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Generalized closed sets in ditopological texture spaces with application in rough set theory

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ABSTRACT

In this paper, the counterparts of generalized open (g -open) and generalized closed (g -closed) sets for ditopological texture spaces are introduced and some of their characterizations are obtained. Some characterizations are presented for generalized bicontinuous difunctions. Also, we introduce new notions of compactness and stability in ditopological texture spaces based on the notion of g -open and g -closed sets. Finally, as an application of g -open and g -closed sets, we generalize the subsystem based definition of rough set theory by using new subsystem, called generalized open-sets to define new types of lower and upper approximation operators, called g -lower and g -upper approximations. These decrease the upper approximation and increase the lower approximation and hence increase the accuracy. Properties of these approximations are studied. An example of multi-valued information systems are given.

Keywords: Ditopological texture space; g -open and closed set; g -continuity and cocontinuity; g -compactness and cocompactness; g -stability and costability; g -lower and upper approximations.

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1. Introduction

The study and research about near open and near closed sets have specific importance, it helps in the modifications of topological spaces via adding new concepts and facts or constructing new classes. Closed sets are fundamental objects in a topological space. For example, one can define the topology on a set by using either the axioms for the closed sets or the Kuratowski closure axioms. One productive area of research in general topology, which has applications to several branches of science, is the investigation of various types of generalized open and generalized closed sets, and the study of their structural properties. In 1970, Levine[1] introduced g-closed sets in topological spaces as a generalization of closed sets.

The notion of texture space was firstly introduced by L. M. Brown in [2, 3] under the name of fuzzy structure, and then it was called as texture space by L. M. Brown and R. Ertürk in [4, 5]. Ditopological texture spaces were introduced by L. M. Brown as a point - based setting for the study fuzzy sets, and this line of investigation continues, see for example [6-9]. It is well known that the concept of ditopology is more general than general topology, fuzzy topology and bitopology. So, it will be more advantage to generalize some various general (fuzzy, bi)-topological concepts to the ditopological texture spaces. An adequate introduction to the theory of texture spaces and ditopological texture spaces, and the motivation for its study may be obtained from [10-14]. On the other hand, textures offer a convenient setting for the investigation of complement-free concept in general, so much of the recent work has proceeded in dependently of the fuzzy setting.

Pawlak is credited with creating the “rough set theory” [15], a mathematical tool for dealing with vagueness or uncertainty. Since 1982, the theory and applications of rough set have impressively developed. There are many applications of rough set theory especially in data analysis, artificial intelligence, and cognitive sciences [16-18]. Rough set theory [19-22] is an extension of set theory in which a subset of a universe is described by a pair of ordinary sets called the lower and upper approximation. Yao [23] classified broadly methods for the development of rough set theory into two classes, namely, the constructive and axiomatic (algebraic) approaches. The main idea of rough sets corresponds to the lower and upper approximations. Pawlak’s definitions for lower and upper approximations were originally introduced with reference to an equivalence relation. However, the equivalence relation appears to be a stringent condition that may limit the applicability of Pawlak’s rough set model. Many extensions have been made in recent years by replacing equivalence relation or partition by notions such as binary relations [24-26], neighborhood systems, and Boolean algebras [25-29].

The theory of rough sets can be generalized in several directions. Within the set theoretic framework, generalizations of the element based definition can be obtained by using non-equivalence binary relations [23, 31, 32-34], generalizations of the granule based definition can be obtained by using coverings [26, 32, 35-37], and generalizations of subsystem based definition can be obtained by using other subsystems [38, 39]. In the standard rough set model, the same subsystem is used to define lower and upper approximation operators. When generalizing the subsystem based definition, one may use two subsystems, one for the lower approximation operator and the other for the upper approximation operator.

In this paper, we introduce and study the concepts of g-closed sets and g-open sets in ditopological textures spaces, which is the extension of the concept closed and open sets. We also, introduce and study the concepts of generalized continuity, generalized compactness and generalized stability in ditopological textures spaces. In the last section, we used the new subsystem, called generalized open-sets to define new types of lower and upper approximation operators, called g-lower approximation and g-upper approximation. These decrease the upper approximation and increase the lower approximation and hence increase the accuracy. Properties of these approximations are studied. Also, we defined the concept of rough membership function using g open sets. It is a generalization of classical rough membership function of Pawlak rough sets. Finally, we give an example from a multi-valued information system showing that the accuracy increased by using the generalized lower and upper approximations. This research not only can form the theoretical basis for further applications of topology on soft sets but also lead to the development of information systems.

2. Preliminaries

In this section, we present the basic definitions and results of ditopological texture space which may found in earlier studies[7, 8, 15].

Definition 1. If S is a set, a texturing φ of S is a subset of $P(S)$ which is a point-separating, complete, completely distributive lattice containing S and ϕ and for which meet coincides with intersection and finite joins with union. The pair (S,φ) is then called a texture. For a texture (S,φ), most properties are conveniently defined in terms of the P-sets and Q-sets

$$P_s = \bigcap\{A \in \varphi : s \in A\}, Q_s = \bigvee\{A \in \varphi : s \notin A\}$$

Since a texturing φ need not be closed under the operation of taking the set complement, the notion of topology is replaced by that of dichotomous topology or ditopology, namely

Definition 2. Let (S,φ) be a texture. A pair (τ,k) of subsets of φ is called a ditopology on (S,φ) iff

$\phi, S \in \tau$

$$G_1, G_2 \in \tau \Rightarrow G_1 \bigcap G_2 \in \tau$$ and

$$G_i \in \tau \bigcap \forall i \in I \Rightarrow \bigvee_i G_i \in \tau$$

and the set of closed sets k satisfies

$$\phi, S \in k$$

$$F_1, F_2 \in k \Rightarrow F_1 \bigcup F_2 \in k$$

$$F_i \in k \quad \forall i \in I \Rightarrow \bigcap_i F_i \in k$$

The elements of τ are called open and those of k are called closed. We refer to τ as the topology and to k as the cotopology of (τ , k).

Hence, a ditopology is essentially a topology which there is no a priori relation between the open and closed sets. For $A \in \varphi$, we define the closure $[A]$ and the interior $]A[$ of A under (τ, k) by the equalities

$$[A] = \bigcap \{F \in k : A \subset F\} \quad and \quad]A[= \bigcup \{G \in \tau : G \subset A\}$$

Definition 3. Let (S, φ) be a texture. A mapping $\sigma : \varphi \to \varphi$ satisfying $A \subset B \Rightarrow \sigma(B) \subset \sigma(A)$ and $\sigma\sigma(A) = A \forall A, B \in \phi$ is called complementation on (S, φ) if τ and k are related by $k = \sigma(\tau)$, we say (τ, k) is a complemented ditopology on (S, φ, σ). In this case $\sigma([A]) =]\sigma(A)[$ and $\sigma(]A[= [\sigma(A)]$.

Example 1.

For any texture (S, φ), a ditopology (τ, k) with $\tau = \varphi$ is called discrete, and one with $k = \varphi$ is called co-discrete.

For any texture (S, φ), a ditopology (τ, k) with $\tau = \{S, \phi\}$ is called indiscrete, and one with $k = \{S, \phi\}$ is called co-indiscrete.

For any topology τ on S, (τ, τ'), $\tau' = \{S - G : G \in \tau\}$, is a complemented ditopology on the usual(crisp) set structure $(S, P(S), \sigma_S)$ of S, where $\sigma_S : P(S) \to P(S)$ defined by $\sigma_S(A) = A'$ where $A' = S - A$ $\forall A \in P(S)$.

For any bitopology (τ_1, τ_2) on S, (τ_1, τ_2') is a ditopology on $(S, P(S))$

Let (S_1, φ_1) and (S_2, φ_2) be textures. In the following definition we consider the product texture $\wp(S_1) \otimes \varphi_2$, and denote by $\overline{P_{s,t}}$, $\overline{Q_{s,t}}$ Then, respectively the $p-sets$ and $q-sets$ for the product texture $(S_1 \times \varphi_2, \wp(S_1) \otimes \varphi_2)$

Definition 4. Let (S_1, φ_1) and (S_2, φ_2) be textures. Then

$r \in \wp(S_1) \otimes \varphi_2$ is called a relation from (S_1, φ_1) to (S_2, φ_2) if it satisfies

$$r \not\subset \overline{Q_{s,t}}, P_{s'} \not\subset Q_s \Rightarrow r \not\subset \overline{Q_{s',t}}.$$

$$r \not\subset \overline{Q_{s,t}} \Rightarrow \exists s' \in S \quad such \quad that \quad P_s \not\subset Q_{s'} \quad and \quad r \not\subset \overline{Q_{s',t}}$$

$R \in \wp(S_1) \otimes \varphi_2$ is called a corelation from (S_1, φ_1) to (S_2, φ_2) if it satisfies

$$\overline{P_{s,t}} \not\subset R, \ P_S \not\subset Q_{s'} \Rightarrow \overline{P_{s',t}} \not\subset R$$

$$\overline{P_{s,t}} \not\subset R \Rightarrow \exists s' \in S \quad such \ that \quad P_{s'} \not\subset Q_s \text{ and } \overline{P_{s',t}} \not\subset R$$

A pair (r, R), where r is a relation and R a corelation from (S_1, φ_1) to (S_2, φ_2) is called a direlation from (S_1, φ_1) to

(S_2,φ_2).

One of the most useful notions of (ditopological) texture spaces is that of difunction. A difunction is a special of direlation.

Definition 5. Let (f,F) be a direlation from (S_1,φ_1) to (S_2,φ_2). Then (f,F) is called a difunction from (S_1,φ_1) to (S_2,φ_2) if it satisfies the following two conditions.

For $s,s'\in S$, $P_s \not\subseteq Q_{s'} \Rightarrow \exists t\in S_2 \quad such \quad that \quad f\not\subseteq \overline{Q_{s,t}}$ and $\overline{P_{s',t}}\not\subseteq F$.

For $t,t'\in S_2$ and $s\in S_1$, $f\not\subseteq \overline{Q_{s,t}}$ and $\overline{P_{s,t'}}\not\subseteq F \Rightarrow P_{t'}\not\subseteq Q_t$

Theorem 1. For a direlation (r,R) from (S_1,φ_1) to (S_2,φ_2) the following are equivalent

(r,R) is a difunction

The following inclusions hold:

$f^{\leftarrow}(F^{\rightarrow}A)\subseteq A\subseteq F^{\leftarrow}(f^{\rightarrow}A)$ for each $A\in\varphi_1$, and

$f^{\rightarrow}(F^{\leftarrow}B)\subseteq B\subseteq F^{\rightarrow}(f^{\leftarrow}B)$ for each $B\in\varphi_2$.

$f^{\leftarrow}B = F^{\leftarrow}B$ for each $B\in\varphi_2$.

Definition 6. Let $(f,F):(S_1,\varphi_1)\to(S_2,\varphi_2)$ be a difunction

For $A\in\varphi_1$, the image $f^{\rightarrow}A$ and the co-image $F^{\rightarrow}A$ are defined by

$$f^{\rightarrow}A = \cap\{Q_t : \forall sf \not\subset \overline{Q_{s,t}} \Rightarrow A\subseteq Q_s\}$$

$$F^{\rightarrow}A = \vee\{P_t : \forall s\overline{P_{s,t}} \not\subset F \Rightarrow P_s \subseteq A\}$$

For $B\in\varphi_2$, the inverse image $f^{\rightarrow}B$ and the inverse co-image $F^{\rightarrow}B$ are defined by

$$f^{\leftarrow}B = \vee\{P_t : \forall f \not\subset \overline{Q_{s,t}} \Rightarrow P_t \subset B\}$$

$$F^{\leftarrow}(B) = \cap\{Q_s : \forall t, \overline{P_{s,t}} \not\subset F \Rightarrow B\subseteq Q$$

Definition 7. The difunction $(f,F):(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ is called.

continuous if $B\in\tau_2 \Rightarrow f^{\rightarrow}(B)\in\tau_1$

co-continuous if $B\in k_2 \Rightarrow f^{\rightarrow}(B)\in k_1$ and it is bicontinuous if it is continuous and cocontinuous.

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K=(S,R)$, where S is a set called the universe and R is an equivalence relation [15]. The equivalence classes of R are also known as the granules, elementary sets, or blocks; we will use $R_s\subseteq S$ to denote the equivalence class containing $s\in S$. In the approximation space, we consider two operators

$$\underline{R}(A) = \{s\in S : R_s \subseteq S\},$$

$$\overline{R}(A) = \{s\in S : R_s \cap S \neq \phi\}$$

called the lower approximation and upper approximation of $A\subseteq S$, respectively. Also let $POS_R(A)=\underline{R}(A)$ denote the positive region of A, $NEG_R(A) = S-\overline{R}(A)$ denote the negative region of A and $BN_R(A)=\overline{R}(A)-\underline{R}(A)$ denote the borderline region of A.

Let S be a finite nonempty universe, $A \subseteq S$, the degree of completeness can also be characterized by the accuracy measure as follows:

$$\alpha(A) = \frac{card\,\underline{R}(A)}{card\,\overline{R}(A)}$$

3. Generalized closed and open sets in a ditopological texture spaces

In the following, we introduce the notion of generalized closed sets in a ditopological texture space.

Definition 8. Let (S, φ, τ, k) be a ditopological texture space. A subset A of a texture φ is said to be generalized closed (g-closed for short) if

$$A \subset G \in \tau \Rightarrow [A] \subset G$$

We denote $gc(S, \varphi, \tau, k)$, or when there can be no confusion by $gc(S)$ or even just gc, the set of g-closed sets in φ.

Theorem 2. Let (S, φ, τ, k) be a ditopological texture space. If $A, B \in gc(S)$, then $A \cup B \in gc(S)$.

Proof: Assume that $A \cup B \subset G \in \tau$, then $A \subset G \in \tau$ and $B \subset G \in \tau$. Since A and B are g-closed, then $[A] \subseteq G$ and $[B] \subseteq G$. Therefore $[A] \cup [B] \subset G$ and hence $[A \cup B] \subset G$. Consequently $A \cup B \in gc(S)$.

Theorem 3. Let (S, φ, τ, k) be a ditopological texture space. If $A \in gc(S)$ and $A \subset B \subset [A]$. Then $B \in gc(S)$.

Proof: Let $A \subset B \subset [A]$, and $B \subset G \in \tau$ Since $A \subset B$, then $[A] \subset G$ because A is a g-closed. Since $[A] \subset [B] \subset [A]$, then $[B] \subset G$. Hence, B is g-closed.

Theorem 4. Let (S, φ, τ, k) be a ditopological texture space. If $A \subset Y \subset S$ and $A \in gc(S)$. Then A is g-closed with respect to the subspace $(Y, \varphi|_Y, \tau_Y, k_Y)$.

Proof: Let $A \subset G^* \in \tau_Y$, where $G^* = G \cap Y, G \in \tau$. Since $A \subset G \cap Y$, then $A \subset G \in \tau$ and hence $[A] \subset G$ because A is g-closed. So, $[A] \cap Y \subset G \cap Y = G^*$, where $[A] \cap Y$, is a closure of A with respect to the subspace $(Y, \varphi|_Y, \tau_Y, k_Y)$. Hence A is g-closed with respect to the subspace $(Y, \varphi|_Y, \tau_Y, k_Y)$.

Definition 9. Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space. A subset A of a texture φ is called a generalized open (g-open for short) if $\sigma(A)$ is g-closed.

We denote $go(S, \varphi, \tau, k, \sigma)$, or when there can be no confusion by $go(S)$ or even just go, the set of g-open sets in φ.

Definition 10. Two subsets A and B of texture space (S, φ) is said to be separated in a ditopological texture space (S, φ, τ, k) if $[A] \cap B = A \cap [B] = \phi$.

Theorem 5. Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space. A subset A of φ is g-open iff

$$F \subset A, F \in k \Rightarrow F \subset]A[$$

Proof: $(\Rightarrow)$ Assume that $A \in \varphi$ is g-open and $F \subset A, F \in k$. Then $\sigma(A)$ is g-closed and $\sigma(A) \subset \sigma(F) \in \tau$. Hence $[\sigma(A)] \subset \sigma(F) \in \tau$ and therefore $F \subset \sigma([\sigma(A)] =]\sigma(\sigma(A))[=]A[$.

$(\Leftarrow)$ We show A is g-open (i.e. $\sigma(A)$ is g-closed). Suppose that $\sigma(A) \subset G \in \tau$, then $\sigma(G) \subset A$, $\sigma(G) \in k$. By hypothesis $\sigma(G) \subset]A[$ and so $\sigma(]A[) \subset G$. Thus $[\sigma(A)] \subset G$ and consequently $\sigma(A)$ is g-closed. Hence A is g-open.

Theorem 6. Let $(S, \varphi, \tau, k, \sigma)$ be a complemented ditopological texture space and A, $B \in go(S)$. If A and B and are separated, then $A \cup B \in go(S)$.

Proof: Suppose that A and B are separated and g -open sets. We show $A\cup B$ is g -open. Let $F\subset A\cup B, F\in k$. Since A and B are separated, then $A\cap[B]=[A]\cap B=\phi$. Hence $F\cap[A]\subset A,\ F\cap[B]\subset B,\ F\cap[A]\in k,\ and\ F\cap[B]\in k$. Since A and B are g -open, then $F\cap[A]\subset]A[$ and $F\cap[B]\subset]B[$. Therefore $(F\cap[A])\cup(F\cap[B])\subset]A[\cup]B[$ and so $F\cap([A]\cup[B])\subset]A[\cup]B[$. Hence $F\cap[A\cup B]\subset]A[\cup]B[\subset]A\cup B[$ and thus $F\subset]A\cup B[$. Consequently, $A\cup B$ is g -open.

In the following example, we show that in general $A\cap B\notin gc(S)$ if $A,B\in gc(S)$.

Example 2. Let $S=\{a,b,c\}$, $\varphi=\{S,\phi,\{b\},\{c\},\{b,c\}\}$, $\tau=\{S,\phi,\{a\}\}$ and $k=\{S,\phi,\{b,c\}\}$. Then $(S,\varphi,\tau,k,\sigma)$ is a complemented ditopological texture space where $\sigma(A)=A'$. Let $A=\{a,c\}$ and $B=\{a,b\}$. Then $A,B\in gc(S)$ but $A\cap B\notin gc(S)$. In fact, $A\cap B=\{a\}\subseteq\{a\}\in\tau$. But $[\{a\}]=S\not\subseteq\{a\}$.

Corollary 7. Let $(S,\varphi,\tau,k,\sigma)$ be a complemented ditopological texture space and $A,B\in gc(S)$. If $\sigma(A)$ and $\sigma(B)$ are separated, then $A\cap B\in gc(S)$.

Proof: Let $A\cap B\subset G\in\tau$. Since A and B are g -closed, then $\sigma(A)$ and $\sigma(B)$ are g -open sets. Since $\sigma(A)$ and $\sigma(B)$ are separated, then $\sigma(A)\cup\sigma(B)$ is g -open set. Therefore $\sigma(\sigma(A)\cup\sigma(B))=\sigma(\sigma(A\cap B))=A\cap B$ is g -closed.

Corollary 8. Let $(S,\varphi,\tau,k,\sigma)$ be a complemented ditopological texture space and $A,B\in go(S)$, then $A\cap B\in go(S)$.

Proof: Let A and B are g -open sets, then $\sigma(A)$ and $\sigma(B)$ are g -closed. Hence $\sigma(A)\cup\sigma(B)$ is g -closed and therefore $A\cap B$ is g -open.

Theorem 9. Let $(S,\varphi,\tau,k,\sigma)$ be a complemented ditopological texture space. If $]A[\subset B\subset A$ and $A,B\in go(S)$, then $B\in go(S)$.

Proof: Suppose $]A[\subset A$ and A is g -open. Then $\sigma(A)$ is g -closed and $\sigma(A)\subset\sigma(B)\subset\sigma(]A[)=[\sigma(A)]$. Then $\sigma(B)$ is g -closed and consequently B is g -open.

Definition 11. Let $(S,\varphi,\tau,k,\sigma)$ be a complemented ditopological texture space. For $A\in\varphi$, we define the generalized closure(g -closure, for short) $[A]_g$ and the generalized interior(g -interior, for short) $]A[_g$ of A under (τ,k) by the equalities

$[A]_g=\cap\{G\in\varphi:G$ is g -closed, $A\subset G\}$ and

$]A[_g=\cup\{G\in\varphi:G$ is g -open, $G\subset A\}$.

Proposition 10. Let $(S,\varphi,\tau,k,\sigma)$ be a complemented ditopological texture space, $A\in\varphi$ then

If $A\in k$, then A is g -closed

If $A\in\tau$, then A is g -open

Proof: (1) Let $A\subset G\in\tau$. Since $A=[A]$ (for $A\in k$), then $[A]\subset G$ and therefore A is g -closed.

(2) Let $A\in\tau$, we show $\sigma(A)$ is g -closed. Assume that $\sigma(A)\subset G\in\tau$. Since $A\in\tau$, then $A=]A[$ and thus $\sigma(]A[)\subset G$. So $\sigma(]A[)=[\sigma(A)]\subset G$ and therefore $\sigma(A)$ is g -closed. Consequently, A is g -open.

Proposition 11. Let $(S,\varphi,\tau,k,\sigma)$ be a complementled ditopological texture space. Th $]A[\subseteq]A[_g\subseteq A\subseteq[A]_g\subseteq[A]$
.

Proof: Obviously, $]A[_g\subseteq A\subseteq[A]_g$. From Proposition 10, we have $k\subset\{G\in\varphi:G$ is g -closed $\}$. Hence

$\bigcap\{G\in\varphi : A\subset G\in k\}\supseteq\bigcap\{G\in\varphi : G$ is g-closed, $A\subset G\}$. Consequently, $[A]_g\subseteq[A]$.

Similarly, from Proposition 10, we have $\tau\subset\{G\in\varphi : G$ is g-open $\}$. Then $\bigcup\{G\in\varphi : G\subset A, G$ is g-open $\}\supseteq\bigcup\{G\in\varphi : G\in\tau, G\subset A\}$ and therefore $]A[\subseteq]A[_g$.

4. Generalized continuity in a ditopological texture spaces

In this section, we introduce new types of continuity in ditopological texture spaces based on the notion of generalized open and closed sets.

Definition 12. The difunction $(f,F):(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ is called

g-continuous (Mg-continuous) if $f^{\leftarrow}B\in go(S_1)$ for all $B\in\tau_2(B\in go(S_2))$.

g-cocontinuous (Mg-cocontinuous) if $F^{\leftarrow}B\in gc(S_1)$ for all $B\in k_2(B\in gc(S_2))$

g-bicontinuous (Mg-bicontinuous) if it is both g-continuous and g-cocontinuous (Mg-continuous and Mg-cocontinuous).

Proposition 12. Let $(f,F):(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ be a difunction:

The following are equivalent

(f,F) is Mg-continuous

For each $A\in\varphi_1$ we have $]F^{\to}A[_g^{S_2}\subseteq F^{\to}]A[_g^{S_1}$

For each $B\in\varphi_2$ we have $f^{\leftarrow}]B[_g^{S_2}\subseteq]f^{\leftarrow}B[_g^{S_1}$

The following are equivalent :

(f,F) is Mg-cocontinuous .

For each $A\in\varphi_1$ we have $f^{\to}[A]_g^{S_1}\subseteq[f^{\to}A]_g^{S_2}$

For each $B\in\varphi_2$ we have $[F^{\leftarrow}B]_g^{S_1}\subseteq F^{\to}[B]_g^{S_2}$

Proof: (1) $\Rightarrow$ (2) Take $A\in\varphi_1$. Then $f^{\leftarrow}]F^{\to}A[_g^{S_2}\subseteq f^{\leftarrow}(F^{\to}A)\subseteq A$ by [7, Theorem 2.24(2a)]. Now $f^{\leftarrow}]F^{\to}A[_g^{S_2}=F^{\leftarrow}]F^{\to}A[_g^{S_2}\in go(S_1)$ by Mg-continuity. So $f^{\leftarrow}]F^{\to}A[_g^{S_2}\subseteq]A[_g^{S_1}$ and applying [7, Theorem 2.24(2b)] gives $]F^{\to}A[_g^{S_2}\subseteq F^{\to}(f^{\leftarrow}]F^{\to}a[_g^{S_2})\subseteq F^{\to}]A[_g^{S_1}$, which is the required inclusion.

(2 $\Rightarrow$ 3). Take $B\in\varphi_2$. Applying inclusion (b) to $A=f^{\leftarrow}B$ and using [7, Theorem 2.24 (2b)] gives $]B[_g^{S_2}\subseteq]F^{\to}(f^{\leftarrow}B)[_g^{S_2}\subseteq F^{\to}]f^{\leftarrow}B[_g^{S_1}$. Hence, $f^{\leftarrow}]B[_g^{S_2}\subseteq f^{\leftarrow}(F^{\to}]f^{\leftarrow}B[_g^{S_1})\subseteq]f^{\leftarrow}B[_g^{S_2}$ by [7, Theorem 2.24(2a)].

(3 $\Rightarrow$ 1) Applying (3) for $B\in go(S_2)$ gives $F^{\leftarrow}B=f^{\leftarrow}]B[_g^{S_2}\subseteq]f^{\leftarrow}B[_g^{S_1}$, so $F^{\leftarrow}B=f^{\leftarrow}B=f^{\leftarrow}]B[_g^{S_1}=]f^{\leftarrow}B[_g^{S_1}\in go(S_1)$. Hence (f,F) is M_g continuous.

The following proposition gives corresponding characterizations for g-continuity and g-cocontinuity.

Proposition 13. Let $(f,F):(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ be a difunction:

The following are equivalent

(f,F) is g-continuous

For each $A \in \varphi_1$ we have $]F^{\to}A[^{S_2} \subseteq F^{\to}]A[_g^{S_1}$

For each $B \in \varphi_2$ we have $f^{\leftarrow}]B[^{S_2} \subseteq]f^{\leftarrow}B[_g^{S_1}$

The following are equivalent :

(f,F) is Mg -cocontinuous .

For each $A \in \varphi_1$ we have $f^{\to}[A]_g^{S_1} \subseteq [f^{\to}A]^{S_2}$

For each $B \in \varphi_2$ we have $[F^{\leftarrow}B]_g^{S_1} \subseteq F^{\to}[B]^{S_2}$

5. g -compactness and g -cocompactness

In this section, we give the definition of g -compactness in ditopological texture spaces. As expected, there is also the dual notion of g -cocompactness.

Definition 13. A ditopological texture space (S,φ,τ,k) is called:

g -compact if every cover of S by g -open sets has a finite subcover.

g -cocomact if every cocover of S by g -closed sets has a finite subcover.

Here we recall that $\rho = \{A_j : j \in J\}$, $A_j \in \varphi$ is a cover of S (a cocover of ϕ) if $\vee \rho = S(\cap \rho = \phi)$.

Proposition 14. Let $(S,\varphi,\tau,k,\sigma)$ be a complemented ditopologica texture space. Then $(S,\varphi,\tau,k,\sigma)$ is g -compact iff it is g -cocompact.

Proof: Suppose that $(S,\varphi,\tau,k,\sigma)$ is g -compact and let $\hbar = \{F_j : j \in J\}$ be a family of g -closed sets with $\cap \hbar = \phi$. Clearly $C = \{\sigma(F_j) : j \in J\}$ is a family of g -open sets. Moreover,

$$\vee C = \bigvee \{\sigma(F_j) j \in J\} = \sigma(\cap \{F_j : j \in J\}) = \sigma(\phi) = S,$$

and so we have $J' \subseteq J$ finite with $\bigvee \{\sigma(F_j) : j \in J'\} = S$. Hence $\cap \{F_j : j \in J'\} = \phi$ and see that $(S,\varphi,\tau,k,\sigma)$ is g -cocompact. Likewise, if $(S,\varphi,\tau,k,\sigma)$ is g -cocompact then it is g -compact.

Theorem 15. Let $(f,F)(S_1,\varphi_1,\tau_1,k_1) \to (S_2,\varphi_2,\tau_2,k_2)$ be an Mg -continuous difunction. If $A \in \varphi_1$ is g -compact then $f^{\to}A \in \varphi_2$ is g -compact.

Proof: Take $f^{\to}A \subseteq \vee_{j \in J} G_j$, where $G_j \in go(S_2), j \in J$. Now by [7, Theorem 2.24 (2a) and Corollary 2.12(2)] we have

$$A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}(\bigvee_{j \in J} G_j) = \bigvee_{j \in J} F^{\leftarrow}G_j.$$

Also, $F^{\leftarrow}G_j \in go(S_1)$. Since (f,F) is Mg -continuous, so by the g -compactness of A there exists $J' \subseteq J$ finite such that $A \subseteq \bigcup_{j \in J'} F^{\leftarrow}G_j$.

Hence

$$f^{\to}A \subseteq f^{\to}(\bigcup_{j \in J'} F^{\leftarrow}G_j) = \bigcup_{j \in J'} f^{\to}(F^{\leftarrow}G_j) \subseteq \bigcup_{j \in J'} G_j$$

by [7, Corollary 2.12(2) and Theorem 2.24(2b)]. This establishes that $f^{\to}A$ is g -compact.

Proposition 16. Let $(f,F)(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ be a surjective Mg -continuous difunction. Then if $(S_1,\varphi_1,\tau_1,k_1)$ is g -compact so is $(S_2,\varphi_2,\tau_2,k_2)$.

Proof: This follows by taking $A=S_1$ in Theorem 15 and noting that $f^{\to}S_1 = f^{\to}(F^{\leftarrow}S_2)=S_2$ by [7, Proposition 2.28 (1c) and Corollary 2.33(1)].

As expected, we have dual results for cocompactness.

Theorem 17. Let $(f,F)(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ be an Mg -cocontinuous difunction. If $A\in\varphi_1$ is g -cocompact then $F^{\to}A\in\varphi_2$ is g -cocompact.

Proposition 18. Let $(f,F):(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ be an surjective Mg -continuous difunction. Then if $(S_1,\varphi_1,\tau_1,k_1)$ is g -cocompact so is $(S_2,\varphi_2,\tau_2,k_2)$.

6. g -stability and g -costability

The notion of stability for bitopological spaces was introduced by Ralph Kopperman [40]. The analogous notion, and its dual, were given for ditoplogies in [6], and studied in greater detail in [10]. We now wish to generalize these concepts for g -open and g -closed sets. The following definition would seem to be appropriate.

Definition 14. A ditopological texture space (S,φ,τ,k) is called g -stable if very g -closed set $F\in\varphi\backslash\{S\}$ is g -compact in S'. That is, whenever $G_j, j\in J$, are g -open sets in (S,φ,τ,k) satisfying $F\subseteq\bigvee_{j\in J}G_j$, there exists a finite subsets J' of J for $F\subseteq\bigcup_{j\in J'}G_j$.

g -costable if every g -open set $G\in\varphi\backslash\phi$ is g -cocompact in S'. That is, whenever $F_j, j\in J$, are g -closed sets in (S',φ,τ,k) satisfying $\bigcap_{j\in J}F_j\subseteq G$, there exists a finite subsets J' of J for which $\bigcap_{j\in J'}F_j\subseteq G$.

Proposition 19. Let $(S,\varphi,\tau,k,\sigma)$ be a complemented ditopological texture space. Then $(S,\varphi,\tau,k,\sigma)$ is g -stable iff it is g -costable.

Proof: Assume that $(S,\varphi,\tau,k,\sigma)$ is g -stable and let G be a g -open set with $G\neq\phi$ and D a g -closed cocover of G. Set $H=\sigma(G)$. Then H is g -closed and satisfies $H\neq S'$. Hence H is g -compact. Let $\rho=\{\sigma(F):F\in D\}$. Since $\cap D\subseteq G$ we have $H\subseteq\vee\rho$, i.e e is g -open cover of H. Hence there exists $F_1,F_2,\ldots F_n\in D$ so tha

$$H\subseteq\sigma(F_1)\cup\sigma(F_2)\cup\ldots\cup\sigma(F_n)=\sigma(F_1\cap F_2\cap\ldots\cap F_n)$$

This gives $F_1\cap F_2\cap\ldots\cap F_n\subseteq\sigma(H)=G$, so G is g -compact in S. Hence (S,φ,τ,k) is g -costable.

The proof that g -costable implies g -stable is the dual of the above.

Theorem 20. Let $(S_1,\varphi_1,\tau_1,k_1),(S_2,\varphi_2,\tau_2,k_2)$ be ditopological texture spaces with $(S_1,\varphi_1,\tau_1,k_1)$ is g -stable, and $(f,F):(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ an Mg -bicontinuous surjective difunction. Then $(S_2,\varphi_2,\tau_2,k_2)$ is g -stable.

Proof: Take $H\in gc(S_2)$ with $H\neq S_2$. Since (f,F) is g -co-continuous, $f^{\leftarrow}H\in gc(S_1)$. Let us prove that $f^{\leftarrow}H\neq S_1$ Assume the contrary. Since $f^{\leftarrow}S_2=S_1$, by [7, Lemma 2.28 (1c)] we have $f^{\leftarrow}S_2\subseteq f^{\leftarrow}H$, whenever $S_2\subseteq H$ by [7, Corollary 2.33(1 ii)] as (f,F) is surjective. This is a contradiction. So $f^{\leftarrow}(H)\neq S_1$. Hence $f^{\leftarrow}(H)$ is g -compact in $(S_1,\varphi_1,\tau_1,k_1)$ by g -stability. As (f,F) is Mg -continuous, $f^{\to}(f^{\leftarrow}H)$ is g -compact for the ditopology (τ_2,k_2) by Theorem 15 and by [7, Corollary 2.33(1)] this set is equal to H. This establishes that

$(S_2,\varphi_2,\tau_2,k_2)$ is g-stable.

Theorem 21. Let $(S_1,\varphi_1,\tau_1,k_1),(S_2,\varphi_2,\tau_2,k_2)$ be ditopological texture spaces with $(S_1,\varphi_1,\tau_1,k_1)$ is g-costable, and $(f,F):(S_1,\varphi_1,\tau_1,k_1)\to(S_2,\varphi_2,\tau_2,k_2)$ an Mg-bicontinuous surjective difunction. Then $(S_2,\varphi_2,\tau_2,k_2)$ is g-costable.

7 New type of rough classification based on generalized open and closed sets

In this section, we introduced and investigated the concept of g-approximation space. Also, we introduce the concepts of g-lower approximation and g-upper approximation for any subset and study their properties.

Remark 22. Let S be any set, then $\wp(S)$ is a texturing on S. Let R be a general relation on S. We use this relation to get a subbase for a complemented ditopology (τ,k,σ) on (S,φ), where $\varphi=\wp(S)$ and $\sigma(A)=A'$ for all $A\in\varphi$, and a class of open sets.

Definition 15. Let (S,R) be a an approximation space. Lower and upper approximation of any nonempty subset A of S is defined as

$$\underline{R}(A)=]A[=\cup\{G\in\tau:G\subset A\}$$ and

$$\overline{R}(A)=[A]=\cap\{F\in k:A\subset F\}$$

We can get the the approximation operator as follows.

(1) Get the right neighborhoods xR from the given relation R as $xR=\{y:xRy\}$.

(2) Using right neighborhoods xR as a sub-base to get the family τ. Then write $k=\sigma(\tau)$, where $\sigma(A)=A'$ for all $A\in\varphi=\wp(S)$

(4) Using the set of all open sets to get approximation operators as Definition 14.

Remark 23. We denote the relation which used to get a subbase for a complemented ditopology (τ,k) on (S,φ) and a class of g-open sets by R_g. Also, we denote g-approximation space by (S,R_g).

Definition 16. Let (S,R_g) be a g-approximation space. g- lower approximation and g-upper approximation of any nonempty subset A of S is defined as $\underline{R}_g(A)=]A[_g=\cup\{G\in\varphi:G$ is g-open, $G\subset A\}$

and $\overline{R}_g(A)=[A]_g=\cap\{G\in\varphi:G$ is g-closed, $A\subset G\}$.

The following proposition shows the properties of g- lower approximation and g-upper approximation of any nonempty subset.

Proposition 24. Let (S,R_g) be a g-approximation space and $A,B\subseteq S$. Then:

$$\underline{R}_g(A)\subseteq A\subseteq\overline{R}_g(A)$$

$$\underline{R}_g(\phi)=\overline{R}_g(\phi)=\phi,\ \underline{R}_g(S)=\overline{R}_g(S)=S$$

If $A\subseteq B$, then $\underline{R}_g(A)\subseteq\underline{R}_g(B)$ and $\overline{R}_g(A)\subseteq\overline{R}_g(B)$

$$\underline{R}_g(A^c)=(\overline{R}_g(A))^c$$

$$\overline{R}_g(A^c)=(\underline{R}_g(A))^c$$

$$\underline{R}_g(\underline{R}_g(A))=\underline{R}_g(A)$$

$$\overline{R}_g(\overline{R}_g(A))=\overline{R}_g(A)$$

$$\underline{R}_g(A \cup B) \supseteq \underline{R}_g(A) \cup \underline{R}_g(B)$$

$$\overline{R}_g(A \cup B) \supseteq \overline{R}_g(A) \cup \overline{R}_g(B)$$

$$\underline{R}_g(A \cap B) \subseteq \underline{R}_g(A) \cap \underline{R}_g(B)$$

$$\overline{R}_g(A \cap B) \subseteq \overline{R}_g(A) \cap \overline{R}_g(B)$$

Definition 17 Let (S, R_g) be a g -approximation space. The Universe S can be divided into 12 regions with respect to any $A \subseteq S$ as follows.

The internal edg of A, $\underline{Edg}(A) = A - \underline{R}(A)$

The g -internal edg of A, $\underline{Edg}_g(A) = A - \underline{R}_g(A)$

The external edg of A, $\overline{Edg}(A) = \overline{R}(A) - A$

The g -external edg of A, $\overline{Edg}_g(A) = \overline{R}_g(A) - A$

The boundary of A, $b(A) = \overline{R}(A) - \underline{R}(A)$

The g -boundary of A, $b_g(A) = \overline{R}_g(A) - \underline{R}_g(A)$

The exterior of A, $ext(A) = S - \overline{R}(A)$

The g -exterior of A, $ext_g(A) = S - \overline{R}_g(A)$

$\overline{R}(A) - \underline{R}_g(A)$

$\overline{R}_g(A) - \underline{R}(A)$

$\overline{R}_g(A) - \underline{R}_g(A)$

$\overline{R}(A) - \overline{R}_g(A)$

Remark 25. As shown in previous proposition, the study of g -approximation spaces is a generalization for study of approximation spaces. Because of the elements of the regions $\underline{R}_g(A) - \underline{R}(A)$ will be defined well in A, while this points was undefinable in Pawlak s approximation spaces. Also, the elements of the region $\overline{R}(A) - \underline{R}_g(A)$ and $\overline{R}(A) - \overline{R}_g(A)$ do not be belong to A, while these elements was not well defined in Pawlak s approximation spaces.

Theorem 26. For any complemented ditopological texture space $(S, \varphi, \tau, k, \sigma)$ generated by a binary relation R on S, we have, $\underline{R}(A) \subseteq \underline{R}_g(A) \subseteq A \subseteq \overline{R}_g(A) \subseteq \overline{R}(A)$.

Definition 18. Let (S, R_g) be a g -approximation space and $A \subseteq S$. Then there are memberships $\underline{\in}_g$ and $\overline{\in}_g$, say, g -strong and g -weak memberships respectively which defined by

$$x \underline{\in}_g A \text{ iff } x \in \underline{R}_g(A)$$

$$x \overline{\in}_g A \text{ iff } x \in \overline{R}_g(A).$$

Remark 27. According to Definition , g -lower and g -upper approximations of a set $A \subseteq S$ can be written as

$\underline{R}_g(A) = \{x \in A : x \underline{\in}_g A\}$

$\overline{R}_g(A) = \{x \in A : x \overline{\in}_g A\}$

Remark 28. Let (S, R_g) be a g -approximation space and $A \subseteq S$. Then

$x \overline{\in} A \Rightarrow x \underline{\in}_g A$

$x \overline{\in}_g A \Rightarrow x \overline{\in} A$

Definition 19. Let S be a finite nonempty universe, $A \subseteq S$, we can characterize the degree of completeness by a new tool named g -accuracy measure defined as follows.

$$\alpha_g(A) = \frac{card\,\underline{R}_g(A)}{card\,\overline{R}_g(A)}$$

Table 1

	r	p	q
1	$\{E\}$	$\{H\}$	$\{S\}$
2	$\{E, G\}$	$\{H\}$	$\{S\}$
3	$\{G\}$	$\{H, B\}$	$\{R\}$
4	$\{G, A\}$	$\{B\}$	$\{R, F\}$
5	$\{A\}$	$\{T\}$	$\{F\}$

Example 3. This example is a small form of multi-valued information table of a file containing some persons $U = \{1, 2, 3, 4, 5\}$. Let $A = \{p, q, r\}$ as in Table 1, where:

r = Languages = { *English; German; Arabic* } = $\{E, G, A\}$

p = Sports = { *Tennis; Handball; Basketball* } = $\{T, H, B\}$

q = Skills = { *Swimming; Running; Fishing* } = $\{S, R, F\}$

Our choice for relation R depends on our view to the choice of objects, where we can choose a level of experience and any of objects having higher levels.

Let xRy iff $R(x) \subseteq R(y)$,

then: $xpy = \{(1,1), (1,2), (1,3), (2,2), (2,1), (2,3), (3,3), (4,4), (4,3), (5,5)\}$, Then the subbase for $p = S_p = \{\{1,2,3\}, \{3\}, \{3,4\}, \{5\}\}$

S0, $\tau_p = \{\{1,2,3\}, \{3\}, \{3,4\}, \{5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{3,5\}, \{3,4,5\}, X, \phi\}$.

Then $\sigma(\tau) = k = \{\{4,5\}, \{1,2,4,5\}, \{1,2,5\}, \{1,2,3,4\}, \{5\}, \{4\}, \{1,2,4\}, \{1,2\}, X, \phi\}$

So $go(U) = \{\{1,2,3\}, \{3\}, \{3,4\}, \{5\}, \{1,2,3,4\}, \{1,2,3,5\}, \{3,5\}, \{3,4,5\}, \{2,3,4,5\}, \{1,3,4,5\}, \{2,3,4\}, \{1,3,4\}, \{1,3,5\}, \{2,3,5\}, \{2,3\}, \{1,3\}, \{3\}, X, \phi\}$.

Let $A = \{2,4,5\}$, then $\underline{R}(A) =]A[= \{5\}$ and $\overline{R}(A) = [A] = \{1,2,4,5\}$. So the accuracy of $X = \frac{card\,\underline{R}\ (A)}{card\,\overline{R}\ (A)} = \frac{1}{4}$.

On the other hand $\underline{R}_g(A) =]A[_g = \{5\}$ and $\overline{R}_g(A) = [A]_g = \{2,4,5\}$. So the accuracy α_g of $A = \frac{card\,\underline{R}_g(A)}{card\,\overline{R}_g(A)} = \frac{1}{3}$.

Therefore $\alpha_g > \alpha$. Also, if $A = \{2,3\}$, then underline $R(A) =]A[= \{3\}$ and $\overline{R}(A) = [A] = \{1,2,3,4\}$.

So the accuracy α of $A = \frac{card\,\underline{R}\,(A)}{card\,\overline{R}\,(A)} = \frac{1}{4}$. On the other hand $\underline{R}_g(A) =]A[_g = \{2,3\}$ and $\overline{R}_g(A) = [A]_g = \{1,2,3,4\}$.

So the accuracy α_g of $A = \frac{\underline{R}_g(A)}{\overline{R}_g(A)} = \frac{1}{2}$. Therefore $\alpha_g > \alpha$.

8. Conclusion

In this paper, we introduced the concepts of g-open and g-closed sets in ditopological texture spaces. We studied the properties of these concepts and the relations between them. Also, We generalized the notions of continuous difunction, compactness and stability in ditopological texture spaces by introducing new notions using g- open and g-closed sets. We used the class of g-open sets to introduce a new type of approximations named g-approximation operator. Also, using g-approximation operators we can obtain 12 dissimilar granules of the universe of discourse. This made the accuracy measures higher than the use of open sets. Some important properties of the classical Pawlaks rough sets are generalized. Also, we defined the concept of rough membership function using g open sets. It is a generalization of classical rough membership function of Pawlak rough sets. The generalized rough membership function can be used to analyze which decision should be made according to a conditional attribute in decision information system. The rough set approach to approximation of sets leads to useful forms of granular computing that are part of computational intelligence.

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