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Common Fixed Point for Weakly Compatible Mappings on Dislocated Metric Spaces

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Abstract: In this paper we prove a common fixed point theorem for multi-valued and single-valued mappings in a dislocated metric space. The set of generalized ψ-φ weak contractive mappings considered in this paper contains the family of generalized φ-weak contractive mappings as a proper subset. The theorem use weakly compatibility and ψ-φ-weak contractivity condition. It extends the work of several authors.

Keywords: Dislocated metric spaces; common fixed point; generalized ψ- φ-weak contractions; weakly compatible mappings; complete dislocated metric spaces.

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1. Introduction.

The notion of dislocated metric spaces (d-metric spaces) was introduced by Pascal Hitzler in [8]. The study of common fixed point mapping in dislocated metric space satisfying certain contractive conditions has been at the center of vigorous research activity..

Weakly contractive mappings in Hilbert spaces setting was first introduced by Alber and Guerr-Delabriere [1] in 1997. Rhoades showed that most results of [1] arew still true for any Banach spaces and Bae considered these type of multi-valued mappings [2]. Kammran [13], Zhang and Song [17], Beg and Abbas [3], Bose and Roychowdhury considered some generalized versions of these mappings and proved some fixed point theorems.

After Jungck has introduced the notion of compatible and weakly compatible mappings for single valued maps [6] many authors have immediately extended the concepts to multi-valued maps.

The purpose of the present paper is to contribute in this field. We have considered a family of generalized ψ-φ weakly contractive mappings, which contains the class of generalized φ-weakly contractive mappings and have proved a fixed point theorem for them when they are weakly compatible maps on dislocated metric spaces, which extends the work of several authors ([7], [4],[16],[12], [5])

2. Preliminaries.

For convenience we start with the following definitions, lemmas, and theorems.

Definition 2.1.[9] Let X be a non-empty and let $d:X\times X\to[0,+\infty[$ be a function, called a distance function if for all $x,y,z\in X$ satisfies:

$d1$: $d(x,x)=0$

$d2$: $d(x,y)=d(y,x)=0 \Rightarrow x=y$

$d3$: $d(x,y)=d(y,x)$

$d4$: $d(x,y)\leq d(x,z)+d(z,y)$

If d satisfies the condition $d1-d4$, then d is called a metric on X. If it satisfies the conditions $d1$, $d2$ and $d4$ it is called a quasi-metric space. If d satisfies conditions $d2$, $d3$ and $d4$ it is called a dislocated metric (or simply d-metric).

A nonempty set X with d-metric d, i. e., (X,d) is called a dislocated metric space.

Example. Let be $X=R^+\cup\{0\}$ and the function $d: X\times X\to[0,\infty[$ where $d(x,y)=x+y$. Then (X,d) is a dislocated metric space.

Definition 2.2 [9] A sequence $\{x_n\}_{n\in N}$ in a d-metric space (X,d) is called aCauchy sequence if for all $\varepsilon>0$, $\exists n_0\in N$ such that $\forall m,n\geq n_0$, we have $d(x_m,x_n)<\varepsilon$.

Definition 2.3 [9] A sequence in d-metric space converges with respect to d, if there exists $x\in X$ such that $\lim_{n\to\infty} d(x_n,x)=0$.

In this case x is called a d-limit of $\{x_n\}_{n\in N}$ and we write $x_n\to x$.

Definition 2.4 [9] A d-metric space (X,d) is called complete if every Cauchy sequence in it is convergent with respect to d.

Lemma 2.5 [9] Limits in a d-metric space are unique.

Lemma 2.6. [12] Let (X,d) be a d-metric space. Then

(A) if $d(x,y)=0$, then $d(x,x)=d(y,y)=0$

(B) if $\{x_n\}_{n\in N}$ is a sequence such that $\lim_{n\to\infty} d(x_n,x_{n+1})=0$, then $\lim_{n\to\infty} d(x_n,x_n)=\lim_{n\to\infty} d(x_{n+1},x_{n+1})=0$

(C) if $x\neq y$, then $d(x,y)>0$

(D) $d(x,x)\leq\frac{2}{n}\sum_{i=1}^{n}d(x,x_i)$ holds for all x, $x_i\in X$, where $1\leq i\leq n$.

Let (X,d) be a d-metric space. We denote the family of all nonempty, bounded subset of X by $B(X)$

Definition 2.7. Let $A, B\in B(X)$, then

$$H(A,B)=\max\left\{\sup_{x\in A}d(x,B),\ \sup_{y\in B}d(y,A)\right\}$$

$$d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$$

$$\delta(A,B) = \sup\{d(x,y) : x \in A, y \in B\}$$

Definition 2.8. A sequence $\{A_n\}$ of nonempty subset of X is said to be convergent towards a subset *A* of X if

i) each point $a \in A$ is a limit of a convergent sequence $\{a_n\}$, $a_n \in A_n$ for $n \in N$.

ii) for arbitrary $\varepsilon > 0$, there is an integer *m* such that for $n > m,\ A_n \subseteq A_\varepsilon$, where $A_\varepsilon = \{x \in X : \exists a \in A / d(x,a) < \varepsilon\}$,

A is then said to be the limit of the sequence $\{A_n\}$.

Lemma 2.9. Let $\{A_n\}$, $\{B_n\}$ be sequences in $B(X)$ converging respectively to *A* and *B* in $B(X)$. Then the sequences of numbers $\{H(A_n, B_n)\}$, $\{d(A_n, B_n)\}$ and $\{\delta(A_n, B_n)\}$ converge to $H(A,B)$, $d(A,B)$ and $\delta(A,B)$, respectively.

Lemma 2.10. Let $\{A_n\}$ be sequences in $B(X)$ and $y \in X$ such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.

Definition 2.11. [10] The self maps *f*, *g* of X are compatible iff $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Several authors ([11], [15]) extend the Definition 2.3 by introducing the compatibility of set-valued mappings as below.

Definition 2.12. The mappings $F : X \to B(X)$ and $f : X \to X$ are δ-compatible iff $\lim_{n\to\infty} \delta(fFx_n, Ffx_n) = 0$, whenever $\{x_n\}$ is a sequence in *X* such that $fFx_n \in B(X), \lim_{n\to\infty} Fx_n = \{t\}$ and $\lim_{n\to\infty} fx_n = t$ for some $t \in X$.

Definition 2.13 [10] The mappings $F : X \to B(X)$ and $f : X \to X$ are weakly compatible (or sub compatible) if they commute at coincidence points that is $\{t \in X / Ft = \{ft\}\} \subseteq \{t \in X / Fft = fFt\}$.

Definition 2.14. Two set-valued mappings $F, G : X \to B(X)$, are called generalized φ-weak contractions if there exist a map

$\varphi : [0,+\infty[\to [0,+\infty[$ with $\varphi(t) > 0$ for all *t>0* and $\varphi(0) = 0$, such that

$H(Fx, Gy) \le M(x,y) - \varphi(M(x,y))$ for all $x, y \in X$, where

$$M(x,y) = \max\left\{d(x,y),\ d(x,Fx),\ d(y,Gy), \frac{d(x,Gy) + d(y,Fx)}{2}\right\}.$$

Definition 2.15. Two set-valued mappings $F, G : X \to B(X)$, are called generalized ψ-φ-weak contractions if for all *x, y*∊*X*,

$\psi(H(Fx,Gy)) \le \psi(M(x,y)) - \varphi(M(x,y))$,

where $M(x,y) = \max\left\{d(x,y),\ d(x,Fx),\ d(y,Gy), \frac{d(x,Gy) + d(y,Fx)}{2}\right\}$ and $\psi,\ \varphi : [0,+\infty[\to [0,+\infty[$, ψ is non-decreasing and continuous, φ is lower semi continuous function such that, $\psi(t) > 0,\ \varphi(t) > 0$ for all *t>0* and $\psi(0) = \varphi(0) = 0$.

The aim of this work is to prove a theorem on the common fixed point for two multi-valued and two single valued mappings not necessary continuous, where the multi-valued mappings are generalized ψ-φ-weak contractions on dislocated metric spaces.

3. MAIN RESULTS.

Theorem 3.1. Let $F, G : X \to B(X)$ be mappings, where (X,d) is a complete dislocated metric space and let $I, J : X \to X$ be self mappings. Suppose that

(1) $\psi(H(Fx,Gy)) \le \psi(N(x,y)) - \varphi(N(x,y))$, where

$$N(x,y) = \max\left\{d(Ix,Jy), d(Ix,Fx), d(Jy,Gy), \frac{d(Ix,Gy) + d(Jy,Fx)}{a}\right\}$$

for all $x, y \in X$, $3 < a \le 4$ and ψ, $\varphi : [0,+\infty[\to [0,+\infty[$, ψ is non-decreasing and continuous, φ is lower semi continuous function, $\psi(t) > 0$, $\varphi(t) > 0$ for all *t>0* and $\psi(0) = \varphi(0) = 0$.

(2) $\cup G(X) \subseteq I(X)$, $\cup F(X) \subseteq J(X)$ and either *I(X)* or *J*(*X*) is closed

(3) the pairs of mappings {*F*, *I*} and {*G*, *J*} are weakly compatible,

then *F, G, I, J* have a unique common fixed point $u \in X$. Moreover, *Fu* = *Gu* = {*u*}.

Proof. Let x_0 be an arbitrary point of *X*. By the condition (2) there exists $x_1 \in X$ such that $Jx_1 \in Fx_0$. Furthermore, for this point x_1 we can choose $x_2 \in X$ such that $Ix_2 \in Gx_1$ and $d\left(Jx_1, Ix_2\right) \le H(Fx_0, Gx_1)$

By the condition (1) of Theorem 3.1 we have $\psi(d\left(Jx_1, Ix_2\right)) \le \psi(H(Fx_0, Gx_1)) \le \psi(N(x_0, x_1)) - \varphi(N(x_0, x_1)) \le \psi(N(x_0, x_1))$.

Continuing this process, we can define inductively the sequence $\{x_n\}$ as follows:

$Jx_{2n+1} \in Fx_{2n}$, $\quad Ix_{2n} \in Gx_{2n-1}$ for $n \in N$ and $d\left(Ix_{2n}, Jx_{2n+1}\right) \le H(Fx_{2n}, Gx_{2n-1})$.

Hence as ψ is monotonically increasing and (1) we have

$$\psi(d(Ix_{2n}, Jx_{2n+1})) \le \psi(H(Fx_{2n}, Gx_{2n-1})) \le \psi(N(x_{2n}, x_{2n-1})) - \varphi(N(x_{2n}, x_{2n-1})) \le \psi(N(x_{2n}, x_{2n-1})) \qquad (4)$$

and $d\left(Ix_{2n}, Jx_{2n+1}\right) \le N(x_{2n-1}, x_{2n})$.

$$N(x_{2n-1}, x_{2n}) = \max\left\{d(Ix_{2n}, Jx_{2n-1}), d(Ix_{2n}, Fx_{2n}), d(Jx_{2n-1}, Gx_{2n-1}), \frac{d(Ix_{2n}, Gx_{2n-1}) + d(Jx_{2n-1}, Fx_{2n})}{a}\right\}$$

$$\le \max\left\{d(Ix_{2n}, Jx_{2n-1}), d(Ix_{2n}, Jx_{2n+1}), d(Jx_{2n-1}, Ix_{2n}), \frac{d(Ix_{2n}, Ix_{2n}) + d(Jx_{2n-1}, Jx_{2n+1})}{a}\right\} \qquad (5)$$

On the ather hand, from lemma 2.6 (D) we have

$$d(Ix_{2n}, Ix_{2n}) \le d(Ix_{2n}, Jx_{2n-1}) + d(Ix_{2n}, Jx_{2n+1})$$

and by definition 2.1 (*d*4) we have ,

$$d(Jx_{2n-1}, Jx_{2n+1}) \le d(Ix_{2n}, Jx_{2n-1}) + d(Ix_{2n}, Jx_{2n+1})$$

So,

$$\frac{d(Ix_{2n}, Ix_{2n}) + d(Jx_{2n-1}, Jx_{2n+1})}{a} \le \frac{2\,[d(Ix_{2n}, Jx_{2n-1}) + d(Ix_{2n}, Jx_{2n+1})]}{a}$$

and for 3<*a*≤4 we have

$$N(x_{2n-1}, x_{2n}) \le \max\left\{d(Ix_{2n}, Jx_{2n-1}), d(Ix_{2n}, Jx_{2n+1})\right\}.$$

Now, if $\max\left\{d(Ix_{2n}, Jx_{2n-1}), d(Ix_{2n}, Jx_{2n+1})\right\} = d(Ix_{2n}, Jx_{2n+1})$, by (4) we have

$\psi(d(Ix_{2n}, Jx_{2n+1})) \le \psi(N(x_{2n}, x_{2n-1})) - \varphi(N(x_{2n}, x_{2n-1})) \le \psi(d(Ix_{2n}, Jx_{2n+1})) - \varphi(d(Ix_{2n}, Jx_{2n+1}))$, a contradiction.

Hence, we have

$$N(x_{2n-1}, x_{2n}) \le \max\left\{d(Ix_{2n}, Jx_{2n-1}), d(Ix_{2n}, Jx_{2n+1})\right\} = d(Ix_{2n}, Jx_{2n-1}),$$

So,

$$d\left(Ix_{2n}, Jx_{2n+1}\right) \le N(x_{2n-1}, x_{2n}) \le d(Ix_{2n}, Jx_{2n-1}) \qquad (6)$$

Also

$d\left(Ix_{2n+2}, Jx_{2n+1}\right) \le H(Fx_{2n+2}, Gx_{2n+1})$, and

$$\psi(d(Ix_{2n+2}, Jx_{2n+1})) \le \psi(H(Fx_{2n}, Gx_{2n+1})) \le \psi(N(x_{2n}, x_{2n+1})) - \varphi(N(x_{2n}, x_{2n+1})) \le \psi(N(x_{2n}, x_{2n+1})) \qquad (7)$$

so $d\left(Ix_{2n+2}, Jx_{2n+1}\right) \le N(x_{2n+1}, x_{2n})$.

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In the same way, we have

$$d\left(Ix_{2n+2},Jx_{2n+1}\right)\le N(x_{2n},x_{2n+1})\le \max\left\{d(Ix_{2n},Jx_{2n+1}),d(Jx_{2n+1},Ix_{2n+2})\right\}=d(Ix_{2n},Jx_{2n+1})\,. \quad (8)$$

Let put for convenience,

$$y_k=\begin{cases} Ix_{2n} & k=2n \\ Jx_{2n-1} & k=2n-1\,. \end{cases}$$

By (6) and (8), we conclude that $d\left(y_{k+1},y_k\right)\le N(y_k,y_{k-1})\le d(y_k,y_{k-1})$, for $k\in N$.

It implies that the sequence $\left\{d\left(y_k,y_{k+1}\right)\right\}$ is monotone non-increasing and bounded below. So there exists $l\ge 0$ such that

$$\lim_{k\to\infty} d(y_k,y_{k+1})=\lim_{k\to\infty} N(y_k,y_{k+1})=l\ .$$

Since ψ is continuous and φ is lower semi-continuous,

$$\psi(l)=\lim_{k\to\infty}\psi(N(y_k,y_{k+1}) \qquad \varphi(l)\le \liminf_{k\to\infty}\varphi(N(y_k,y_{k+1})\ .$$

By (4), (7) and the property of the function ψ and φ we conclude that

$\psi(l)\le\psi(l)-\varphi(l)$ i.e. $\varphi(l)=0$ and $l=0$, and so $\lim_{k\to\infty} d(y_k,y_{k+1})=\lim_{k\to\infty} N(y_k,y_{k+1})=0$. (9)

Next we show that $\{y_k\}$ is a Cauchy sequence.

Let, $C_k=\sup\{d(y_i,y_j):i,j>k\}$.

Obviously $\{C_k\}$ is decreasing. So there exists $C\ge 0$ such that $\lim_{k\to\infty} C_k=C$.

For every $p\in N$, there exist $k(p)$, $s(p)\in N$ such that $k(p)$, $s(p)\ge p$ and

$$C_p-\frac{1}{p}\le d(y_{k(p)},y_{s(p)})\le C_p\ .$$

So, $\lim_{p\to\infty} d(y_{k(p)},y_{s(p)})=C$.

Let us to prove that $C=0$.

Case 1: If $k(p)$ is even and $s(p)$ is odd, so $k(p)=2q$ and $s(p)=2t-1$, then

$$d(y_{k(p)+1},y_{s(p)+1})=d(y_{2q+1},y_{2t})=d(Jx_{2q+1},Ix_{2t})\le H(Fx_{2q},Gx_{2t-1})$$

By (1), we have

$$\psi(d(y_{k(p)+1},y_{s(p)+1})=\psi(d(y_{2q+1},y_{2t}))=\psi(d(Jx_{2q+1},Ix_{2t}))\le\psi(H(Fx_{2q},Gx_{2t-1}))\le\psi(N(x_{2q},x_{2t-1}))-\varphi(N(x_{2q},x_{2t-1})) \quad (10)$$

So,

$$d(y_{k(p)+1},y_{s(p)+1})\le N(x_{2q},x_{2t-1})=N(y_{k(p)},y_{s(p)})$$

$$N(x_{2q},x_{2t-1})=\max\left\{d(Ix_{2q},Jx_{2t-1}),d(Ix_{2q},Fx_{2q}),d(Jx_{2t-1},Gx_{2t-1}),\frac{d(Ix_{2q},Gx_{2t-1})+d(Jx_{2t-1},Fx_{2q})}{a}\right\}$$

$$\le\max\left\{d(Ix_{2q},Jx_{2t-1}),d(Ix_{2q},Jx_{2q+1}),d(Jx_{2t-1},Ix_{2t}),\frac{d(Ix_{2q},Ix_{2t})+d(Jx_{2t-1},Jx_{2q+1})}{a}\right\}$$

$$=\max\left\{d(y_{k(p)},y_{s(p)}),d(y_{k(p)},y_{k(p)+1}),d(y_{s(p)},y_{s(p)+1}),\frac{d(y_{k(p)},y_{s(p)+1})+d(y_{s(p)},y_{k(p)+1})}{a}\right\} \quad (11)$$

As $p\to+\infty$ in inequality (11), we have $\lim_{p\to\infty} N(x_{2q},x_{2t-1})=\lim_{p\to\infty} N(y_{k(p)},y_{s(p)})=\max\{C,0,0,\frac{C+C}{a}\}=C$. (12)

Taking $p\to+\infty$ in inequality (10), since ψ is continuous and φ is lower semi-continuous and (9) and (12) hold, we have $\psi(C)\le\psi(C)-\varphi(C)$. Hence $\varphi(C)=0$ and so $C=0$.

So, $\lim_{p\to\infty} d(y_{k(p)}, y_{s(p)}) = 0$, for $k(p)$ is even and $s(p)$ is odd. (13)

Similarly, if $k(p)$ is odd and $s(p)$ is even.

Case 2: If $k(p)$ and $s(p)$ is even, so $k(p) = 2q$ and $s(p) = 2t$, then

$$d(y_{k(p)}, y_{s(p)}) = d(y_{2q}, y_{2t}) = d(Ix_{2q}, Ix_{2t}) \le d((Ix_{2q}, Jx_{2q+1}) + d(Jx_{2q+1}, Ix_{2t}) = d(y_{k(p)}, y_{k(p)+1}) + d(y_{k(p)+1}, y_{s(p)}),$$

Taking $p\to+\infty$ in this inequality, since (9) and (13) hold, we have $\lim_{p\to\infty} d(y_{k(p)}, y_{s(p)}) = 0$

Similarly, if $k(p)$ and $s(p)$ is odd.

Therefore, $\{y_k\}$ is a Cauchy sequence.

Since (X, d) is d-complete, there exists $u \in X$ such that $\lim_{k\to\infty} y_k = u$.

This implies $\lim_{n\to\infty} Ix_{2n} = \lim_{n\to\infty} Jx_{2n-1} = u$ and sequences $\{Fx_{2n}\}, \{Gx_{2n+1}\}$ converge at $\{u\}$. (14)

Suppose now that the set $J(X)$ is closed. Then there is, by condition (2), a point $v \in X$ such that $Jv = u$.

Using (1) we obtain

$$\psi(d(Fx_{2n}, Gv)) \le \psi(H(Fx_{2n}, Gv)) \le \psi(N(x_{2n}, v)) - \varphi(N(x_{2n}, v)) \ . \quad (15)$$

$$N(x_{2n}, v) = \max\left\{d(Ix_{2n}, Jv), d(Ix_{2n}, Fx_{2n}), d(Jv, Gv), \frac{d(Ix_{2n}, Gv) + d(Jv, Fx_{2n})}{a}\right\}$$

$$= \max\{d(Ix_{2n}, u), d(Ix_{2n}, Fx_{2n}), d(u, Gv), \frac{d(Ix_{2n}, Gv) + d(u, Fx_{2n})}{a}\}$$

Letting $n \to \infty$ in the above inequality and by (14) we get

$$\lim_{n\to\infty} N(x_{2n}, v) = \max\left\{d(u,u), d(u,\{u\}), d(u, Gv), \frac{d(u,\{u\}) + d(u, Gv)}{2}\right\} = \max\left\{d(u,u), d(u, Gv), \frac{d(u,Gv) + d(u,Gv) + d(u,Gv)}{a}\right\} = \max\{d(u,u), d(u,Gv)\} \quad (16)$$

Taking $n \to \infty$ in (5) and by (14), we have $0 = \max\{d(u,u), d(u,u)\} = d(u,u)$ and by (16) we have

$$\lim_{n\to\infty} N(x_{2n}, v) = d(u, Gv), \quad (17)$$

Letting $n \to \infty$ in (15) and by (17) we get, so $\varphi(d(u, Gv)) = 0$ and $d(u, Gv) = 0$.

Hence $Gv = \{u\} = \{Jv\}$. But the pair of maps $\{G, J\}$ are weakly compatible, thus $GJv = JGv$, i.e. $Gu = \{Ju\}$. (18)

Now we show that u is a fixed point for G and for J.

$$N(x_{2n}, u) = \max\left\{d(Ix_{2n}, Ju), d(Ix_{2n}, Fx_{2n}), d(Ju, Gu), \frac{d(Ix_{2n}, Gu) + d(Ju, Fx_{2n})}{a}\right\}.$$

Letting $n \to \infty$ in the above inequality and by (14), (18) and equality $0 = d(u,u)$ we get

$$\lim_{n\to\infty} N(x_{2n}, u) = \max\left\{d(u, Gu), d(u,\{u\}), d(Ju,\{Ju\}), \frac{d(u,Gu) + d(u,Gu)}{a}\right\} = d(u, Gu) \ . \quad (19)$$

By (1) we have $\psi(d(Jx_{2n+1}, Gu)) \le \psi(H(Fx_{2n}, Gu)) \le \psi(N(x_{2n}, u)) - \varphi(N(x_{2n}, u))$

Letting $n \to \infty$ in this inequality and by (19) we get $\psi(d(u,Gu)) \le \psi(H(u,Gu) \le \psi(d(u,Gu)) - \varphi(d(u,Gu))$, so $\varphi(d(u,Gu)) = 0$ and $d(u, Gu) = 0$.

Thus we have $Gu = \{u\} = \{Ju\}$ and u is a fixed point for G and J.

Now, $\cup G(X) \subseteq I(X)$ implies that exists $w \in X$ such that $Gu = \{Iw\}$. Hence $\{u\} = Gu = \{Ju\} = \{Iw\}$ and $H(Fw, u) = H(Fw, Gu)$

Using (1) we get

$$\psi(H(Fw,u)) = \psi(H(Fw,Gu)) \le \psi(N(w,u)) - \varphi(N(w,u))$$

and $N(w,u)=\max\left\{d(Iw,Ju),d(Iw,Fw),d(Ju,Gu),\dfrac{d(Iw,Gu)+d(Ju,Fw)}{a}\right\}=d(u,Fw)$.

So, $\psi(H(Fw,u))\le\psi(d(Fw,u))-\varphi(d(Fw,u))$ and $\varphi(d(Fw,u))=0$ and $d(Fw,u)=0$ and $\{u\}=Fw=\{Iw\}$.

Hence,

$\{u\}=Gu=\{Ju\}=Fw=\{Iw\}$.

But the pair of maps {*F,I*} are weakly compatible, thus $FIw=IFw,$ i.e. $Fu=\{Iu\}$.

Moreover,

$\psi(d(Fu,u))\le\psi(H(Fu,u))=\psi(H(Fu,Gu))\le\psi(N(u,u))-\varphi(N(u,u))$

and $N(u,u)=\max\left\{d(Iu,Ju),d(Iu,Fu),d(Ju,Gu),\dfrac{d(Iu,Gu)+d(Ju,Fu)}{a}\right\}=d(u,Fu)$.

So, $\psi(d(Fu,u))\le\psi(H(Fu,u))\le\psi(d(Fu,u))-\varphi(d(Fu,u))$ and $\varphi(d(Fu,u))=0$ and $d(Fu,u)=0$ and

$\{u\}=Gu=\{Ju\}=Fu=\{Iu\}$ i.e. *u* is e common fixed point for *F, G, I, J.*

Similarly, one can reach the above fact by assuming that *I*(*X*) is closed.

Furthermore, we can prove that *u* is unique. In fact, if *u* and *v* are two common fixed points for *F, G, I, J,* then by (1) we have

$\psi(d(u,v))\le\psi(H(u,v))=\psi(H(Fu,Gv))\le\psi(N(u,v))-\varphi(N(u,v))$

and $N(u,v)=\max\left\{d(Iu,Jv),d(Iu,Fu),d(Jv,Gv),\dfrac{d(Iu,Gv)+d(Jv,Fu)}{a}\right\}=d(u,v)$

So, $\psi(d(u,v))\le\psi(d(u,v))-\varphi(d(u,v))$ and $\varphi(d(u,v))=0$ and $d(u,v)=0$ and *u*=*v*. The proof is complete.

Remark. The above theorem extends Theorem 3.1 of [7]. If we take $\psi(t)=t$ for all $t\ge 0$ then the above theorem reduces to theorem 3.1. of [7].

Theorem 3.2. Let $F,G:X\to B(X)$ be mappings, where (X,d) is a complete dislocated metric space and let $J:X\to X$ be self mappings. Suppose that

(1) $\psi(H(Fx,Gy))\le\psi(N(x,y))-\varphi(N(x,y))$, where

$$N(x,y)=\max\left\{d(Jx,Jy),d(Jx,Fx),d(Jy,Gy),\frac{d(Jx,Gy)+d(Jy,Fx)}{a}\right\}$$

for all $x,y\in X$, $3<a\le 4$ and $\psi,\ \varphi:[0,+\infty[\to[0,+\infty[$, ψ is non-decreasing and continuous, φ is lower semi continuous function, $\psi(t)>0,\ \varphi(t)>0$ for all *t>0* and $\psi(0)=\varphi(0)=0$.

(2) $\cup G(X)\subseteq J(X)$, $\cup F(X)\subseteq J(X)$ and *J*(*X*) is closed

(3) the pairs of mappings {*F*, *J*} and {*G*, *J*} are weakly compatible,

then *F, G, J* have a unique common fixed point $u\in X$.

Proof. *F*, *G*, *J* complete the condition in Theorem 3.1 by taking $I=J$. Hence *F*, *G*, *J* have a common fixed point $u\in X$

Remark. This theorem extends the Ciric and Ume theorem (Theorem 2.1.[5]

Theorem 3.3. Let (X,d) be a complete d-metric space, and let $F,G:X\to B(X)$ be two multi-valued mappings such that for all $x,y\in X$

(1) $\psi(H(Fx,Gy))\le\psi(M(x,y))-\varphi(M(x,y)$

where $M(x,y)=\max\left\{d(x,y),d(x,Fx),d(y,Gy),\dfrac{d(x,Gy)+d(y,Fx)}{a}\right\}$

for all $x,y\in X$, $3<a\le 4$ and $\psi,\ \varphi:[0,+\infty[\to[0,+\infty[$, ψ is non-decreasing and continuous, φ is lower semi continuous function, $\psi(t)>0,\ \varphi(t)>0$ for all *t>0* and $\psi(0)=\varphi(0)=0$.

then *F, G,* have a unique common fixed point $u\in X$. Moreover, *Fu* = *Gu* = {*u*}.

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Proof. By taking $I = J = I_X$ we have $\cup G(X) \subseteq I(X) = X$, $\cup F(X) \subseteq J(X) = X$ and *X* is closed. Hence *F*, *G*, *I*, *J* complete the condition in Theorem 3.1 and *F*, *G*, have a common fixed point $u \in X$.

Remark. This theorem extends the Bose and Roychowdhury theorem (Theorem 51. [4]).

Theorem 3.4. Let (X,d) be a complete d-metric space, and let $F,G,I,J : X \to X$ four mappings such that for all $x,y \in X$

(1) $\psi(H(Fx,Gy)) \le \psi(N(x,y)) - \varphi(N(x,y)$

where $N(x,y) = \max\left\{d(Ix,Jy), d(Ix,Fx), d(Jy,Gy), \frac{d(Ix,Gy)+d(Jy,Fx)}{a}\right\}$

for all $x,y \in X$, $3 < a \le 4$ and ψ, $\varphi : [0,+\infty[\to [0,+\infty[$, ψ is non-decreasing and continuous, φ is lower semi continuous function, $\psi(t) > 0$, $\varphi(t) > 0$ for all *t>0* and $\psi(0) = \varphi(0) = 0$.

(2) $G(X) \subseteq I(X)$, $F(X) \subseteq J(X)$ and either *I*(*X*) or *J*(*X*) is closed

(3) the pairs of mappings {*F*, *I*} and {*G*, *J*} are weakly compatible,

then *F, G,I, J* have a unique common fixed point $u \in X$.

Proof. The proof of this theorem is similar to the Theorem 3.1 for $Gx = \{Gx\}$, $Fx = \{Fx\}$.

Remark. If we take $\psi(t)=t$ and $\varphi(t)=(1-\alpha)t$ for all $t\geq 0$ then the above theorem reduces to theorem 3.1. of Kumari and Kumar and Sarma theorem (Theorem 3.1. [14]).

Theorem 3.5. Let (X,d) be a complete d-metric space, and let $F,G : X \to X$ two mappings such that for all $x,y \in X$

(1) $\psi(H(Fx,Gy)) \le \psi(M(x,y)) - \varphi(M(x,y)$

where $M(x,y) = \max\left\{d(x,y), d(x,Fx), d(y,Gy), \frac{d(x,Gy)+d(y,Fx)}{a}\right\}$

for all $x,y \in X$, $3 < a \le 4$ and ψ, $\varphi : [0,+\infty[\to [0,+\infty[$, ψ is non-decreasing and continuous, φ is lower semi continuous function, $\psi(t) > 0$, $\varphi(t) > 0$ for all *t>0* and $\psi(0) = \varphi(0) = 0$.

then *F, G* have a unique common fixed point $u \in X$.

Proof. The proof of this theorem is similar to the Theorem 3.1 for $Gx = \{Gx\}$, $Fx = \{Fx\}$ and $I = J = I_X$

Remark. This theorem extends the Shrivastava and Ansari and Sharma theorem (Theorem 3.7.[16])

Theorem 3.6. Let (X,d) be a complete d-metric spaces and *F: X* $\to$ *X* a self-map, such that for all $x,y \in X$

(1) $\psi(H(Fx,Fy)) \le \psi(M(x,y)) - \varphi(M(x,y)$

where $M(x,y) = \max\left\{d(x,y), d(x,Fx), d(y,Fy), \frac{d(x,Fy)+d(y,Fx)}{a}\right\}$

for all $x,y \in X$, $3 < a \le 4$ and ψ, $\varphi : [0,+\infty[\to [0,+\infty[$, ψ is non-decreasing and continuous, φ is lower semi continuous function, $\psi(t) > 0$, $\varphi(t) > 0$ for all *t>0* and $\psi(0) = \varphi(0) = 0$.

then *F* have a unique common fixed point $u \in X$.

Proof. The proof of this theorem is similar to the Theorem 3.1 for $Fx = Gx = \{Fx\}$ and $I = J = I_X$.

Remark. Theorem 3.5 extends the Karapinar and Salimi theorem (Theorem1.9. [12]).

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