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On the solutions of the heat, wave and Laplace equations with non-classical conditions

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ABSTRACT

In this paper, we present a new approach to solve nonlocal initial-boundary value problems for heat, wave and Laplace equations subject to initial and nonlocal boundary conditions of integral type. We first transform the given nonlocal initial-boundary value problems of integral type and then apply the method of separation of variables.

Keywords: Nonlocal initial-boundary value problem; heat equation; wave equation; Laplace equations; method of separation of variables.

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1- Introduction

The problem of determining a solution to a partial differential equation when both initial data and nonlocal boundary conditions are specified is called a non- local initial-boundary value problem. Various problems arising in heat conduction [1, 2, 3], chemical engineering [4], thermo-elasticity [5] and plasma physics [6] can be modeled by nonlocal initial-boundary value problems with integral boundary conditions. This class of boundary value problems has been investigated in [1, 2, 3, 4, 7, 8] for parabolic and in [9, 10] for hyperbolic partial differential equations. In [11, 12, 13, 14, 15], these problems were investigated and appropriate existence and uniqueness theorems were proved. As a continuation of the research paper [16], we will present a similarly approach for solving nonlocal initial-boundary value problems for heat, wave and Laplace equation by the method of eigen function expansions. We first transform the given nonlocal initial-boundary value problems for the heat, wave and Laplace equations subject to initial and nonlocal homogeneous boundary conditions of integral type into local initial-boundary value problems. Then we apply the method of separation of variables.

2- Nonlocal boundary value problem for the homogeneous heat equation

We begin this section by considering the linear heat equation.

$$u_t - ku_{xx} = 0,\ k > 0,\quad a \le x \le b,\ t \ge 0, \tag{2.1}$$

subject to the initial condition

$$u(0,x) = \alpha(x) \tag{2.2}$$

and the nonlocal homogeneous boundary conditions of integral type

$$\int_a^b \varphi_1(x)u(t,x)dx = 0 \text{ and } \int_a^b \varphi_2(x)u(t,x)dx = 0, \tag{2.3}$$

where $\varphi_i, i = 1,2$ and α are specified as continuous functions on the interval $[a,b]$.

We begin our approach by converting Eqs. (2.1)-(2.3) to a local initial-boundary value problem by introducing a new function $v(t,x)$ such that

$$v(t,x) = \int_a^x \varphi(x)u(t,x)dx, \tag{2.4}$$

Where $\varphi = \varphi_1 + \varphi_2$. Hence we have

$$u(t,x) = \frac{v_x(t,x)}{\varphi(x)},\ u_x(t,x) = \frac{1}{\varphi(x)}v_{xx}(t,x) + \left(\frac{1}{\varphi(x)}\right)' v_x(t,x), u_t(t,x) = \frac{v_{tx}(t,x)}{\varphi(x)}$$

and

$$u_{xx}(t,x) = \left(\frac{1}{\varphi(x)}\right)'' v_x(t,x) + 2\left(\frac{1}{\varphi(x)}\right)' v_{xx}(t,x) + \left(\frac{1}{\varphi(x)}\right) v_{xxx}(t,x).$$

Thus we have transformed Eq. (2.1) into the following linear partial differential equation

$$v_{tx} - k\left(\frac{1}{\varphi(x)}\right)'' \varphi(x)v_x - 2k\left(\frac{1}{\varphi(x)}\right)' \varphi(x)v_{xx} - kv_{xxx} = 0, \tag{2.5}$$

subject to the initial and local homogeneous Dirichlet boundary conditions

$$v(0,x) = h(x),\ \text{where } h(x) = \int_a^x \varphi(x)\alpha(x)dx,$$

and

$$v(t,a) = 0 \text{ and } v(t,b) = 0.$$

Thus we deduce

Lemma 1. The general nonlocal initial-boundary value problem for the linear heat equation (2.1)-(2.3) can always be reduced to a local initial-boundary value problem of the form

$$\begin{cases} v_{tx} - k\left(\frac{1}{\varphi(x)}\right)''\varphi(x)v_x - 2k\left(\frac{1}{\varphi(x)}\right)'\varphi(x)v_{xx} - kv_{xxx} &= 0,\ a < x < b,\ t > 0, \\ v(0,x) &= h(x), \\ v(t,a) = 0 \ and \ v(t,b) &= 0. \end{cases} \quad (2.6)$$

A solution of this problem will lead to a solution of the original Problem (2.1)-(2.3), where

$$u(t,x) = \frac{1}{\varphi(x)}v_x(t,x). \quad (2.7)$$

2.1- Separation of variables

The idea of the method of separation of variables is to assume a solution in the form

$$v(t,x) = X(t)Y(x). \quad (2.8)$$

Differentiate (2.8) and substitute into (2.5) to obtain

$$\frac{X'(t)}{kX(t)} = \frac{Y'''(x)}{Y'(x)} + 2\left(\frac{1}{\varphi(x)}\right)'\varphi(x)\frac{Y''(x)}{Y'(x)} + \left(\frac{1}{\varphi(x)}\right)''\varphi(x). \quad (2.9)$$

This means that each side is really a constant. We denote the so-called separation constant by $-\gamma^2$. Now we have two ordinary differential equations

$$X'(t) + k\lambda^2 X(t) = 0, \quad (2.10)$$

subject to the initial condition $X(0) = h(x)$ and

$$Y'''(x) + 2\left(\frac{1}{\varphi(x)}\right)'\varphi(x)Y''(x) + \left[\left(\frac{1}{\varphi(x)}\right)''\varphi(x) + \lambda^2\right]Y'(x) = 0, \quad (2.11)$$

subject to the local homogeneous Dirichlet boundary conditions $Y(a) = Y(b) = 0$. We note that the method of separation of variables replaces Pr. (2.6) by a pair of ordinary differential equations (2.10)-(2.11). In order to solve Eq. (2.11), we introduce a new dependent variable $Z(x)$ such that $Z(x) = Y'(x)$. Therefore, Eq. (2.11) becomes a second-order differential equation

$$Z''(x) + 2\left(\frac{1}{\varphi(x)}\right)'\varphi(x)Z'(x) + \left[\left(\frac{1}{\varphi(x)}\right)''\varphi(x) + \lambda^2\right]Z(x) = 0. \quad (2.12)$$

Once the function $Z(x)$s determined, we can readily return to the original dependent variable $Y(x)$

We will not be able to give the solution without the explicit knowledge of $\varphi(x)$. Therefore this nonlocal initial-boundary value problem will not be fully solved until the following section, in which we discuss various special cases of $\varphi(x)$.

3- Nonlocal boundary value problem for the homogeneous wave equation

For the nonlocal wave problem

$$\begin{cases} u_{tt} - ku_{xx} &= 0,\ a \le x \le b,\ t \ge 0, \\ u(0,x) = \alpha_1(x),\ u_t(0,x) &= \alpha_2(x),\ \text{and} \\ \int_a^b \varphi_1(x)u(t,x)dx &= 0 \text{ and } \int_a^b \varphi_2(x)u(t,x)dx = 0, \end{cases} \quad (3.1)$$

we deduce

Lemma 2. The general nonlocal initial-boundary value problem for the linear wave equation (3.1) can always be reduced to a local initial-boundary value problem of the form

$$\begin{cases} v_{ttx} - k\left(\frac{1}{\varphi(x)}\right)''\varphi(x)v_x - 2k\left(\frac{1}{\varphi(x)}\right)'\varphi(x)v_{xx} - kv_{xxx} &= 0, \\ v(a,x) = h_1(x),\ v_t(a,x) &= h_2(x),\ and \\ v(t,a) = 0\ and\ v(t,b) &= 0, \end{cases} \tag{3.2}$$

where

$h_i(x) = \int_a^x \varphi(x)\alpha_i(x)dx,\ i = 1, 2.$

3.1-Separation of variables

The same method of separation of variables that we discussed for the nonlocal initial-boundary value problem for the heat equation can also be applied to the wave equation.

Substituting the separable solution $u(t,x) = X(t)Y(x)$ into the initial-boundary value problem (3.2), gives the same equation for $Y(x)$ as before, while the equation for $X(t)$ instead has the form

$$X''(t) + k\lambda^2 X(t) = 0. \tag{3.3}$$

4 - Nonlocal boundary value problem for the Laplace equation

Consider the Laplace equation

$$u_{xx} + u_{yy} = 0,\ 0 \le x \le a,\ 0 \le y \le b, \tag{4.1}$$

subject to the initial condition

$$u(0,y) = \alpha_1(y), \tag{4.2}$$

final condition

$$u(a,y) = \alpha_2(y) \tag{4.3}$$

and the nonlocal homogeneous boundary conditions of integral type

$$\int_0^b \varphi_1(y)u(x,y)dy = 0 \text{ and } \int_0^b \varphi_2(y)u(x,y)dy = 0, \tag{4.4}$$

Introducing a new function v(x; y) such that

$$v(x,y) = \int_c^x \varphi(y)u(x,y)dy, \tag{4.5}$$

Proceeding as before, thus we have transformed Eq. (4.1)-Eq. (4.4) into the following local initial-boundary value problem of linear partial differential equation

Lemma 3. The general nonlocal initial-boundary value problem for the linear elliptic equation (4.1)- (4.4) can always be reduced to a local initial-boundary value problem of the form

$$\begin{cases} v_{xxy} + \left(\frac{1}{\varphi(y)}\right)''\varphi(y)v_y + 2\left(\frac{1}{\varphi(y)}\right)'\varphi(y)v_{yy} + v_{yyy} &= 0, \\ v(x,0) = 0\ and\ v(x,b) &= 0,\ and \\ v(0,y) = h_1(y),\ v(a,y) &= h_2(y), \end{cases} \tag{4.6}$$

where

$h_i(y) = \int_0^y \varphi(x)\alpha_i(y)dy,\ i = 1, 2.$

A solution of this problem will lead to a solution of the original Problem (4.1)-(4.4), where

$$u(x,y) = \frac{1}{\varphi(y)}v_y(x,y). \tag{4.7}$$

4.1- Separation of variables

Assume a solution in the form

$$v(x,y) = X(x)Y(y). \tag{4.8}$$

Differentiate (4.8) and substitute into the first equation of (4.6) to obtain

$$-\frac{X''(x)}{X(x)} = \frac{Y'''(y)}{Y'(y)} + 2\left(\frac{1}{\varphi(y)}\right)'\varphi(y)\frac{Y''(y)}{Y'(y)} + \left(\frac{1}{\varphi(y)}\right)''\varphi(y). \tag{4.9}$$

This means that each side is really a constant. We have two ordinary differential equations

$$X''(x) - \lambda^2 X(x) = 0 \tag{4.10}$$

and

$$Y'''(y) + 2\left(\frac{1}{\varphi(y)}\right)'\varphi(y)Y''(y) + \left[\left(\frac{1}{\varphi(y)}\right)''\varphi(y) + \lambda^2\right]Y'(y) = 0. \tag{4.11}$$

In order to solve Eq. (4.11), we introduce a new dependent variable $Z(y)$ such that $Z(y) = Y'(y)$. Therefore, Eq. (4.11) becomes a second-order differential equation

$$Z''(y) + 2\left(\frac{1}{\varphi(y)}\right)'\varphi(y)Z'(y) + \left[\left(\frac{1}{\varphi(y)}\right)''\varphi(y) + \lambda^2\right]Z(y) = 0. \tag{4.12}$$

Once the function $Z(x)$ is determined, we can readily return to the original dependent variable $Y(y)$.

In the following, we discuss an example to illustrate the procedure as outlined above.

Example 1. Let $\varphi_1(x) = \sin\left(\frac{n\pi}{L}x\right)$, $\varphi_2(x) = c - \sin\left(\frac{n\pi}{L}x\right)$ for n = 1, 2,…, where c is a constant, $a = 0$ and $b = L$, then $\varphi(x) = c$ and the equation for the dependent variable Z is

$$Z''(x) + \lambda^2 Z(x) = 0, \tag{5.1}$$

which has a solution

$$Z(x) = C_1 \cos\lambda x + C_2 \sin\lambda x. \tag{5.2}$$

Integrating this equation yields the solution

$$Y(x) = \int Z(x)dx + C_3, \tag{5.3}$$

Thus

$$Y(x) = \frac{C_1}{\lambda}\sin\lambda x - \frac{C_2}{\lambda}\cos\lambda x + C_3, \tag{5.4}$$

so that the homogeneous Dirichlet boundary conditions give

$$Y(0) = \frac{C_2}{\lambda} + C_3 = 0 \text{ and } Y(L) = \frac{C_1}{\lambda}\sin\lambda L - \frac{C_2}{\lambda}\cos\lambda L + C_3 = 0. \tag{5.5}$$

Since the constants of integration C_i for i = 1,2,3 cannot all be zero.

Thus the eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \frac{n\pi}{L},\ Y_n(x) = \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right),\ \text{for } n = 1, 2, \ldots. \tag{5.6}$$

The solutions $X_n(t)$ corresponding to $\frac{n\pi}{L}$ for the heat and wave problems are given as

$$X_n(t) = e^{-\lambda_n^2 kt}, \text{ for } n = 1, 2, ... \tag{5.7}$$

and

$$X_n(t) = \cos \sqrt{k}\lambda_n t + \sin \sqrt{k}\lambda_n t, \text{ for } n = 1, 2, ..., \tag{5.8}$$

respectively, thus

$$v(t, x) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 kt} \sin\left(\frac{n\pi}{L}x\right), \tag{5.9}$$

for the above nonlocal heat problem.

Upon substituting $t = 0$ in the above equation and using $v(0, x) = h(x)$, we obtain

$$B_n = \frac{2}{L}\int_0^L h(x) \sin\left(\frac{n\pi}{L}x\right) dx. \tag{5.10}$$

Returning to the original dependent variable by Eq. (2.4), we obtain the solution to the nonlocal heat problem as given by the series

$$u(t, x) = \frac{1}{c}v_x(t, x) = \sum_{n=1}^{\infty} B_n^* e^{-\lambda_n^2 kt} \cos\left(\frac{n\pi}{L}x\right), \tag{5.11}$$

where

$B_n^* = \frac{n\pi}{cL} B_n.$

Similarly, for the above nonlocal wave problem, we have

$$v(t, x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(C_n \cos \sqrt{k}\lambda_n t + D_n \sin \sqrt{k}\lambda_n t\right). \tag{5.12}$$

Upon substituting $t = 0$ in Eq. (5.12) and using v(0; x) = h1(x) $v(0, x) = h_1(x)$ and $v_t(0, x) = h_2(x)$, we obtain

$$v(0, x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = h_1(x) \tag{5.13}$$

and

$$v_t(0, x) = \sum_{n=1}^{\infty} D_n \sqrt{k}\lambda_n \sin\left(\frac{n\pi}{L}x\right) = h_2(x). \tag{5.14}$$

Thus

$$C_n = \frac{2}{L}\int_0^L h_1(x) \sin\left(\frac{n\pi}{L}x\right) dx \tag{5.15}$$

$$D_n = \frac{2}{n\pi\sqrt{k}}\int_0^L h_2(x) \sin\left(\frac{n\pi}{L}x\right) dx. \tag{5.16}$$

Consequently, the solution to the nonlocal wave problem is given by the series

$$u(t, x) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left(C_n^* \cos \sqrt{k}\lambda_n t + D_n^* \sin \sqrt{k}\lambda_n t\right), \tag{5.17}$$

where

$C_n^* = \frac{n\pi}{cL} C_n$ and $D_n^* = \frac{n\pi}{cL} D_n.$

Example 2. Let $\varphi_1(x) = \frac{n\pi}{2}\sin\left(\frac{n\pi}{L}x\right) + \cos\left(\frac{n\pi}{L}x\right)$, for n = 1, 2,..., $\varphi_2(x) = e^{\frac{x}{2}} - \varphi_1(x)$, $a = 0$ and $b = L$, then $\varphi_2(x) = e^{\frac{x}{2}}$ and the equation for the dependent variable Z is

$$Z''(x) - Z'(x) + (\frac{1}{4} + \lambda^2)Z(x) = 0, \qquad (5.18)$$

which has a solution

$$Z(x) = C_1 e^{\frac{x}{2}} \sin\lambda x + C_2 e^{\frac{x}{2}} \cos\lambda x. \qquad (5.19)$$

Integrating this equation yields the solution

$$Y(x) = \int Z(x)dx + C_3. \qquad (5.20)$$

Thus

$$Y(x) = \left(\frac{2C_1}{4\lambda^2+1} + \frac{4\lambda C_2}{4\lambda^2+1}\right) e^{\frac{x}{2}} \sin\lambda x + \left(\frac{2C_2}{4\lambda^2+1} - \frac{4\lambda C_1}{4\lambda^2+1}\right) e^{\frac{x}{2}} \cos\lambda x + C_3, \qquad (5.21)$$

so that the homogeneous Dirichlet boundary $Y(0)= Y(L) = 0$ conditions give

$$\frac{2C_2}{4\lambda^2+1} - \frac{4\lambda C_1}{4\lambda^2+1} + C_3 = 0 \qquad (5.22)$$

and

$$\left(\frac{2C_1}{4\lambda^2+1} + \frac{4\lambda C_2}{4\lambda^2+1}\right) e^{\frac{L}{2}} \sin\lambda L + \left(\frac{2C_2}{4\lambda^2+1} - \frac{4\lambda C_1}{4\lambda^2+1}\right) e^{\frac{L}{2}} \cos\lambda L + C_3 = 0. \qquad (5.23)$$

Since ¸$\lambda \neq 0$, the constants of integration C_i for $i = 1,2,3$, cannot all be zero; if we choose $C_3 = 0$ then we have $2\lambda C_1 = C_2$, and $Y(L) = 2C_1 e^{\frac{L}{2}} Sin\lambda L = 0$, so that $Sin\lambda L = 0$. In order to satisfy the last boundary condition, we must have $\lambda L = n\pi$ for $n = 1,2,\ldots$.. Thus the eigenvalues and the corresponding eigen functions are

$$\lambda_n = \frac{n\pi}{L},\ Y_n(x) = e^{\frac{x}{2}} \sin\left(\frac{n\pi}{L}x\right),\ \text{for } n = 1, 2, \ldots. \qquad (5.24)$$

Consequently, the solution to the nonlocal heat and wave problems can be given by the series

$$u(t,x) = \sum_{n=1}^{\infty} B_n^* e^{-\lambda_n^2 kt} \cos\left(\frac{n\pi}{L}x\right) \qquad (5.25)$$

and

$$u(t,x) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left(C_n^* \cos\sqrt{k}\lambda_n t + D_n^* \sin\sqrt{k}\lambda_n t\right), \qquad (5.26)$$

respectively, where $B_n^* = \frac{n\pi}{L}B_n$, $C_n^* = \frac{n\pi}{L}C_n$ and $D_n^* = \frac{n\pi}{L}D_n$.

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