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Blow-up of weak solutions for nonlinear hyperbolic equations with variable exponents

Yunzhu Gao
Department of Mathematics and Statistics, Beihua University,
Jilin City, PR China
yzgao_2008@163.com

ABSTRACT

This work is concerned with a viscoelastic equation with strongly damping and variable exponents. The existence of weak solutions is established to the initial and boundary value problem under suitable assumptions by using the Faedo-Galerkin method and embedding theory.

Keywords：

Existence; Wave equations; Strongly damping; Variable exponents.

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Introduction

Let $\Omega \subset R^N\ (N \geq 2)$ be a bounded Lipschitz domain and $0 < T < \infty$. Consider the following nonlinear strongly damping viscoelastic wave problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_{tt} - \alpha \Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = a(x,t)|u|^{p(x)-2}u,\ (x,t) \in Q_T, \\ u(x,t) = 0,\ (x,t) \in S_T, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1,\ x \in \Omega. \end{cases} \quad , \tag{1}$$

Where α, β are positive parameters, $Q_T = \Omega \times (0,T]$, S_T denotes the lateral boundary of the cylinder Q_T.

It will be assumed throughout the paper that the coefficient $a(x,t)$ is measurable and the exponent $p(x)$ is continuous in Ω with logarithmic module of continuity:

$$0 < a^- = \inf_{(x,t)\in Q_T} a(x,t) \leq a(x,t) \leq a^+ = \sup_{(x,t)\in Q_T} a(x,t) < \infty, \tag{2}$$

$$1 < p^- = \inf_{x\in\Omega} p(x) \leq p(x) \leq p^+ = \sup_{x\in\Omega} p(x) < \infty, \tag{3}$$

$$\forall\, z, \zeta \in \Omega, |z-\zeta| < 1, |p(z) - p(\zeta)| \leq \omega(|z-\zeta|), \tag{4}$$

where

$$\limsup_{(x,t)\in Q_T} \omega(\tau) \ln \frac{1}{\tau} = C < +\infty .$$

And we also assume

(H1) $g: R_+ \to R_+$ is C^1 function and satisfies

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = l > 0 ,$$

(H2) there exists $\eta > 0$ such that

$$g'(t) < \eta\, g(t), \quad t \geq 0 .$$

There have been many results about the existence and blow-up properties of the solutions when p is constant and $a(x,t) = 1$. We refer the readers to the bibliography given in [1,2.3.4.5,6].

In recent years, much attention has been paid to the study of mathematical models of electro-rheological fluids. These models include hyperbolic and parabolic equations or systems which are nonlinear with respect to gradient of the thought solution and with variable exponents of nonlinearity. See [7,8,9,10] and references therein. Besides, another important application is the image processing where the anisotropy and nonlinearity of the diffusion operator and convection terms are used to underline the borders of the distorted image and to eliminate the noise[11,12].

To the best of our knowledge, there are only a few works about viscoelastic hyperbolic equations with variable exponents of nonlinearity. In [13], The authors studied the finite time blow-up a of solutions for viscoelastic hyperbolic equations and in [1], the authors discussed only the viscoelastic hyperbolic problem with constant exponents. Motivated by the works of [1,13], we shall study the existence of the solutions to Problem (1) and state some properties to the solutions.

The outline of this paper is the following: In Section 2, we shall introduce the function spaces of $\rm{Orlicz-Sobolev}$ type, give the definition of the weak solution to the problem and prove the existence of weak solutions for Problem (1.1).

Existence of weak solutions

In this section, the existence of weak solutions will be studied. Firstly, we introduce some Banach spaces

$$L^{p(x)}(\Omega) = \{u(x) : u \text{ is measurable in } \Omega, \quad A_{p(\cdot)}(u) = \int_\Omega |u|^{p(x)} dx < \infty\},$$

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0,\ A_{p(\cdot)}(u/\lambda) \le 1\}.$$

Lemma 2.1. [14] For $u \in L^{p(x)}(\Omega)$, the following relations hold:

(1) $\|u\|_{p(\cdot)} < 1 (=1; >1) \Leftrightarrow A_{p(\cdot)}(u) < 1 (=1; >1)$;

(2) $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \le A_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^-}; \|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \le A_{p(\cdot)}(u) \le \|u\|_{p(\cdot)}^{p^+}$;

(3) $\|u\|_{p(\cdot)} \to 0 \Leftrightarrow A_{p(\cdot)} \to 0; \|u\|_{p(\cdot)} \to \infty \Leftrightarrow A_{p(\cdot)} \to \infty$.

Lemma 2.2. [15,16] For \$ $u \in w_0^{1,p(\cdot)}(\Omega)$, if p satisfies the condition (3), the \$ $p(\cdot) - Poincare'$ inequality

$$\|u\|_{p(x)} \le C\|\nabla u\|_{p(x)}$$

holds, where the positive constant C depends on p and Ω.

Remark 2.1. Note that the following inequality

$$\int_\Omega |u|^{p(x)} dx \le C\int_\Omega |\nabla u|^{p(x)} dx$$

does not in general hold.

Lemma 2.3. [17] Let Ω be an open domain (that may be unbounded) in R^N with cone property. If $p(x): \overline{\Omega} \to R$ is a Lipschitz continuous function satisfying $1 < p^- \le p^+ < \frac{N}{k}$ and $r(x): \overline{\Omega} \to R$ is measurable and satisfies

$$p(x) \le r(x) \le p^*(x) = \frac{Np(x)}{N - kp(x)}, \quad a.e. \quad x \in \overline{\Omega},$$

then there is a continuous embedding $W^{k,p(x)}(\Omega) \to L^{r(x)}(\Omega)$.

The main result in this section is the following theorem.

Theorem 2.1. Let $u_0, u_1 \in H_0^1(\Omega)$, (H1)-(H2) hold, the exponents $a(x,t), p(x)$ satisfy Conditions （2）-(4) and $a_t(x,t) \ge 0, p^- > 2$ such that $p^+ \le \frac{2(n-1)}{\text{n - 2}}, n \ge 3$. Then Problem （1）\$ has at least one weak solution $u: \Omega \times (0,\infty) \to R$ in the class

$$u \in L^\infty(0,\infty; H_0^1(\Omega)), u' \in L^\infty(0,\infty; H_0^1(\Omega)), u'' \in L^2(0,\infty; H_0^1(\Omega)).$$

Proof: Let $\{w_j\}_{j=1}^\infty$ be an orthogonal basis of $H_0^1(\Omega)$ with w_j

$$-\Delta w_j = \lambda_j w_j, x \in \Omega, \ w_j = 0, x \in \partial\Omega.$$

$V_k = span\{w_1, \cdots, w_k\}$ is the subspace generated by the first k vectors of the basis $\{w_j\}_{j=1}^\infty$. By normalization we have $\|w_j\|_2 = 1$. For $\Phi \in V_k$, let us define the operator

$$\langle Lu, \Phi\rangle = \int_\Omega [u_{tt}\Phi + \nabla u\nabla\Phi + \nabla u_{tt}\nabla\Phi + \alpha\nabla u_t\nabla\Phi - \int_0^t g(t-\tau)\nabla u(\tau)\nabla\Phi d\tau - a(x,t)|u|^{p(x)-2}u\Phi]dx.$$

For any given integer k, we consider the approximate solution

$$u_k = \sum_{i=1}^{k} c_i^k(t) w_i \,,$$

which satisfies

$$\begin{cases} \langle Lu_k, w_i \rangle = 0, i = 1,2,\cdots k, \\ u_k(0) = u_{0k}, u_{kt}(0) = u_{1k}, \end{cases} \tag{5}$$

here $u_{0k} = \sum_{i=1}^{k}(u_0, w_i)w_i$, $u_{1k} = \sum_{i=1}^{k}(u_1, w_i)w_i$ and $u_{0k} \to u_0$, $u_{1k} \to u_1$ in $H_0^1(\Omega)$. Here we denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$.

Problem (1) generates the system of k ordinary differential equations

$$\begin{cases} (c_i^k(t))'' = -\lambda_i c_i^k(t) - \lambda_i (c_i^k(t))'' - \lambda_i \alpha (c_i^k(t))' + \lambda_i \int_0^t g(t-\tau) c_i^k(\tau) d\tau \\ \qquad - a(x,t) | (\sum_{i=1}^{k} c_i^k(t), w_i) |^{p(x)-2} (\sum_{i=1}^{k} c_i^k(t), w_i), \\ c_i^k(0) = (u_0, w_{i,}), (c_i^k(0))' = (u_1, w_{i,}), i = 1,2,\cdots k. \end{cases} \tag{6}$$

By the standard theory of the ODE system, we infer that problem (6) admits a unique solution $c_i^k(t)$ in $[0, t_k]$, where $t_k > 0$. Then we can obtain an approximate solution $u_k(t)$ for (1), in V_k, over $[0, t_k)$. And the solution can be extended to $[0,T]$, for any given $T > 0$, by the estimate below. Multiplying (6) $(c_i^k(t))'$ and summing with respect to i we conclude that

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 - \int_0^t g(t-\tau)\int_\Omega (\nabla u_k(\tau) \nabla u_k'(t)) dx d\tau - \int_\Omega a(x,t)\frac{1}{p(x)}|u_k|^{p(x)} dx\Big) \\ &\qquad + \alpha \int_\Omega |\nabla u_k'(\tau)|^2 dx + \int_\Omega a_t(x,t) \frac{1}{p(x)} |u_k|^{p(x)} dx = 0. \end{aligned} \tag{7}$$

By simple calculation, we have

$$\begin{aligned} &-\int_0^t g(t-\tau)\int_\Omega (\nabla u_k(\tau) \nabla u_k'(t)) dx d\tau \\ &= \frac{1}{2}\frac{d}{dt}((g \circ \nabla u_k)(t)) - \frac{1}{2}(g' \circ \nabla u_k)(t) - \frac{1}{2}\frac{d}{dt}\int_0^t g(s) ds \|\nabla u_k\|_2^2 + \frac{1}{2} g(t) \|\nabla u_k\|_2^2, \end{aligned} \tag{8}$$

here

$$(\varphi \circ \psi)(t) = \int_0^t \varphi(t-\tau) \|\nabla \psi(t) - \nabla \psi(\tau)\|^2 d\tau \;.$$

Combining (7)-(8) and (H1)-(H2), we get

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}\big(1 - \int_0^t g(s) ds\big)\|\nabla u_k\|_2^2 + \frac{1}{2}(g \circ \nabla u_k)(t) - \int_\Omega a(x,t)\frac{1}{p(x)}|u_k|^{p(x)} dx\Big) \\ &\quad = \frac{1}{2}(g' \circ \nabla u_k)(t) - \frac{1}{2} g(t)\|\nabla u_k\|_2^2 - \alpha \int_\Omega |\nabla u_k'(\tau)|^2 dx - \int_\Omega a_t(x,t)\frac{1}{p(x)}|u_k|^{p(x)} dx \le 0. \end{aligned} \tag{9}$$

Integrating (9) over $(0,t)$, and using assumptions (2)-(4), we have

$$\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}\big(1 - \int_0^t g(s) ds\big)\|\nabla u_k\|_2^2 + \frac{1}{2}(g \circ \nabla u_k)(t) - \int_\Omega a(x,t)\frac{1}{p(x)}|u_k|^{p(x)} dx \le C1,$$

where $C1$ is a positive constant depending only on $\|u_0\|_{H_0^1(\Omega)}, \|u_1\|_{H_0^1(\Omega)}$.

Hence, by Lemma 2.1, we also have

$$\frac{1}{2}\|u_k'\|_2^2 + \frac{1}{2}\|\nabla u_k'\|_2^2 + \frac{1}{2}(1-\int_0^t g(s)ds)\|\nabla u_k\|_2^2 + \frac{1}{2}(g \circ \nabla u_k)(t) - \max\{\frac{a^+}{p^-}\|u_k\|_{p(x)}^{p^-}, \frac{a^+}{p^-}\|u_k\|_{p(x)}^{p^+}\} \le C1. \tag{10}$$

In view of (H1)-(H2), we also have

$$\|u_k'\|_2^2 + \|\nabla u_k'\|_2^2 + \|\nabla u_k\|_2^2 + (g \circ \nabla u_k)(t) \le C2, \tag{11}$$

where $C2$ is a positive constant depending only on $\|u_0\|_{H_0^1(\Omega)}, \|u_1\|_{H_0^1(\Omega)}, l, p^-, p^+$. It follows from (11) that

$$u_k \text{ is uniformly bounded in } L^\infty(0,\infty; H_0^1(\Omega)), \tag{12}$$

$$u_k' \text{ is uniformly bounded in } L^\infty(0,\infty; H_0^1(\Omega)). \tag{13}$$

Next, multiplying (1) by $(c_i^k(t))''$ and then summing with respect to i, we get the following holds:

$$\begin{aligned}&\int_\Omega |u_k''|_2^2 dx + \|\nabla u_k''\|_2^2 + \alpha\frac{d}{dt}(\frac{1}{2}\|\nabla u_k'\|_2^2)\\ &= -\int_\Omega \nabla u_k(t)\nabla u_k'' dx + \int_0^t g(t-\tau)\int_\Omega \nabla u_k(\tau)\nabla u_k''(t)dxd\tau + \int_\Omega a(x,t)|u_k|^{p(x)-2}u_k u_k'' dx.\end{aligned} \tag{14}$$

Note that

$$\left|-\int_\Omega \nabla u_k(t)\nabla u_k'' dx\right| \le \varepsilon\|\nabla u_k''(t)\|_2^2 + \frac{1}{4\varepsilon}\|\nabla u_k(t)\|_2^2, \quad \varepsilon > 0. \tag{15}$$

$$\begin{aligned}&\left|\int_0^t g(t-\tau)\int_\Omega \nabla u_k(\tau)\nabla u_k''(t)dxd\tau\right| \le \varepsilon\|\nabla u_k''(t)\|_2^2 + \frac{1}{4\varepsilon}\int_\Omega(\int_0^t g(t-\tau)\nabla u_k(\tau)d\tau)^2 dx\\ &\le \varepsilon\|\nabla u_k''(t)\|_2^2 + \frac{1}{4\varepsilon}\int_0^t g(s)ds\int_0^t g(t-\tau)\int_\Omega|\nabla u_k(\tau)|^2 dxd\tau \le \varepsilon\|\nabla u_k''(t)\|_2^2 + \frac{(1-l)g(0)}{4\varepsilon}\int_0^t\|\nabla u_k(t)\|_2^2 d\tau.\end{aligned} \tag{16}$$

$$\left|\int_\Omega a(x,t)|u_k|^{p(x)-2}u_k u_k'' dx\right| \le a^+\varepsilon\|u_k''\|_2^2 + \frac{a^+}{4\varepsilon}\left\||u_k|^{p(x)-2}u_k\right\|_2^2 \le a^+\varepsilon\|u_k''\|_2^2 + \frac{a^+}{4\varepsilon}\int_\Omega(|u_k|^{p(x)-2}u_k)^2 dx. \tag{17}$$

From Lemma 2.2, we have

$$\|u_k''\|_2^2 \le C^2\|\nabla u_k''\|_2^2. \tag{18}$$

$$\begin{aligned}&\int_\Omega(|u_k|^{p(x)-2}u_k)^2 dx = \int_\Omega(|u_k|^{2(p(x)-1)} dx \le \max\{\int_\Omega(|u_k|^{2(p^--1)} dx, \int_\Omega(|u_k|^{2(p^+-1)} dx\}\\ &\le \max\{C^{*\frac{1}{2(p^--1)}}\|\nabla u_k\|^{\frac{2}{2(p^--1)}}, C^{*\frac{1}{2(p^+-1)}}\|\nabla u_k\|^{\frac{2}{2(p^+-1)}}\},\end{aligned} \tag{19}$$

where C, C^* are embedding constants. From (14)-(19), we obtain that

$$\begin{aligned}&\int_\Omega|u_k''|_2^2 dx + (1-2\varepsilon-a^+\varepsilon C^2)\|\nabla u_k''\|_2^2 + \alpha\frac{d}{dt}(\frac{1}{2}\|\nabla u_k'\|_2^2)\\ &\le \frac{1}{4\varepsilon}\|\nabla u_k(t)\|_2^2 + \frac{(1-l)g(0)}{4\varepsilon}\int_0^t\|\nabla u_k(t)\|_2^2 d\tau + \max\{C^{*\frac{1}{2(p^--1)}}\|\nabla u_k\|^{\frac{2}{2(p^--1)}}, C^{*\frac{1}{2(p^+-1)}}\|\nabla u_k\|^{\frac{2}{2(p^+-1)}}\}.\end{aligned} \tag{20}$$

Integrating (20) over $(0,t)$ and using (11), Lemma 2.3, we get

$$\int_0^t \|u_k''\|_2^2 d\tau + (1-2\varepsilon - a^+\varepsilon C^2)\int_0^t \|\nabla u_k''\|_2^2 d\tau + \frac{\alpha}{2}\|\nabla u_k'\|_2^2 \le \frac{1}{4\varepsilon}(C2 + (1-l)g(0)T) + C3, \qquad (21)$$

where $C3$ is a positive constant depending only on $\|u_1\|_{H_0^1(\Omega)}$.

Taking ε small enough in (21), we obtain the estimate

$$\int_0^t \|u_k''\|_2^2 d\tau + \frac{\alpha}{2}\|\nabla u_k'\|_2^2 \le C4, \qquad (22)$$

where $C4$ is a positive constant depending only on $\|u_0\|_{H_0^1(\Omega)}, \|u_1\|_{H_0^1(\Omega)}, l, g(0), T$.

From estimate (22), we get

$$u_k'' \text{ is uniformly bounded in } L^2(0,T;H_0^1(\Omega)). \qquad (23)$$

By (12)-(13) and (23), we infer that there exist a subsequence u_i of u_k and a function u such that

$$u_i \to u \text{ weakly star in } L^\infty(0,T;H_0^1(\Omega)), \qquad (24)$$

$$u_i \to u \text{ weakly in } L^{p^-}(0,T;W^{1,p(x)}(\Omega)), \qquad (25)$$

$$u_i' \to u' \text{ weakly star in } L^\infty(0,T;H_0^1(\Omega)), \qquad (26)$$

$$u_i'' \to u'' \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \qquad (27)$$

Next, we will deal with the nonlinear term. From the Aubin-Lions theorem, see Lions [18] (pp. 57-58), it follows from (26) and (27) that there exists a subsequence of u_i, still represented by the same notation, such that

$$u_i' \to u' \text{ strongly in } L^2(0,T;L^2(\Omega)),$$

which implies $u_i' \to u'$ almost everywhere in $\Omega\times(0,T)$. Hence, by (24)-(27)

$$|u_i|^{p(x)-2}u_i \to |u|^{p(x)-2}u \text{ weakly in } \Omega\times(0,T). \qquad (28)$$

Multiplying (6) by $\phi(t)\in \ell(0,T)$ (which $\ell(0,T)$ is the space of C^∞ function with compact support in $(0,T)$ and integrating the obtained result over $(0,T)$, we obtain that

$$\langle Lu_k, w_i\phi(t)\rangle = 0, \quad i=1,2,\cdots k. \qquad (29)$$

Note that $\{w_i\}_{i=1}^\infty$ is a basis of $H_0^1(\Omega)$. Convergence (24)-(28) is sufficient to pass to the limit in (29) in order to get

$$u_{tt} - \Delta u - \Delta u_{tt} - \alpha\Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = a(x,t)|u|^{p(x)-2}u, u\in L^2(0,T;H^{-1}(\Omega))$$

for arbitrary $T>0$. In view of (24)-(27) and Lemma 3.3.17 in [19], we obtain

$$u_k(0)\to u(0) \text{ weakly in } H_0^1(\Omega), \quad u_k'(0)\to u'(0) \text{ weakly in } H_0^1(\Omega).$$

Hence, we get $u(0)\to u_0, u_1(0)\to u_1$. Then, the existence of weak solutions is established.

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