k-1}x_1(t) & \dots & D_{q,\omega}^{n-k-1}x_n(t) \\ D_{q,\omega}^{n-k+1}x_1(t) & \dots & D_{q,\omega}^{n-k+1}x_n(t) \\ \vdots & \ddots & \vdots \\ D_{q,\omega}^{n}x_1(t) & \dots & D_{q,\omega}^{n}x_n(t) \end{vmatrix} \tag{3.3}$$

holds. Indeed, clearly (3.3) is true at $n=1$. Assume that (3.3) is true for n=m. From Lemma 3.2,

$$\begin{aligned} D_{q,\omega}W_{q,\omega}(x_1,\dots,x_{m+1}) &= \begin{vmatrix} x_1(h(t)) & \dots & x_{m+1}(h(t)) \\ D_{q,\omega}x_1(h(t)) & \dots & D_{q,\omega}x_{m+1}(h(t)) \\ \vdots & \ddots & \vdots \\ D_{q,\omega}^{m-1}x_1(h(t)) & \dots & D_{q,\omega}^{m-1}x_{m+1}(h(t)) \\ D_{q,\omega}^{m+1}x_1(t) & \dots & D_{q,\omega}^{m+1}x_{m+1}(t) \end{vmatrix} \\ &= \sum_{j=1}^{m+1}(-1)^{j+1}x_j(h(t))W^{*(j)}_{q,\omega}(t), \end{aligned}$$

where

$$W^{*(j)}_{q,\omega}=\begin{cases} D_{q,\omega}W_{q,\omega}(D_{q,\omega}x_2,\dots,D_{q,\omega}x_{m+1}),\ j=1 \\ D_{q,\omega}W_{q,\omega}(D_{q,\omega}x_1,\dots,D_{q,\omega}x_{j-1},D_{q,\omega}x_{j+1},\dots,D_{q,\omega}x_{m+1}),\ 2\le j\le m \\ D_{q,\omega}W_{q,\omega}(D_{q,\omega}x_1,\dots,D_{q,\omega}x_m),\ j=m+1. \end{cases}$$

One can see that

$$W^{*(j)}_{q,\omega}(t)=\sum_{k=1}^{m}(-1)^{k-1}(t-h(t))^{k-1}S_{jk},$$

where

$$S_{jk}=\begin{vmatrix} D_{q,\omega}x_1(t) & \dots & D_{q,\omega}x_{j-1}(t) & D_{q,\omega}x_{j+1}(t) & \dots & D_{q,\omega}x_{m+1}(t) \\ D_{q,\omega}^2x_1(t) & \dots & D_{q,\omega}^2x_{j-1}(t) & D_{q,\omega}^2x_{j+1}(t) & \dots & D_{q,\omega}^2x_{m+1}(t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_{q,\omega}^{m-k}x_1(t) & \dots & D_{q,\omega}^{m-k}x_{j-1}(t) & D_{q,\omega}^{m-k}x_{j+1}(t) & \dots & D_{q,\omega}^{m-k}x_{m+1}(t) \\ D_{q,\omega}^{m-k+2}x_1(t) & \dots & D_{q,\omega}^{m-k+2}x_{j-1}(t) & D_{q,\omega}^{m-k+2}x_{j+1}(t) & \dots & D_{q,\omega}^{m-k+2}x_{m+1}(t) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_{q,\omega}^{m+1}x_1(t) & \dots & D_{q,\omega}^{m+1}x_{j-1}(t) & D_{q,\omega}^{m+1}x_{j+1}(t) & \dots & D_{q,\omega}^{m+1}x_{m+1}(t) \end{vmatrix}, 2\le j\le m,$$

$$S_{jk}=\begin{vmatrix} D_{q,\omega}x_2(t) & \dots & D_{q,\omega}x_{m+1}(t) \\ D^2_{q,\omega}x_2(t) & \dots & D^2_{q,\omega}x_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D^{m-k}_{q,\omega}x_2(t) & \dots & D^{m-k}_{q,\omega}x_{m+1}(t) \\ D^{m-k+2}_{q,\omega}x_2(t) & \dots & D^{m-k+2}_{q,\omega}x_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D^{m+1}_{q,\omega}x_2(t) & \dots & D^{m+1}_{q,\omega}x_{m+1}(t) \end{vmatrix}, \ j=1,$$

and

$$S_{jk}=\begin{vmatrix} D_{q,\omega}x_1(t) & \dots & D_{q,\omega}x_{m}(t) \\ D^2_{q,\omega}x_1(t) & \dots & D^2_{q,\omega}x_{m}(t) \\ \vdots & \ddots & \vdots \\ D^{m-k}_{q,\omega}x_1(t) & \dots & D^{m-k}_{q,\omega}x_{m}(t) \\ D^{m-k+2}_{q,\omega}x_1(t) & \dots & D^{m-k+2}_{q,\omega}x_{m}(t) \\ \vdots & \ddots & \vdots \\ D^{m+1}_{q,\omega}x_1(t) & \dots & D^{m+1}_{q,\omega}x_{m}(t) \end{vmatrix}, \ j=m+1.$$

It follows that

$$\begin{aligned} D_{q,\omega}W_{q,\omega}(x_1,\dots,x_{m+1})(t) &= \sum_{j=1}^{m+1}(-1)^{j+1}(x_j(t)-(t-h(t))D_{q,\omega}x_j(t)) \\ &\quad\times\sum_{k=1}^{m}(-1)^{k-1}(t-h(t))^{k-1}S_{jk} \\ &= \sum_{k=1}^{m}(-1)^{k-1}(t-h(t))^{k-1}\sum_{j=1}^{m+1}(-1)^{j+1}x_j(t)S_{jk} \\ &\quad+\sum_{k=1}^{m}(-1)^{k}(t-h(t))^{k}\sum_{j=1}^{m+1}(-1)^{j+1}D_{q,\omega}x_j(t)S_{jk} \\ &= \sum_{k=1}^{m}(-1)^{k-1}(t-h(t))^{k-1}L(k) \\ &\quad+\sum_{k=1}^{m}(-1)^{k}(t-h(t))^{k}M(k), \end{aligned} \tag{3.4}$$

where

$$L(k)=\sum_{j=1}^{m+1}(-1)^{j+1}x_j(t)S_{jk}=\begin{vmatrix} x_1(t) & \dots & x_{m+1}(t) \\ D_{q,\omega}x_1(t) & \dots & D_{q,\omega}x_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D^{m-k}_{q,\omega}x_1(t) & \dots & D^{m-k}_{q,\omega}x_{m+1}(t) \\ D^{m-k+2}_{q,\omega}x_1(t) & \dots & D^{m-k+2}_{q,\omega}x_{m+1}(t) \\ \vdots & \ddots & \vdots \\ D^{m+1}_{q,\omega}x_1(t) & \dots & D^{m+1}_{q,\omega}x_{m+1}(t) \end{vmatrix}, \tag{3.5}$$

and

$$M(k)=\sum_{j=1}^{m+1}(-1)^{j+1}D_{q,\omega}x_j(t)S_{jk}=\begin{cases}0, \quad if \quad (k=1,\dots,m-1),\\ \begin{vmatrix} D_{q,\omega}x_1(t) & \dots & D_{q,\omega}x_{m+1}(t)\\ D^2_{q,\omega}x_1(t) & \dots & D^2_{q,\omega}x_{m+1}(t)\\ \vdots & \ddots & \vdots\\ D^{m+1}_{q,\omega}x_1(t) & \dots & D^{m+1}_{q,\omega}x_{m+1}(t)\end{vmatrix} & if \quad k=m.\end{cases} \tag{3.6}$$

Using relations (3.5) and (3.6) and substituting in (3.4) , we obtain relation (3.3) at $n=m+1$ Since $D^n_{q,\omega}x_j(t)=-\sum_{i=1}^{n}(a_i(t)/a_0(t))D^{n-i}_{q,\omega}x_j(t)$, it follows that

$$\begin{aligned}D_{q,\omega}W_{q,\omega}(t) &= \sum_{k=1}^{n}(-1)^{k-1}(t-h(t))^{k-1}\left(\frac{-a_k(t)}{a_0(t)}\right)\begin{vmatrix} x_1(t) & \dots & x_n(t)\\ D_{q,\omega}x_1(t) & \dots & D_{q,\omega}x_n(t)\\ \vdots & \ddots & \vdots\\ D^{n-k-1}_{q,\omega}x_1(t) & \dots & D^{n-k-1}_{q,\omega}x_n(t)\\ D^{n-k+1}_{q,\omega}x_1(t) & \dots & D^{n-k+1}_{q,\omega}x_n(t)\\ \vdots & \ddots & \vdots\\ D^{n-1}_{q,\omega}x_1(t) & \dots & D^{n-1}_{q,\omega}x_n(t)\\ D^{n-k}_{q,\omega}x_1(t) & \dots & D^{n-k}_{q,\omega}x_n(t)\end{vmatrix}\\ &= \sum_{k=1}^{n}(-1)^{2(k-1)}(t-h(t))^{k-1}\left(\frac{-a_k(t)}{a_0(t)}\right)W_{q,\omega}(t)\\ &= -\sum_{k=0}^{n-1}(t-h(t))^{k}\frac{a_{k+1}(t)}{a_0(t)}W_{q,\omega}(t)\\ &= -R(t)W_{q,\omega}(t),\end{aligned}$$

which is the desired result.

The following theorem gives us Liouville's formula for Hahn difference equations.

Theorem 3.4. Assume that $(h(t)-t)R(t)\neq 1, t\in J$. Then, the $q,\omega-$ Wronskian of any set of solution $\{\psi_i(t)\}_{i=1}^n$, valid in J, is given by

$$W_{q,\omega}(t)=\frac{W_{q,\omega}(\theta)}{\prod_{k=0}^{\infty}(1+q^k(t(1-q)-\omega)R(h^k(t)))} \quad ,t\in J. \tag{3.7}$$

Proof: Relation (3.2) implies that

$$W_{q,\omega}(h(t))=(1+(t-h(t))R(t))W_{q,\omega}(t) \quad ,t\in J\setminus\{\theta\}.$$

Hence,

$$\begin{aligned} W_{q,\omega}(t) &= \frac{W_{q,\omega}(h(t))}{1+(t-h(t))R(t)} \\ &= \frac{W_{q,\omega}(h^m(t))}{\prod_{k=0}^{m-1}(1+q^k(t(1-q)-\omega)R(h^k(t)))}, m\in \mathbb{N}. \end{aligned}$$

Taking $m\to\infty$, we get

$$W_{q,\omega}(t)=\frac{W_{q,\omega}(\theta)}{\prod_{k=0}^{\infty}(1+q^k(t(1-q)-\omega)R(h^k(t)))}, t\in J.$$

An interesting result which can be deduced directly from Theorems 2.4 and 3.4 is the following.

Corollary 3.5. Let $\{\psi_i\}_{i=1}^n$ be a set of solutions of Equation (1.4) in J. Then, $W_{q,\omega}(t)$ has two possibilities:

(i) $W_{q,\omega}(t)\neq 0$ in J if and only if $\{\psi_i\}_{i=1}^n$ is a fundamental set of Equation (1.4) valid in J.

(ii) $W_{q,\omega}(t)=0$ in J if and only if $\{\psi_i\}_{i=1}^n$ is not a fundamental set of Equation (1.4) valid in J.

Example 3.6. We calculate the q,ω-Wronskian of the Hahn difference equation

$$D^2_{q,\omega}x(t)+x(t)=0. \tag{3.8}$$

The functions $x_1(t)=\cos_{q,\omega}(1,t)$ and $x_2(t)=\sin_{q,\omega}(1,t)$ where $|t-\theta|<\dfrac{1}{1-q}$ are solutions of Equation (3.8) subject to the initial conditions $x_1(\theta)=1, D_{q,\omega}x_1(\theta)=0$ and $x_2(\theta)=0, D_{q,\omega}x_2(\theta)=1$ respectively. Here, $R(t)=(t-h(t))$. So, $(h(t)-t)R(t)\neq 1\ \forall\, t\neq\theta$. Consequently,

$$\prod_{k=0}^{\infty}(1+q^k(t(1-q)-\omega)(h^k(t)-h^{k+1}(t)))=\prod_{k=0}^{\infty}(1+q^{2k}(t(1-q)-\omega)^2),$$

which implies

$$\begin{aligned} W_{q,\omega}(\theta) &= \begin{vmatrix} \cos_{q,\omega}(1,\theta) & \sin_{q,\omega}(1,\theta) \\ \sin_{q,\omega}(1,\theta) & \cos_{q,\omega}(1,\theta) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \end{aligned}$$

Therefore,

$$W_{q,\omega}(t)=\frac{1}{\prod_{k=0}^{\infty}(1+q^{2k}(t(1-q)-\omega)^2)}.$$

4. First order linear Hahn difference equations

In [4], M. H. Annaby, A. E. Hamza and K. A. Aldwoah solved the first order linear Hahn difference equations with constant coefficients. This result was stated as follows.

Lemma 4.1. For fixed $z\in\mathbb{C}$, the q,ω-exponential functions $e_z(t)$ and $E_{-z}(t)$ are the unique solutions of the initial value problems

$$D_{q,\omega}x(t) = zx(t),\ \ x(\theta) = 1,\ \ |t-\theta| < \frac{1}{|z(1-q)|},$$

and

$$D_{q,\omega}x(t) = -zx(qt+\omega),\ \ x(\theta) = 1,\quad t \in \mathbb{C},$$

respectively.

In the following theorem, we generalize lemma (4.1) when we replace the complex fixed number z by a complex function $p(t)$ which is continuous at θ. We define the exponential functions $e_p(t)$ and $E_p(t)$ by

$$e_p(t) = \frac{1}{\prod_{k=0}^{\infty}(1-p(h^k(t))q^k(t(1-q)-\omega))} \tag{4.1}$$

and

$$E_p(t) = \prod_{k=0}^{\infty}(1+p(h^k(t))q^k(t(1-q)-\omega)) \tag{4.2}$$

whenever the first product is convergent to a nonzero number for every $t \in I$. It is worth noting that the two products are convergent since $\sum_{k=0}^{\infty}|p(h^k(t))|q^k(t(1-q)-\omega)$ is convergent, see [4].

Theorem 4. 2. The q,ω-exponential functions $e_p(t)$ and $E_{-p}(t)$ are the unique solutions of the initial value problems

$$D_{q,\omega}x(t) = p(t)x(t), x(\theta) = 1, \tag{4.3}$$

and

$$D_{q,\omega}x(t) = -p(t)x(qt+\omega),\ \ x(\theta) = 1, \tag{4.4}$$

respectively.

Proof: First, $e_p(t)$ is a solution of Equation (3). Indeed, we have for $t \neq \theta$

$$\begin{aligned} D_{q,\omega}e_p(t) &= \frac{1}{h(t)-t}\Big(\frac{1}{\prod_{k=0}^{\infty}(1-p(h^{k+1}(t))q^k(h(t)(1-q)-\omega))} \\ &\quad -\frac{1}{\prod_{k=0}^{\infty}(1-p(h^k(t))q^k(t(1-q)-\omega))}\Big) \\ &= \frac{1}{h(t)-t}\Big(\frac{1}{\prod_{k=1}^{\infty}(1-p(h^k(t))q^k(t(1-q)-\omega))} \\ &\quad -\frac{1}{\prod_{k=0}^{\infty}(1-p(h^k(t))q^k(t(1-q)-\omega))}\Big) \\ &= \frac{1}{h(t)-t}\Big(\frac{-p(t)(t(1-q)-\omega)}{\prod_{k=0}^{\infty}(1-p(h^k(t))q^k(t(1-q)-\omega))}\Big) \\ &= p(t)e_p(t). \end{aligned}$$

By the Existence and Uniqueness Theorem of solutions, this solution is unique.

Finally, $E_{-p}(t)$ is a solution of Equation (4), since

$$\begin{aligned} D_{q,\omega}x(t) &= D_{q,\omega}E_{-p}(t) = D_{q,\omega}(\frac{1}{e_p(t)}) \\ &= \frac{-p(t)e_p(t)}{e_p(t)e_p(h(t))} = -p(t)x(h(t)). \end{aligned}$$

The uniqueness of the solution can be deduced again by the Existence and Uniqueness Theorem of solutions.

We can see that $E_p(t)$ is continuous at θ. Indeed, the uniform convergence of $\sum_{k=0}^{n} p(h^k(t))q^k(t(1-q)-\omega)$ implies the uniform convergence of $T_n = \prod_{k=0}^{n}(1+p(h^k(t))q^k(t(1-q)-\omega))$. Since T_n is continuous at θ for every n, then $E_p(t)$ is continuous at θ.

In the following theorem, we give a closed formula for solutions the non-homogeneous first order linear Hahn difference equations of the form

$$D_{q,\omega}x(t) = p(t)x(t) + f(t), \quad x(\theta) = x_\theta \in X. \tag{4.5}$$

Theorem 4.3. Assume that $f: I \to X$ is continuous at θ. Then the solution of Equation (5) has the form

$$x(t) = e_p(t)(x_\theta + \int_\theta^t f(\tau)E_{-p}(q\tau+\omega)d_{q,\omega}\tau). \tag{4.6}$$

Proof: The function $x(t)$ given in (4.6) solves equation (4.5). Indeed, we have

$$\begin{aligned} D_{q,\omega}x(t) &= p(t)e_p(t)x_\theta + p(t)e_p(t)(\int_\theta^t f(\tau)E_{-p}(q\tau+\omega)d_{q,\omega}\tau) \\ &+ f(t)E_{-p}(qt+\omega)e_p(h(t)) \\ &= p(t)x(t) + f(t). \end{aligned}$$

In the following theorem, we prove some useful properties about the exponential function $e_p(t)$. Throughout the remainder of this paper we put $\xi(t) = h(t) - t = t(q-1)+\omega$.

Theorem 4.4. Assume that $r, s: I \to \mathbb{C}$ are continuous at θ. The following properties are true.

(i) $\frac{1}{e_r(t)} = e_{-r/(1+\xi r)}(t)$,

(ii) $e_r(t)e_s(t) = e_{r+s+\xi rs}(t)$,

(iii) $e_r(t)/e_s(t) = e_{(r-s)/(1+\xi s)}(t)$.

Proof: (i) The function $e_{-r/(1+\xi r)}(t)$ is a solution of the initial value problem

$$D_{q,\omega}x(t) = \frac{-r}{1+\xi r}x(t),\ x(\theta) = 1.$$

Also, $\frac{1}{e_r(t)}$ is another solution. Indeed, we conclude that

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$$\begin{aligned} D_{q,\omega}\left(\frac{1}{e_r(t)}\right) &= -\frac{re_r(t)}{e_r(t)e_r(h(t))} \\ &= \frac{-r}{1+\xi r}\left(\frac{1}{e_r(t)}\right). \end{aligned}$$

Clearly, $\frac{1}{e_r(\theta)} = 1$. By the Uniqueness Theorem of solutions, statement (i) is true.

(ii) The function $e_r(t)e_s(t)$ is a solution of the initial value problem

$$D_{q,\omega}x(t) = (r+s+\xi rs)(t)x(t),\ x(\theta)=1.$$

This is because

$$\begin{aligned} D_{q,\omega}(e_r(t)e_s(t)) &= re_r(t)e_s(h(t)) + e_r(t)se_s(t) \\ &= (r+s+\xi rs)(t)e_r(t)e_s(t). \end{aligned}$$

Also, $e_r(\theta)e_s(\theta)=1$. Again by the Uniqueness Theorem of solutions, we obtain the desired result.

(iii) This follows directly by using items (i) and (ii). In fact, we have

$$\begin{aligned} e_r(t)/e_s(t) &= e_r(t)\times e_{-s/1+\xi s}(t) \\ &= e_{(r-s)/(1+\xi s)}(t). \end{aligned}$$

5. Second order linear Hahn difference equations

In general, there is no method to solve second order linear Hahn difference equations with arbitrary non-constant coefficients. Therefore, we will try to solve special cases of second order linear Hahn difference equations. In [4], M. H. Annaby, A. E. Hamza and K. A. Aldwoah deduced the following lemma.

Lemma 5.1. The functions $\cos_{q,\omega}(z,.), \sin_{q,\omega}(z,.), \mathrm{Cos}_{q,\omega}(z,.)$ and $\mathrm{Sin}_{q,\omega}(z,.)$ solve the initial value problems

$$\begin{aligned} &D^2_{q,\omega}x(t) = -z^2x(t),\ \ x(\theta)=1, D_{q,\omega}x(\theta)=0,\ |t-\theta| < \frac{1}{|z(1-q)|}, \\ &D^2_{q,\omega}x(t) = -z^2x(t),\ \ x(\theta)=0, D_{q,\omega}x(\theta)=z,\ |t-\theta| < \frac{1}{|z(1-q)|}, \\ &D^2_{q,\omega}x(t) = -z^2x(h^2(t)),\ \ x(\theta)=1,\ D_{q,\omega}x(\theta)=0,\ t\in \mathbb{R}, \end{aligned}$$

and

$$D^2_{q,\omega}x(t) = -z^2x(h^2(t)),\ \ x(\theta)=0,\ D_{q,\omega}x(\theta)=-z,\ t\in \mathbb{R},$$

respectively.

In the following result we prove a useful formula of a solution of second order linear Hahn difference equations.

In the following theorem, $[a,b]$ is a closed interval containing θ and $p:[a,b]\to C$ is continuous at θ.

Theorem 5. 2. Any solution ψ of the equation

$$D^2_{q,\omega}x(t) + p(t)x(t) = 0,\quad t\in[a,b]$$

satisfies the following relation

$$\psi(t)=c_1(b-t)+c_2(t-a)+\frac{b-t}{b-a}\int_a^t(\tau-a)p(\frac{\tau-\omega}{q})\psi(\frac{\tau-\omega}{q})d_{q,\omega}\tau$$
$$+\frac{t-a}{b-a}\int_t^b(b-\tau)p(\frac{\tau-\omega}{q})\psi(\frac{\tau-\omega}{q})d_{q,\omega}\tau.$$

Proof: It follows by direct computations.

Now, we turn to a very special case which we call it Euler-Cauchy Hahn difference equation. It takes the form

$$t(qt+\omega)D^2_{q,\omega}x(t)+atD_{q,\omega}x(t)+bx(t)=0,\, t\in I\setminus\{\theta\}. \tag{5.1}$$

The characteristic equation of (1) is given by

$$\lambda^2+(a-1)\lambda+b=0. \tag{5.2}$$

Theorem 5.3. If the characteristic equation (5.2) has two distinct roots λ_1 and λ_2, then a fundamental set of solutions of (5.1) is given by $e_{\lambda_1/t}(t)$ and $e_{\lambda_2/t}(t)$.

Proof: Let $x(t)=e_{\lambda/t}(t)$, where λ is a root of Equation (5.2). It follows that

$$D_{q,\omega}x(t)=\frac{\lambda}{t}x(t) \quad and \quad D^2_{q,\omega}x(t)=\frac{(\lambda^2-\lambda)}{t\,h(t)}x(t).$$

Consequently, we have

$$\begin{aligned} t(qt+\omega)D^2_{q,\omega}x(t)+atD_{q,\omega}x(t)+bx(t) &= (\lambda^2+(a-1)\lambda+b)x(t) \\ &= 0. \end{aligned}$$

Now, assume that λ_1 and λ_2 are distinct roots of the characteristic equation (5.2). So, we have

$$\lambda_1+\lambda_2=1-a \quad and \quad \lambda_1\lambda_2=b.$$

Moreover, the Wronskian of the two solutions $e_{\lambda_1/t}(t)$ and $e_{\lambda_2/t}(t)$ is given by

$$\begin{vmatrix} e_{\lambda_1/t}(t) & e_{\lambda_2/t}(t) \\ \frac{\lambda_1}{t}e_{\lambda_1/t}(t) & \frac{\lambda_2}{t}e_{\lambda_2/t}(t) \end{vmatrix}=\frac{1}{t}(\lambda_2-\lambda_1)e_{\lambda_1/t}(t)e_{\lambda_2/t}(t)$$

which does not vanish since $\lambda_1\neq\lambda_2$. Hence, the exponential functions $e_{\lambda_1/t}(t)$ and $e_{\lambda_2/t}(t)$ form a fundamental set of solutions of (5.1).

Now we are concerning with the Euler-Cauchy Hahn difference equation in the double root case. Consider the second order Hahn difference operator

$$Lx(t)=D^2_{q,\omega}x(t)+p(t)D_{q,\omega}x(t)+r(t)x(t). \tag{5.3}$$

We need the following two Lemmas in establishing the general solution in the double root case. Their proofs are direct, so they will be omitted.

Lemma 5.4. Let $x_1(t)$ and $x_2(t)$ be twice q,ω-differentiable. Then, we have

$$(i)\quad W_{q,\omega}(x_1,x_2)(t)=\begin{vmatrix} x_1(h(t)) & x_2(h(t)) \\ D_{q,\omega}x_1(t) & D_{q,\omega}x_2(t) \end{vmatrix},$$

$$(ii)\quad D_{q,\omega}W_{q,\omega}(x_1,x_2)(t)=\begin{vmatrix} x_1(h(t)) & x_2(h(t)) \\ D^2_{q,\omega}x_1(t) & D^2_{q,\omega}x_2(t) \end{vmatrix},$$

$$(iii)\quad D_{q,\omega}W_{q,\omega}(x_1,x_2)(t)=\begin{vmatrix} x_1(h(t)) & x_2(h(t)) \\ Lx_1(t) & Lx_2(t) \end{vmatrix}+(-p(t)+\xi(t)r(t))W_{q,\omega}(x_1,x_2)(t),$$ where

$\xi(t)=h(t)-t$.

Lemma 5. 5. Assume that $x_1(t)$ and $x_2(t)$ are two solutions of

$$Lx(t)=0.$$

Then, their q,ω-Wronskian W satisfies

$$W_{q,\omega}(t)=e_{-p+r\xi}(t)W_{q,\omega}(\theta),\quad t\in I.$$

The following theorem gives us the general solution of the Euler-Cauchy Hahn difference equation in the double root case.

Theorem 5.6. Assume that I does not contain 0 and $1/h(t)$ is bounded on I. Then, the general solution of the Euler-Cauchy Hahn difference equation

$$t(qt+\omega)D^2_{q,\omega}x(t)+(1-2\alpha)tD_{q,\omega}x(t)+\alpha^2x(t)=0,\ t\in I \tag{5.4}$$

is given by

$$x(t)=c_1e_{\frac{\alpha}{t}}(t)+c_2e_{\frac{\alpha}{t}}(t)\int_\theta^t\frac{e_{-\frac{1}{h(\tau)}}}{1+\frac{\alpha}{\tau}\xi(\tau)}d_{q,\omega}\tau.$$

Proof: The characteristic equation of (5.4) is

$$\lambda^2-2\alpha\lambda+\alpha^2=0.$$

Consequently, the characteristic roots are $\lambda_1=\lambda_2=\alpha$. Hence one linearly independent solution of Equation (5.4) is

$$x_1(t)=e_{\alpha/t}(t).$$

Now, we will look for the second linearly independent solution. We can rewrite Equation (5.4) in the form

$$D^2_{q,\omega}x(t)+p(t)D_{q,\omega}x(t)+r(t)x(t)=0,$$

with

$$p(t)=\frac{1-2\alpha}{h(t)},\quad r(t)=\frac{\alpha^2}{th(t)}.$$

Consequently,

$$-p+\xi r=\frac{\alpha^{2}}{t}-\frac{(\alpha-1)^{2}}{h(t)}.$$

Let u be a solution of Equation (5.4) such that $u(\theta)=0, D_{q,\omega}u(\theta)=1$. Then, the Wronskian of the two solutions $x_1(t)=e_{\alpha/t}(t)$ and $u(t)$ is given by

$$W_{q,\omega}(e_{\frac{\alpha}{t}},u)(t)=e_{-p+\xi r}(t)=e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{h(t)}}(t).$$

By the quotient rule, we find that u satisfies the following Hahn difference equation

$$D_{q,\omega}\Big(\frac{u}{e_{\frac{\alpha}{t}}}\Big)(t)=\frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{h(t)}}(t)}{e^2_{\frac{\alpha}{t}}(t)(1+\frac{\alpha}{t}\xi(t))}.$$

Indeed, simple calculations show that

$$\begin{aligned}D_{q,\omega}\Big(\frac{u}{e_{\frac{\alpha}{t}}}\Big)(t) &= \frac{W_{q,\omega}(e_{\frac{\alpha}{t}},u)(t)}{e_{\frac{\alpha}{t}}(t)e_{\frac{\alpha}{h(t)}}(h(t))}\\ &=\frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{h(t)}}(t)}{e^2_{\frac{\alpha}{t}}(t)(1+\frac{\alpha}{t}\xi(t))},\end{aligned}$$

Integrating both sides from θ to t and using $u(\theta)=0$ we deduce that

$$u(t)=e_{\frac{\alpha}{t}}(t)\int_{\theta}^{t}\frac{e_{\frac{\alpha^2}{\tau}-\frac{(\alpha-1)^2}{h(\tau)}}(\tau)}{e^2_{\frac{\alpha}{\tau}}(\tau)(1+\frac{\alpha}{\tau}\xi(\tau))}d_{q,\omega}\tau$$

is a solution of Equation (5.4). On the other hand, simple calculations show that

$$\frac{e_{\frac{\alpha^2}{t}-\frac{(\alpha-1)^2}{h(t)}}(t)}{e^2_{\frac{\alpha}{t}}(t)}=e_{-\frac{1}{h(t)}}(t).$$

Therefore, the general solution of Equation (5.4) is given by

$$x(t)=c_1e_{\frac{\alpha}{t}}(t)+c_2e_{\frac{\alpha}{t}}(t)\int_{\theta}^{t}\frac{e_{-1/h(\tau)}}{1+\frac{\alpha}{\tau}\xi(\tau)}d_{q,\omega}\tau.$$

6. Construction of a fundamental set of solutions

In this section, we are concerned with constructing a fundamental set of solutions for (1.4). Since such a construction is not in general possible for $n>1$, we found that it is more convenient to deal with (1.4) when the coefficients are constants except for $n=1$ which was given in Section 3. Now, Equation (1.4) can be written as

$$Lx(t) = a_0 D^n_{q,\omega} x(t) + a_1 D^{n-1}_{q,\omega} x(t) + ... + a_n x(t) = 0, \tag{6.1}$$

where $a_j, 0 \le j \le n$ are constants. The characteristic polynomial of Equation (6.1) is given by

$$p(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + ... + a_n.$$

Let $\lambda_i, 1 \le i \le k$ denote the distinct roots of $p(\lambda) = 0$ of multiplicity m_i, so that $\sum_{i=1}^{k} m_i = n$. The following theorem is the key for constructing a fundamental set of solutions of Equation (6.1). Its proof is an analogue to the $q-$difference operator case $[5]$. So it will be omitted.

Theorem 6.1. The initial value problem

$$D_{q,\omega}\psi_{0,i}(t) = \lambda_i \psi_{0,i}(t), \quad \psi_{0,i}(\theta) = 1$$
$$D_{q,\omega}\psi_{r,i}(t) = \lambda_i \psi_{r,i}(t) + \psi_{r-1,i}(t), \psi_{r,i}(\theta) = 0, r = 1,..., m_i - 1$$

has the solution

(1)

$$\psi_{r,i}(t) = \begin{cases} e_{\lambda_i}(t) = \sum_{k=0}^{\infty} \frac{(\lambda_i(t(1-q)-\omega))^k}{(q;q)_k}, r = 0 \\ \frac{1}{\lambda_i^r} \sum_{k=r}^{\infty} \frac{k(k-1)...(k-r+1)(\lambda_i(t-h(t)))^k}{r!(q;q)_k}, r = 1,2,..., m_i - 1, \end{cases} \tag{6.2}$$

if $\lambda_i \ne 0$.

(2)

$$\psi_{r,i}(t) = \frac{(t-h(t))^r}{(q;q)_r}, r = (0,1,2,..., m_i - 1), \tag{6.3}$$

if $\lambda_i = 0$.

It is worth to mention that the Hahn difference operator L can be written as follows:

$$Lx(t) = \prod_{i=1}^{k} (D_{q,\omega} - \lambda_i I)^{m_i} x(t). \tag{6.4}$$

Here, I is the identity operator. For each λ_i, the function $\psi_{r,i}$ satisfies the following equation

$$(D_{q,\omega} - \lambda_i I)^{m_i} \psi_{r,i}(t) = 0, \quad r = 0,..., m_i - 1.$$

This leads us to state the following theorem.

Theorem 6. 2. The set $\{\psi_{r,i}, r = 0,..., m_i - 1, 1 \le i \le k\}$ which is given by (6.2) and (6.3) when $\lambda_i \ne 0$ and $\lambda_i = 0$ respectively forms a fundamental set of solutions of Equation (6.1).

Example 6.3. The Hahn difference equation

$$D^3_{q,\omega} x(t) - 4D^2_{q,\omega} x(t) + 5D_{q,\omega} x(t) - 2x(t) = 0$$

has the functions $e_2(t), e_1(t)$ and $\sum_{k=1}^{\infty} \frac{k(t-h(t))^k}{(q;q)_k}$, as a fundamental set of solutions.

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7. Non-Homogeneous Hahn difference equations

In this Section, we are interesting in finding the general solution of the nonhomogeneous Hahn difference Equation (1.5) where the coefficients $a_j(t)$ and $b(t)$ are assumed to satisfy the conditions of Existence and Uniqueness Theorem 1.6. As in the theory of differential equations, one can see that: If $\psi_1(t)$ and $\psi_2(t)$ are two solutions of (1.5), then $\psi_1(t)-\psi_2(t)$ is a solution of the corresponding homogeneous Equation (1.4). Based on the above-mentioned note and Theorem 2.5, we get the following: If $\psi_1(t),\psi_2(t),\ldots,\psi_n(t)$ form a fundamental set for (1.4) and $\psi_0(t)$ is a solution of Equation (1.5), then for any solution of Equation (1.5), there are unique constants $c_1,\ldots,c_n$ such that

$$\psi(t)=c_1\psi_1(t)+\ldots+c_n\psi_n(t)+\psi_0(t). \tag{7.1}$$

Therefore, if we can find any particular solution $\psi_0(t)$ of Equation (1.5), then (7.1) gives a general formula for all solutions of Equation (1.5).

7.1 Method of Variation of parameters

We aim to obtain a particular solution $\psi_0(t)$ by the method of variation of parameters. This method depends on replacing the constants c_r in relation (2.1) by the functions $c_r(t)$. Hence, we try to find a solution of the form

$$\psi_0(t)=c_1(t)\psi_1(t)+\ldots+c_n(t)\psi_n(t). \tag{7.2}$$

Now, our objective is to determine the functions $c_r(t)$. We have

$$D^{i-1}_{q,\omega}\psi_0(t)=\sum_{j=1}^{n}c_j(t)D^{i-1}_{q,\omega}\psi_j(t), 1\le i\le n \tag{7.3}$$

provided that

$$\sum_{j=1}^{n}D_{q,\omega}c_j(t)D^{i-1}_{q,\omega}\psi_j(h(t))=0, 1\le i\le n-1. \tag{7.4}$$

Putting $i=n$ in (7.3) and operating on it by $D_{q,\omega}$, we obtain

$$D^{n}_{q,\omega}\psi_0(t)=\sum_{j=1}^{n}(c_j(t)D^{n}_{q,\omega}\psi_j(t)+D_{q,\omega}c_j(t)D^{n-1}_{q,\omega}\psi_j(h(t))) \tag{7.5}$$

Since $\psi_0(t)$ satisfies Equation (1.5), it follows that

$$a_0(t)D^{n}_{q,\omega}\psi_0(t)+a_1(t)D^{n-1}_{q,\omega}\psi_0(t)+\ldots+a_n(t)\psi_0(t)=b(t). \tag{7.6}$$

Substitute by (7.3) and (7.5) in (7.6) and in view of Equation (1.4), we obtain

$$\sum_{j=1}^{n}D_{q,\omega}c_j(t)D^{n-1}_{q,\omega}\psi_j(h(t))=\frac{b(t)}{a_0(t)}.$$

Thus, we get the following system

$$\left.\begin{array}{l}
D_{q,\omega}c_1(t)\psi_1(h(t))+\ldots+D_{q,\omega}c_n(t)\psi_n(h(t))=0\\
\vdots\\
D_{q,\omega}c_1(t)D^{n-2}_{q,\omega}\psi_1(h(t))+\ldots+D_{q,\omega}c_n(t)D^{n-2}_{q,\omega}\psi_n(h(t))=0\\
D_{q,\omega}c_1(t)D^{n-1}_{q,\omega}\psi_1(h(t))+\ldots+D_{q,\omega}c_n(t)D^{n-1}_{q,\omega}\psi_n(h(t))=\dfrac{b(t)}{a_0(t)}.
\end{array}\right\} \tag{7.7}$$

Consequently,

$$D_{q,\omega}c_r(t)=\frac{W_r(h(t))}{W_{q,\omega}(h(t))}\times\frac{b(t)}{a_0(t)}\quad ,t\in I.$$

where $1\le r\le n$ and $W_r(h(t))$ is the determinant obtained from $W_{q,\omega}(h(t))$ by replacing the r th column by $(0,...,0,1)$. It follows that

$$c_r(t)=\int_\theta^t\frac{W_r(h(\tau))}{W_{q,\omega}(h(\tau))}\times\frac{b(\tau)}{a_0(\tau)}d_{q,\omega}\tau, r=1,\dots,n$$

Example 7.1. Consider the equation

$$D^2_{q,\omega}x(t)+z^2x(t)=b(t), \tag{7.8}$$

where $z\in\mathbb{C}\setminus\{0\}$. It is known that $\cos_{q,\omega}(z,.)$ and $\sin_{q,\omega}(z,.)$ are the solutions of the corresponding homogeneous equation of (8). We can easily show that

$$\psi_0(t)\quad=\frac{1}{z}(\sin_{q,\omega}(z,t)\int_\theta^t b(\tau)\mathrm{Cos}_{q,\omega}(z,h(\tau))d_{q,\omega}\tau-\cos_{q,\omega}(z,t)\int_\theta^t b(\tau)\mathrm{Sin}_{q,\omega}(z,h(\tau))d_{q,\omega}\tau).$$

It follows that every solution of Equation (7.8) has the form

$$\begin{aligned}\psi(t)\quad&=c_1\cos_{q,\omega}(z,t)+c_2\sin_{q,\omega}(z,t)\\&+\frac{1}{z}(\sin_{q,\omega}(z,t)\int_\theta^t b(\tau)\mathrm{Cos}_{q,\omega}(z,h(\tau))d_{q,\omega}\tau-\cos_{q,\omega}(z,t)\int_\theta^t b(\tau)\mathrm{Sin}_{q,\omega}(z,h(\tau))d_{q,\omega}\tau).\end{aligned}$$

7.2 Annihilator method

Sometimes, we use another method which is called annihilator method instead of the variation of parameter technique. We believe that, unlike the variation of parameters method, the annihilator method is usually easier to apply but it can not be applied in all cases.

Definition 2. We say that $f:I\to\mathbb{C}$ can be annihilated provided that we can find an operator of the form

$$L(D)=\alpha_nD^n_{q,\omega}+\alpha_{n-1}D^{n-1}_{q,\omega}+...+\alpha_0I\quad suchthat$$

$$L(D)f(t)=0, t\in I$$

where $\alpha_i, 0\le i\le n$ are constants, not all zero.

Example 3. Since

$$(D_{q,\omega}-5I)e_5(t)\quad=0,$$

$D_{q,\omega}-5I$ is an annihilator for $e_5(t)$.

The following table indicates a list of some functions and their annihilators.

function	annihilator
1	$D_{q,\omega}$
t	$D^2_{q,\omega}$
$e_p(t)$	$D_{q,\omega} - p(t)I$
$\cos_{q,\omega}(z,t)$	$D^2_{q,\omega} + z^2 I$
$\sin_{q,\omega}(z,t)$	$D^2_{q,\omega} + z^2 I$

We solve the following equation by using the annihilator method.

Example 7. 4. Consider the equation

$$D^2_{q,\omega}x(t) - 5D_{q,\omega}x(t) + 6x(t) = e_4(t). \tag{7.9}$$

Equation (7.9) can be rewritten in the form

$$(D_{q,\omega} - 3I)(D_{q,\omega} - 2I)x(t) = e_4(t).$$

Multiplying both sides by the annihilator $D_{q,\omega} - 4I$, we get that if $x(t)$ is a solution of (7.9) , then $x(t)$ satisfies

$$(D_{q,\omega} - 4I)(D_{q,\omega} - 3I)(D_{q,\omega} - 2I)x(t) = 0.$$

Hence,

$$x(t) = c_1 e_4(t) + c_2 e_3(t) + c_3 e_2(t).$$

One can see that $\psi_0(t) = (1/2)e_4(t)$. is a solution of Equation (7.9) . Therefore, the general solution of Equation (7.9) has the following form

$$x(t) = c_2 e_3(t) + c_3 e_2(t) + \frac{1}{2}e_4(t).$$

8. Conclusion and Perspectives

The aim of this paper is to establish the theory of linear Hahn difference equations and solve its corresponding first order with non-constant coefficient as well as Euler-Cauchy as a special case of the second order equations. However, there is a lot of work ahead of us. The most interesting work is to study the stability and the oscillation of linear and non-linear Hahn difference equations.

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