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New Two-step predictor-corrector method with ninth-order convergence for solving nonlinear equations

MOHAMED. S. M. BAHGAT[*,**]
[*]Department of mathematics, Faculty of Science
Minia University, Minia 61111 Egypt.
msmbahgat66@hotmail.com
M.A. Hafiz[**]
[**]Department of mathematics, Faculty of Arts and Sciences
Najran University, Najran 1988, Saudi Arabia.
admin@mhafiz.net

Abstract

In this paper, we suggest and analyze a new two-step predictor-corrector type iterative method for solving nonlinear equations of the type $f(x)=0$. This method based on a Halley and Householder iterative method and using predictor–corrector technique. The convergence analysis of our method is discussed. It is established that the new method has convergence order nine. Numerical tests show that the new methods are comparable with the well known existing methods and gives better results.

Mathematics Subject Classification: 41A25, 65H05, 65D99

Keywords: Nonlinear equations; Predictor-corrector; Convergence analysis; Iterative method; Efficiency index.

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1. Introduction

Finding iterative methods for solving nonlinear equations is an important area of research in numerical analysis at it has interesting applications in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x)=0$. Due to their importance, several numerical methods have been suggested and analyzed under certain condition. These numerical methods have been constructed using different techniques such as Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variational iteration method, and decomposition method. For more details, see [1-10]. In this paper, based on a Halley and Householder iterative method and using predictor–corrector technique, we construct modification of Newton's method with higher-order convergence for solving nonlinear equations. The error equations are given theoretically to show that the proposed technique has ninth-order convergence. Commonly in the literature the efficiency of an iterative method is measured by the *efficiency index* defined as $I \approx p^{1/d}$ [11], where p is the order of convergence and d is the total number of functional evaluations per step. Therefore this method has efficiency index $9^{1/6} \approx 1.442$ which is higher than $2^{1/2} \approx 1.4142$ of the Steffensen's method (SM) [12], $3^{1/4} \approx 1.3161$ of the DHM method [13]. Several examples are given to illustrate the efficiency and performance of this method.

2. Development of method and convergence analysis

For the sake completeness, we recall Newton, Halley, Traub, and Householder methods. These methods as follows:

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

It is well known that algorithm 2.1 has a quadratic convergence.

Algorithm 2.2. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}.$$

This is known as Halley's method and has cubic convergence [6].

Algorithm 2.3. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x)f''(x_n)}{2f'(x_n)^3}.$$

This is known as Householder's method and has cubic convergence [14]. Now using the technique of updating the solution, therefore, using Algorithm 2.2 as a predictor and Algorithm 2.3 as a corrector, we suggest and analyze a new two-step iterative method for solving the nonlinear equation, which is the main motivation of this paper.

Algorithm 2.4. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}.$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y)f''(y_n)}{2f'(y_n)^3}.$$

Algorithm 2.4 is called the predictor-corrector Halley's method (PCH) and has ninth-order convergence. Now we consider the convergence criteria of Algorithm 2. 4.

Theorem 2.1. Let r be a sample zero of sufficient differentiable function $f :\subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to r, then the two-step method defined by Algorithm 2. 4 has ninth-order convergence.

Proof: Consider to

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}. \tag{1}$$

$$\begin{aligned} x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y)f''(y_n)}{2f'(y_n)^3} \\ &= y_n - f(y_n)\left(\frac{2f'^2(y_n) + f(y)f''(y_n)}{2f'(y_n)^3}\right). \end{aligned} \tag{2}$$

Let r be a simple zero of f . Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about r , we get

$$f(x_n) = f(r) + (x_n - r)f'(r) + \frac{(x_n - r)^2}{2!}f^{(2)}(r) + \frac{(x_n - r)^3}{3!}f^{(3)}(r) + \frac{(x_n - r)^4}{4!}f^{(4)}(r) + \cdots,$$

then

$$f(x_n) = f'(r)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + \cdots], \tag{3}$$

and

$$f'(x_n) = f'(r)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + \cdots], \tag{4}$$

where $c_k = \dfrac{1}{k!}\dfrac{f^{(k)}(r)}{f'(r)}, k = 2, 3, \ldots$ and $e_n = x_n - r$.

From Equations (3) and (4), we have:

$$y_n = r + (c_2^2 - c_3)e_n^3 + (-3c_2^3 + 6c_2c_3 - 3c_4)e_n^4 + (148c_3c_2^2 + 5c_2^4 + 12c_2c_4 + 6c_3^2)e^5 + \cdots \tag{5}$$

The equation (5) can be written as the form

$$Y = (c_2^2 - c_3)e_n^3 + (-3c_2^3 + 6c_2c_3 - 3c_4)e_n^4 + (148c_3c_2^2 + 5c_2^4 + 12c_2c_4 + 6c_3^2)e^5 + \cdots, \tag{6}$$

where $Y = y_n - r$,

Now expanding $f(y_n)$, $f'(y_n)$, $f''(y_n)$ about r and using (5), we get

$$f(y_n) = f'(r)[Y + c_2Y^2 + c_3Y^3 + \cdots], \tag{7}$$

$$f'(y_n) = f'(r)[1 + 2c_2Y + 3c_3Y^2 + 4c_4Y^3 + \cdots], \tag{8}$$

and

$$f''(y_n) = f'(r)[2c_2 + 6c_3Y + 12c_4Y^2 + \cdots] \tag{9}$$

Combining (7) - (9), we have

$$2f'^2(y_n) + f(y_n)f''(y_n) = f'^2(r)[2 + 10c_2Y + (18c_3 + 10c_2^2)Y^2 + \cdots] \tag{10}$$

and

$$2f'^3(y_n) = f'^3(r)[2 + 12c_2Y + (18c_3 + 24c_2^2)Y^2 + \cdots] \tag{11}$$

From (10) and (11), we have

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$$
\begin{aligned}
x_{n+1} &= y_n - [Y + c_2 Y^2 + c_3 Y^3 + \cdots].[1 - c_2 Y - c_2^2 Y^2 + \cdots]. \\
&= y_n - [Y + (c_3 - 2c_2^2)Y^3 + \cdots]. \\
&= r + (2c_2^2 - c_3)(c_2^2 - c_3)^3 e_n^9 + \cdots
\end{aligned}
\tag{12}
$$

From (12), and $e_{n+1} = x_{n+1} - r$ finally, we have

$$e_{n+1} = (2c_2^2 - c_3)(c_3 - c_2^2)^3 e_n^9 + O(e^{10})$$

from which it follows that Algorithm 2.4 has ninth-order convergence. □

Remark. The order of convergence of the iterative method 2.4 is 9. Per iteration of the iterative method 2.4 requires two evaluations of the function, two evaluations of first derivative, and two evaluations of second derivative. We take into account the definition of efficiency index [11], if we suppose that all the evaluations have the same cost as function one, we have that the efficiency index of the method 2.3 is $9^{1/6} \approx 1.4422$ which is better $2^{1/2} \approx 1.4142$ of the Steffensen's method (SM) [12], $3^{1/4} \approx 1.3161$ of the DHM method [13].

3. Numerical examples

For comparisons, we have used the fourth-order Jarratt method [15] (JM) and Ostrowski's method (OM) [11] defined respectively by

$$
\begin{aligned}
y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= x_n - \left(1 - \frac{3}{2}\frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}
\end{aligned}
$$

and

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
x_{n+1} &= y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)}\frac{f(y_n)}{f'(x_n)}.
\end{aligned}
$$

We consider here some numerical examples to demonstrate the performance of the new modified iterative method, namely (PCH). We compare the classical Newton's method (NM), Jarratt method (JM), the Ostrowski's method (OM) and (PCH), in this paper. In the Tables 1, 2 the number of iteration is $n = 3$ for all our examples. But in Table 1 our examples are tested with precision $\varepsilon = 10^{-1000}$. The following stopping criteria is used for computer programs: $|x_{n+1} - x_n| + |f(x_{n+1})| < \varepsilon$. And the computational order of convergence (*COC*) can be approximated using the formula,

$$COC \approx \frac{\ln|(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln|(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}$$

Table 1 shows the difference of the root r and the approximation x_n to r, where r is the exact root computed with 2000 significant digits, but only 25 digits are displayed for x_n. In Table 2, we listed the number of iterations for various methods. The absolute values of the function $f(x_n)$ and the computational order of convergence (*COC*) are also shown in Tables 2, 3. All the computations are performed using Maple 15. The following examples are used for numerical testing:

$f_1(x) = x^3 + 4x^2 - 10, \quad x_0 = 1.$ $\qquad f_2(x) = \sin^2 x - x^2 + 1, \quad x_0 = 1.3.$

$f_3(x) = x^2 - e^x - 3x + 2, \quad x_0 = 2.$ $\qquad f_4(x) = \cos x - x, \quad x_0 = 1.7.$

$f_5(x) = (x-1)^3 - 1, \quad x_0 = 2.5.$ $\qquad f_6(x) = x^3 - 10, \quad x_0 = 2.$

$f_7(x) = e^{x^2 + 7x - 30} - 1, \quad x_0 = 3.1.$

Table 1. Comparison of Number of iterations for various methods required such that $|f(x_{n+1})| < 10^{-200}$.

Method	f_1	f_2	f_3	f_4	f_5	f_6	f_7
Guess	1	1.3	2	1.7	2.5	2	3.1
NM	12	11	12	11	13	11	13
JM	7	6	7	7	7	6	7
OM	7	6	7	6	7	6	7
PCH	5	5	5	5	5	5	5

Results are summarized in Table 1, 2 and Table 3 as it shows, new algorithm is comparable with all of the methods and in most cases gives better or equal results.

Table 2. Comparison of different methods

Method	x_0	x_3	*COC*	$\|x_3 - x_2\|$	$\|f(x_3)\|$
f_1	1				
NM		1.3652366002021159462369662	1.88	3.66E-03	1.09E-04
JM		1.3652300134140968457610286	4.10	4.50E-12	5.95E-46
OM		1.3652300134140968457610286	4.10	4.50E-12	5.95E-46
PCH		1.3652300134140968457610286	9.09	7.12E-58	1.90E-516
f_2	1.3				
NM		1.4044916527111965739297374	1.98	7.57E-05	1.12E-08
JM		1.4044916482153412260350868	4.03	5.09E-18	6.61E-70
OM		1.4044916482153412260350868	4.03	5.96E-18	1.29E-69
PCH		1.4044916482153412260350868	9.02	1.69E-86	4.72E-773
f_3	2				
NM		0.2575292578013089584442857	7.68	3.31E-03	3.88E-06
JM		0.2575302854398607604553673	4.35	6.21E-06	3.44E-23
OM		0.2575302854398607604553673	4.55	8.79E-06	1.02E-22
PCH		0.2575302854398607604553673	8.76	1.39E-44	3.19E-400
f_4	1.7				
NM		0.7390851658032147634513238	1.53	3.84E-04	5.45E-08
JM		0.7390851332151606416553121	3.66	1.47E-12	1.85E-49
OM		0.7390851332151606416553121	3.67	3.34E-12	5.32E-48
PCH		0.7390851332151606416553121	8.87	4.78E-43	5.55E-385
f_5	2.5				
NM		2.0003266792741527249601052	1.98	1.80E-02	9.80E-04
JM		2	3.73	2.55E-08	8.43E-31
OM		2	3.73	2.55E-08	8.43E-31
PCH		2	8.73	3.79E-37	2.39E-328

4. Conclusions

In this paper, we have suggested and analyzed a new higher-order iterative method for solving nonlinear equations. This method based on a Halley and Householder iterative method and using predictor–corrector technique. The error equations are given theoretically to show that the proposed technique has ninth-order convergence. The new method attain efficiency index of 1.442, which makes it competitive. In addition, the proposed method has been tested on a series of examples published in the literature and show good results when compared it with the previous literature

Table 3. Comparison of different methods

Method	x_0	x_3	*COC*	$\lvert x_3 - x_2 \rvert$	$\lvert f(x_3) \rvert$
f_6	2				
NM		2.1544346922369133091005011	1.97	6.89E-05	3.07E-08
JM		2.1544346900318837217592936	4.02	2.71E-19	4.98E-75
OM		2.1544346900318837217592936	4.02	2.71E-19	4.98E-75
PCH		2.1544346900318837217592936	9.01	6.47E-95	2.92E-850
f_7	3.1				
NM		3.0007511637578020952127918	2.24	1.02E-02	9.81E-03
JM		3	3.91	1.46E-07	6.17E-25
OM		3	3.92	9.81E-08	1.12E-25
PCH		3	8.92	9.96E-32	2.01E-293

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