



## Derivation of Yang – Mills equations from Maxwell Equations and Exact solutions

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### Abstract

In this paper we derived the Yang-Mills equations from Maxwell equations. Consequently we find a new form for self-duality equations. In addition exact solution class of the classical SU(2) Yang-Mills field equations in four-dimensional Euclidean space and two exact solution classes for SU(2) Yang- Mills equations when is  $\rho$  a complex analytic function are also obtained.

**Keywords :** Self-dual SU(2); Yang-Mills fields; Gauge theory



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## 1. Introduction

The self-dual Yang-Mills equations (a system of equations for Lie algebra valued functions of  $C^4$ ) play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics [1-4].

In addition the self-dual Yang-Mills equations are of great importance in their own right and have found a remarkable number of applications in physics and mathematics as well. These equations arise in the context of gauge theory [5], in classical general relativity [6,7], and can be used as a powerful tool in the analysis of 4-manifolds [8].

Non-Abelian gauge theories first appeared in the seminal work of Yang and Mills (1954) as a non-Abelian generalization of Maxwell's equations [9]. The fact that the Yang-Mills equations have a natural geometric interpretation was recognized early on in the history of gauge theory [10,11].

The Yang-Mills equations are a set of coupled, second-order partial differential equations in four dimensions for the Lie algebra-valued gauge potential functions  $A_\mu$ , and are extremely difficult to solve in general. The self-dual Yang-Mills equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space [12,13].

A very important property of the theory of non-abelian gauge fields is that the action functional has local minima in the Euclidean domain with non-vanishing field strength  $F_{\mu\nu}$  [14]. The corresponding field configurations, which are often called pseudoparticles, have the self-dual or anti-self-dual field strength, and fall into topologically inequivalent classes labelled by an integer  $n$ , the Pontryagin index. The existence of these non-local minima was first pointed out by Belavin et al. (1975) who also exhibited the solution of the self-duality equation with  $n = 1$  for an  $SU(2)$  gauge group [15]. Solutions of the self-duality equations with an arbitrary number of pseudoparticles were discovered by Witten (1979) and 't Hooft (1979) [16].

In this paper we found a new representation for self-duality equations. In addition exact solution class of the classical  $SU(2)$  Yang-Mills field equations in four-dimensional Euclidean space and two exact solution classes for  $SU(2)$  Yang-Mills equations when  $\rho$  is a complex analytic function are also obtained.

This paper is organized as follows: This introduction followed by the derivation of Yang – Mills equations from Maxwell Equations in section 2. A new representation of the self-duality equations in section 3. In section 4 we found an exact solution class of the classical  $SU(2)$  Yang-Mills field equations. Moreover two exact solution classes for self-dual  $SU(2)$  gauge fields on Euclidean space when  $\rho$  is a complex analytic function are given in section 5.

### Derivation of Yang –Mills equations from Maxwell Equations

The classical equations of Maxwell describing electromagnetic phenomena are

$$\begin{aligned}\nabla \cdot E &= 4\pi\rho, & \nabla \times B &= 4\pi J + \frac{\partial E}{\partial t}, \\ \nabla \cdot B &= 0, & \nabla \times E &= -\frac{\partial B}{\partial t},\end{aligned}$$

We would like to formulate these equations in the language of differential forms. Let  $x_\mu = (t, x^1, x^2, x^3)$  be local coordinates in Minkowski's space  $M_{1,3}$ . Define the Maxwell 2-form  $F$  by the equation

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu, \quad (\mu, \nu = 0, 1, 2, 3), \quad (2)$$

where

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \quad (3)$$

Written in complete detail, Maxwell's 2-form is given by

$$\begin{aligned}F &= -E_x dt \wedge dx^1 - E_y dt \wedge dx^2 - E_z dt \wedge dx^3 + \\ &B_z dx^1 \wedge dx^2 - B_y dx^1 \wedge dx^3 + B_x dx^2 \wedge dx^3.\end{aligned} \quad (4)$$

We also define the source current 1-form

$$J = J_\mu dx^\mu = \rho dt + J_1 dx^1 + J_2 dx^2 + J_3 dx^3. \quad (5)$$

**Proposition 1:** Maxwell's Equations (1) are equivalent to the equations

$$dF = 0,$$



$$d * F = 4\pi * J. \tag{6}$$

**Proof:** The proof is by direct computation using the definitions of the exterior derivative and the Hodge-\* operator.

$$\begin{aligned} dF = & -\frac{\partial E_x}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^1 - \frac{\partial E_x}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^1 - \frac{\partial E_y}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^2 + \\ & -\frac{\partial E_y}{\partial x^3} \wedge dx^3 \wedge dt \wedge dx^2 - \frac{\partial E_z}{\partial x^1} \wedge dx^1 \wedge dt \wedge dx^3 - \frac{\partial E_z}{\partial x^2} \wedge dx^2 \wedge dt \wedge dx^3 + \\ & \frac{\partial B_z}{\partial t} \wedge dt \wedge dx^1 \wedge dx^2 - \frac{\partial B_z}{\partial x^3} \wedge dx^3 \wedge dx^1 \wedge dx^2 - \frac{\partial B_y}{\partial t} \wedge dt \wedge dx^1 \wedge dx^3 + \\ & -\frac{\partial B_y}{\partial x^2} \wedge dx^2 \wedge dx^1 \wedge dx^3 + \frac{\partial B_x}{\partial t} \wedge dt \wedge dx^2 \wedge dx^3 + \frac{\partial B_x}{\partial x^1} \wedge dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

Collecting terms and using the anti-symmetry of the wedge operator, we get

$$\begin{aligned} dF = & \left( \frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} \right) dx^1 \wedge dx^2 \wedge dx^3 + \left( \frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial t} \right) dx^2 \wedge dt \wedge dx^3 + \\ & \left( \frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial x^3} - \frac{\partial B_y}{\partial t} \right) dt \wedge dx^1 \wedge dx^3 + \left( \frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial t} \right) dx^1 \wedge dt \wedge dx^2. \end{aligned}$$

Therefore,  $dF = 0$  iff

$$\frac{\partial B_x}{\partial x^1} + \frac{\partial B_y}{\partial x^2} + \frac{\partial B_z}{\partial x^3} = 0,$$

which is the same as

$$\nabla \cdot B = 0,$$

and

$$\frac{\partial E_y}{\partial x^3} - \frac{\partial E_z}{\partial x^2} - \frac{\partial B_x}{\partial t} = 0,$$

$$\frac{\partial E_z}{\partial x^1} - \frac{\partial E_x}{\partial x^3} - \frac{\partial B_y}{\partial t} = 0,$$

$$\frac{\partial E_x}{\partial x^2} - \frac{\partial E_y}{\partial x^1} - \frac{\partial B_z}{\partial t} = 0,$$

which means that

$$-\nabla \times E - \frac{\partial B}{\partial t} = 0, \tag{7}$$

To verify the second set of Maxwell equations, we first compute the dual of the current density 1-form (5) using the results of

$$* dt = -dx^1 \wedge dx^2 \wedge dx^3$$

$$* dx^1 = dx^2 \wedge dt \wedge dx^3$$

$$* dx^2 = dt \wedge dx^1 \wedge dx^3$$

$$* dx^3 = dx^1 \wedge dt \wedge dx^2.$$

We get

$$* J = -\rho dx^1 \wedge dx^2 \wedge dx^3 + J_1 dx^2 \wedge dt \wedge dx^3 + J_2 dt \wedge dx^1 \wedge dx^3 + J_3 dx^1 \wedge dt \wedge dx^2. \tag{8}$$

We could now proceed to compute  $d * F$ , but perhaps it is more elegant to notice that  $F \in \Lambda^2(M)$ , and  $F$  splits into  $F = F^+ + F^-$ . In fact, we see from (3) that the components of  $F^+$  are those of  $-E$  and the components of  $F^-$  constitute the magnetic field vector  $B$ . Using the above results, we can immediately write the components of  $* F$ :



$$F = B_x dt \wedge dx^1 + B_y dt \wedge dx^2 + B_z dt \wedge dx^3 + E_z dx^1 \wedge dx^2 - E_y dx^1 \wedge dx^3 + E_x dx^2 \wedge dx^3, \tag{9}$$

or equivalently,

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}. \tag{10}$$

Since the effect of the dual operator amounts to exchanging

$$E \mapsto -B \\ B \mapsto +E,$$

we can infer from equations (7) and (8) that

$$\nabla \cdot E = 4\pi\rho,$$

and

$$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi J.$$

## 2. New form of the self-duality equations

The essential idea of Yang and Mills (1954) is to consider an analytic continuation of the gauge potential  $A_\mu$  into complex space where  $x_1, x_2, x_3$  and  $x_4$  are complex. The self-duality equations  $F_{\mu\nu} = *F_{\mu\nu}$  are then valid also in complex space, in a region containing real space where the  $x$ 's are real. Now consider four new complex variables,  $\bar{y}, Z$  and  $\bar{Z}$  defined by

$$\begin{aligned} \sqrt{2}\bar{y} &= x_1 + ix_2, & \sqrt{2}\bar{y} &= x_1 - ix_2, \\ \sqrt{2}Z &= x_3 - ix_4, & \sqrt{2}\bar{Z} &= x_3 + ix_4, \end{aligned} \tag{11}$$

it is simple to check that the self-duality equations  $F_{\mu\nu} = *F_{\mu\nu}$  reduces to

$$F_{yZ} = 0, \quad F_{\bar{y}\bar{Z}} = 0, \quad F_{y\bar{y}} + F_{Z\bar{Z}} = 0. \tag{12}$$

Equations (12) can be immediately integrated, since they are pure gauge, to give [17,18,19]

$$A_y = D^{-1}D_y, \quad A_Z = D^{-1}D_Z, \quad A_{\bar{y}} = \bar{D}^{-1}\bar{D}_{\bar{y}}, \quad A_{\bar{Z}} = \bar{D}^{-1}\bar{D}_{\bar{Z}}, \tag{13}$$

where  $D$  and  $\bar{D}$  are arbitrary  $2 \times 2$  complex matrix functions of  $y, \bar{y}, Z$  and  $\bar{Z}$  with determinant = 1 (for  $SU(2)$  gauge group) and  $D_y = \partial_y D$ , etc. For real gauge fields  $A_\mu \doteq -A_\mu^+$  (the symbol  $\doteq$  is used for equations valid only for real values of  $x_1, x_2, x_3$  and  $x_4$ ), we require

$$\bar{D} \doteq (D^+)^{-1}. \tag{14}$$

Gauge transformations are the transformations

$$D \rightarrow U D, \quad \bar{D} \rightarrow \bar{D} U, \quad U^+ U \doteq I, \tag{15}$$

where  $U$  is a  $2 \times 2$  matrix function of  $y, \bar{y}, Z, \bar{Z}$  with determined = 1. Under transformation (15), equation (14) remains unchanged.. We now define the hermitian matrix  $J$  as [20-22]

$$J \equiv D \bar{D}^{-1} \doteq D D^+. \tag{16}$$

$J$  has the very important property of being invariant under the gauge transformation (15). The only nonvanishing field strengths in terms of  $J$  becomes

$$F_{u\bar{v}} = -\bar{D}^{-1} (J^{-1} J_u)_{\bar{v}} \bar{D}, \tag{17}$$

( $u, v = y, Z$ ) and the remaining self-duality equation (12) takes the form:

$$(J^{-1} J_y)_{\bar{y}} + (J^{-1} J_Z)_{\bar{Z}} = 0. \tag{18}$$

The action density in terms of  $J$  [23] is



$$\phi(\mathcal{J}) \equiv -\frac{1}{2} \text{Tr} F_{\mu\nu} F_{\mu\nu} = -2\text{Tr} ( F_{y\bar{y}} F_{z\bar{z}} + F_{y\bar{z}} F_{\bar{y}z} ), \tag{19}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \tag{20}$$

Our construction begins by explicit parametrization of the matrix  $\mathcal{J}$

$$\mathcal{J} = \begin{pmatrix} \phi & \rho \\ \bar{\rho} & \phi(1 + \rho\bar{\rho}) \end{pmatrix}, \tag{21}$$

and for real gauge fields  $A_\mu \doteq -A_\mu^\dagger$ , we require  $\phi \doteq \text{real}$ ,  $\bar{\rho} \doteq \rho^*$  ( $\rho^* \equiv$  complex conjugate of  $\rho$ ). The self-duality equations (18) take the form

$$\begin{aligned} \frac{1}{2}(1 + \rho\bar{\rho})\partial_\mu\partial_\mu \ln\phi + \frac{1}{2}\rho\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{\phi}[\phi_{y\rho\bar{y}} + \phi_{z\rho\bar{z}}] + \frac{\rho}{\phi}[\phi_{y\bar{\rho}\bar{y}} + \phi_{z\bar{\rho}\bar{z}}] + \\ + [\rho_{\bar{y}}\bar{\rho}_y + \rho_{\bar{z}}\bar{\rho}_z] = 0, \end{aligned} \tag{22}$$

$$\frac{\phi}{2}\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{2}\partial_\mu\partial_\mu\phi - \frac{2\bar{\rho}}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\phi_{y\bar{\rho}\bar{y}} + \phi_{z\bar{\rho}\bar{z}} - \phi_{\bar{y}\rho_y} - \phi_{\bar{z}\rho_z}) = 0, \tag{23}$$

$$\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\phi - \frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\phi_{\bar{y}\rho_y} + \phi_{\bar{z}\rho_z} - \phi_{y\rho\bar{y}} - \phi_{z\rho\bar{z}}) = 0, \tag{24}$$

where  $\partial_\mu\partial_\mu = 2(\partial_y\partial_{\bar{y}} + \partial_z\partial_{\bar{z}})$ .

The positive definite Hermitian matrix  $\mathcal{J} = D D^\dagger$  can be factored into a product upper and lower (or vice versa) triangular matrices as follows

$$\begin{aligned} \mathcal{J} &= R R^\dagger = R^I R^{I\dagger}, \\ R &= \begin{pmatrix} \frac{1}{\sqrt{\phi}} & 0 \\ \bar{\rho}\sqrt{\phi} & \sqrt{\phi} \end{pmatrix}, \quad R^I = \begin{pmatrix} \sqrt{\phi^I} & \bar{\rho}^I\sqrt{\phi^I} \\ 0 & \frac{1}{\sqrt{\phi^I}} \end{pmatrix}, \\ \phi &\doteq \text{real}, \quad \bar{\rho} \doteq \rho^*, \quad \bar{\rho}^I \doteq \rho^{I*}. \end{aligned} \tag{25}$$

It is evident from (25) that one can choose a gauge so that  $D = R$  or  $D = R^I$  and it is easy to check that in both gauges the self-duality equations (22)-(24) (in the case of  $D = R^I$  all the  $\phi, \rho, \bar{\rho}$  are replaced by  $\phi^I, \rho^I, \bar{\rho}^I$ ).

From equation (2.25) we see that  $R^{-1}R^I$  is a unitary matrix so that we can always make a gauge transformation from the  $R$  gauge to the  $R^I$  gauge.

**Theorem 1:** If  $(\phi, \rho, \bar{\rho})$  satisfy equations (22)-(24) then so do  $(\phi^I, \rho^I, \bar{\rho}^I)$  defined by [24]

$$\phi^I = \frac{\phi}{1 + \rho\bar{\rho}}, \quad \rho^I = \frac{\bar{\rho}}{1 + \rho\bar{\rho}}, \quad \bar{\rho}^I = \frac{\rho}{1 + \rho\bar{\rho}},$$

### 3. Exact solution class of the classical SU(2) Yang-Mills field equations

To obtain an exact solution class of the classical SU(2) Yang-Mills field equations in four-dimensional Euclidean space, consider the system.

$$\begin{aligned} \frac{1}{2}(1 + \rho\bar{\rho})\partial_\mu\partial_\mu \ln\phi + \frac{1}{2}\bar{\rho}\partial_\mu\partial_\mu\rho + \frac{\rho}{\phi}[\phi_{\bar{y}\bar{\rho}\bar{y}} + \phi_{\bar{z}\bar{\rho}\bar{z}}] + \frac{\bar{\rho}}{\phi}[\phi_{\bar{y}\rho_y} + \phi_{\bar{z}\rho_z}] + \\ + [\bar{\rho}_y\rho_{\bar{y}} + \bar{\rho}_z\rho_{\bar{z}}] = 0, \end{aligned} \tag{26}$$

$$\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\phi - \frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\phi_{\bar{y}\rho_y} + \phi_{\bar{z}\rho_z} - \phi_{y\rho\bar{y}} - \phi_{z\rho\bar{z}}) = 0.$$

Let us make the ansatz [25]

$$\phi = \phi(g), \quad \rho = e^{ia} \sigma(g). \tag{27}$$



Where  $g = g(x_1, x_2, x_3, x_4)$  is a real function of  $x_\mu$ ,  $\mu = 1, 2, 3, 4$ ,  $\phi$  and  $\sigma$  are real functions of  $g$ , and  $a$  is a real constant. Then equation (26) give the relations

$$(g_{y\bar{y}} + g_{z\bar{z}})((1 + \sigma^2)\phi^2)' - 2(g_y g_{\bar{y}} + g_z g_{\bar{z}})\phi^2[(1 + \sigma^2)\frac{\phi'}{\phi} + \sigma\sigma']' = 0, \tag{28}$$

$$(g_{y\bar{y}} + g_{z\bar{z}})(\phi\sigma)' + (g_y g_{\bar{y}} + g_z g_{\bar{z}})\phi^2\left(\frac{(\phi\sigma)'}{\phi^2}\right)' = 0. \tag{29}$$

Where the prime means differentiation with respect to  $g$ . The above relations imply that the determinant of the coefficients of  $(g_{y\bar{y}} + g_{z\bar{z}})$  and  $(g_y g_{\bar{y}} + g_z g_{\bar{z}})$  is zero i.e.

$$((1 + \sigma^2)\phi^2)' \left(\frac{(\phi\sigma)'}{\phi^2}\right)' + 2[(1 + \sigma^2)\frac{\phi'}{\phi} + \sigma\sigma']'(\phi\sigma)' = 0. \tag{30}$$

We shall determine  $\phi$  and  $\sigma$  from the above equation (30), let  $(\phi\sigma) = c$ , where  $c$  is constant, then  $(\phi\sigma)' = 0$ ,

$$\begin{aligned} ((1 + \sigma^2)\phi^2)' \left(\frac{(\phi\sigma)'}{\phi^2}\right)' &= 0, \\ [(1 + \sigma^2)\frac{\phi'}{\phi} + \sigma\sigma']'(\phi\sigma)' &= 0. \end{aligned} \tag{31}$$

We suppose

$$\phi = \sqrt{ce^{-g}}, \quad \sigma = \sqrt{ce^g}, \quad \text{then } \rho = \sqrt{ce^{g+ia}}. \tag{32}$$

Applying theorem (1) to  $\phi$  and  $\rho$  of equation (32), then we get

$$\phi^I = \frac{\sqrt{ce^{-g}}}{1+ce^{2g}}, \quad \rho^I = \frac{\sqrt{ce^{g-ia}}}{1+ce^{2g}}, \quad \bar{\rho}^I = \frac{\sqrt{ce^{g+ia}}}{1+ce^{2g}}. \tag{33}$$

Equations (32) and (33) is a new class of solutions of Yang-Mills equations for self-dual SU(2) gauge fields.

#### 4. Exact solutions for self-dual SU(2) gauge fields on Euclidean space when $\rho$ is a complex analytic function

We reduce the equations for self-dual SU(2) gauge fields on Euclidean space to the following equations

$$\begin{aligned} \frac{1}{2}(1 + \rho\bar{\rho})\partial_\mu\partial_\mu \ln\phi + \frac{1}{2}\rho\partial_\mu\partial_\mu\bar{\rho} + \frac{\bar{\rho}}{\phi}[\phi_y\rho_{\bar{y}} + \phi_z\rho_{\bar{z}}] + \frac{\rho}{\phi}[\phi_y\bar{\rho}_{\bar{y}} + \phi_z\bar{\rho}_{\bar{z}}] + \\ + [\bar{\rho}_y\rho_{\bar{y}} + \bar{\rho}_z\rho_{\bar{z}}] = 0, \end{aligned} \tag{34}$$

$$\frac{\phi}{2}\partial_\mu\partial_\mu\rho + \frac{\rho}{2}\partial_\mu\partial_\mu\phi - \frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\phi_{\bar{y}}\rho_y + \phi_{\bar{z}}\rho_z - \phi_y\rho_{\bar{y}} - \phi_z\rho_{\bar{z}}) = 0.$$

When  $\rho$  is a complex analytic function of  $y$  and  $z$ , then we have

$$\rho_{\bar{y}} = \rho_{\bar{z}} = 0, \quad \rho_y\bar{y} + \rho_z\bar{z} = 0. \tag{35}$$

Then, the self-dual Yang-Mills equations (34) takes the form

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0, \tag{36}$$

$$\rho(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - \frac{2\rho}{\phi}(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) + (\rho_y\phi_{\bar{y}} + \rho_z\phi_{\bar{z}}) = 0. \tag{37}$$

We consider now two cases:

(a) Let  $\rho = \rho(\phi)$ , then we find

$$\rho_y = \rho' \phi_y, \quad \rho_z = \rho' \phi_z. \tag{38}$$

Then the two equations (36) and (37) becomes

$$\phi(\phi_{y\bar{y}} + \phi_{z\bar{z}}) - (\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0, \tag{39}$$

$$\rho(\phi_{y\bar{y}} + \phi_{z\bar{z}}) + \left(\rho' - \frac{2\rho}{\phi}\right)(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0. \tag{40}$$



If we do not consider the case  $(\phi_{y\bar{y}} + \phi_{z\bar{z}}) = 0$  and  $(\phi_y\phi_{\bar{y}} + \phi_z\phi_{\bar{z}}) = 0$ , then we have

$$\phi\rho' - \rho = 0, \quad (41)$$

by integration we obtain

$$\rho = c\phi, \quad \text{where } c \text{ is complex constant.} \quad (42)$$

Both equations (39) and (40) reduce to the same equation. A solution is given by

$$\phi_y = \phi_z, \quad \phi_{\bar{y}} = -\phi_{\bar{z}}. \quad (43)$$

The solution class is given by [26]

$$\phi = F(\mathcal{Y} + \mathcal{Z}, \bar{\mathcal{Y}} - \bar{\mathcal{Z}}), \quad (44)$$

where  $F$  is an arbitrary function, equations (42) and (44) gives a new class of solutions of Yang-Mills equations for self-dual  $SU(2)$  gauge fields. Applying theorem (1) to  $\phi$  and  $\rho$  of equations (42) and (44) then we get

$$\phi^I = \frac{F}{1+c\bar{c}F^2}, \quad \rho^I = \frac{\bar{c}F}{1+c\bar{c}F^2}, \quad \bar{\rho}^I = \frac{cF}{1+c\bar{c}F^2}. \quad (45)$$

(b) Let us make the ansatz

$$\phi = \phi(g), \quad \rho = e^{ia}\sigma(g). \quad (46)$$

Where  $g = g(x_1, x_2, x_3, x_4)$  is a real function of  $x_\mu$ ,  $\mu = 1, 2, 3, 4$ ,  $\phi$  and  $\sigma$  are real functions of  $g$ , and  $a$  is a real constant. Then equation (36) and (37) gives the relations

$$\phi\phi'(g_y\bar{y} + g_z\bar{z}) + (g_y g_{\bar{y}} + g_z g_{\bar{z}})[\phi\phi'' - \phi'^2] = 0, \quad (47)$$

$$\sigma\phi'(g_y\bar{y} + g_z\bar{z}) + (g_y g_{\bar{y}} + g_z g_{\bar{z}})\left(\sigma\phi'' - \frac{2\sigma\phi'^2}{\phi} + \phi'\sigma'\right) = 0. \quad (48)$$

Where the prime means differentiation with respect to  $g$ . The above relations imply that the determinant of the coefficients of  $(g_y\bar{y} + g_z\bar{z})$  and  $(g_y g_{\bar{y}} + g_z g_{\bar{z}})$  is zero i.e.

$$\frac{\sigma'}{\sigma} = \frac{\phi'}{\phi}, \quad (49)$$

by integration (49) we obtain

$$\sigma(g) = c\phi(g), \quad \rho = ce^{ia}\phi(g). \quad (50)$$

Applying theorem (1) to  $\phi$  and  $\rho$  of equation (50), then we get

$$\phi^I = \frac{\phi(g)}{1+c^2\phi^2(g)}, \quad \rho^I = \frac{ce^{-ia}\phi(g)}{1+c^2\phi^2(g)}, \quad \bar{\rho}^I = \frac{ce^{ia}\phi(g)}{1+c^2\phi^2(g)}. \quad (51)$$

Equations (50) and (51) is a new class of solutions of Yang-Mills equations for self-dual  $SU(2)$  gauge fields.

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