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The gap functions and error bounds of solutions of a class of set-valued mixed variational inequality and related algorithms

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ABSTRACT

This paper generalizedizes a class of generalizedized mixed variational inequality problem, and we study the gap functions, error bounds of solutions and related algorithms of a class of set-valued mixed variational inequalities. In order tosolve our problem, we establish the corresponding generalized resolvent equations and prove the equivalence. Finally, we give three iterative algorithms and analyze the convergence of algorithms.

Keywords: variational inequality; gap functions; error bounds; resolvent equations; iterative algorithms

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INTRODUCTION

Let H be a real Hilbert space, whose inner product and norm are denoted by $<.,.>$ and $\|.\|$ respectively.Let S be a non-empty closed convex set in H and $T:H\to C(H)$ be a multivalued operator, where $C(H)$ is the nonempty compact set in H, $g:H\to H$ is a

single-valued operator. Let $\partial\varphi$ be the subdifferential of a proper, convex and lower

semicontinuous function $\varphi:H\to R\cup\{+\infty\}$ with dom $\partial\varphi\neq\phi$.

We consider the problem of finding $u\in H, w\in T(u)$ such that

$$<w,v-g(u)>+\varphi(v)-\varphi(g(u))\geq 0\text{, for all } v\in H. \quad (1.1)$$

The problem (1.1) is called the generalizedized set-valued mixed variational inequality problem, the problem (1.1) is denoted by $GSMVIP(T,g,\varphi)$.

The problem (1.1) contains several typical variational inequality problems.

(1) If T: $H\to H$ is a single-valued mapping,the problem (1.1) is equivalent to finding $u\in H$ such that

$$<T(u),v-g(u)>+\varphi(v)-\varphi(g(u))\geq 0\text{, for all } v\in H. \quad (1.2)$$

The problem (1.2) is known as the generalizedized mixed variational inequality problem.

Hence, our work generalizedizes the problem (1.2).

(2) If g is a identity mapping, the problem (1.1) is equivalent to finding $u\in S, w\in T(u)$ such that

$$<w,v-u>+\varphi(v)-\varphi(u)\geq 0\text{,for all } v\in S. \quad (1.3)$$

The problem (1.3) is called set-valued mixed variational inequality problem.

(3) If φ is the indicator function of S such that

$$\varphi(u)=\begin{cases}0 & ,u\in S\\ +\infty, & u\notin S\end{cases}$$

the problem (1.1) is equivalent to finding $u\in S, w\in T(u)$ such that

$$<w,v-g(u)>-\varphi(g(u))\geq 0\text{,for all } v\in S.$$

(4) If φ is the indicator function of S and g is a identity mapping, the problem (1.1) is equivalent to finding $u\in S, w\in T(u)$ such that

$$<w,v-u>\geq 0\text{,for all } v\in S. \quad (1.4)$$

(5) If T: $H\to H$ is a single-valued mapping, φ is the indicator function of S and g is a identity mapping, the problem (1.1) is equivalent to finding $u\in S$ such that

$$<T(u),v-u>\geq 0,\forall v\in S \quad (1.5)$$

The problem (1.5 is the typical variational inequality problem of $VIP(T,S)$.

In order to solve the the problem of $VIP(T,S)$, in the reference of[6], Wu、$Florian$ and $Marcotte$ define a gap function $G_\alpha:H\to R$ such that

$$G_\alpha(u)=\max_{v\in S}\psi_\alpha(u,v)=\max_{v\in S}\{<T(u),u-v>-\alpha\phi(u,v)\},$$

Where, $\alpha>0$,the function $\phi:H\times H\to R$ satisfies the following conditions:

$C.1$ ϕ is continuously differentiable on $H \times H$;

$C.2$ ϕ is nonnegative on $H \times H$;

$C.3$ $\phi(u,.)$ is strongly convex uniformly on H .i.e. there exists a constant $\lambda > 0$ such that

$$\phi(u,v_1) - \phi(u,v_2) \geq < \nabla_2 \phi(u,v_2), v_1 - v_2 > + \lambda \|v_1 - v_2\|^2, \forall v_1, v_2 \in H,$$

for any $u \in H$.

$C.4$ $\phi(u,v) = 0$ if and only if $u = v$;

Wu proved that the function G_α is equivalent to the optimization of $VIP(T,S)$,and $G_\alpha(u) = 0$, $u \in S$ if and only if u is the solution of $VIP(T,S)$.

In order to solve the problem (1.1), this paper will construct the gap function and $D-$ gap function of $GSMVIP(T,g,\varphi)$ by using the function ϕ which satisfies $C.1 - C.5$, and we will study the properties of gap functions and give the error bounds of the problem (1.1)

One of the most difficult and important problem in variational inequality theory is the development of finding efficient algorithm. There are a lot of algorithms including the projection technique, principle technique, Newton and descent framework and so on.

In the reference [17], Noor considers the following problem:

Let H be a real Hilbert space, whose inner product and norm are denoted by $<.,.>$ and $\|.\|$ respectively. $T, A, g : H \to H$ are nonlinear operators. Let $\partial\varphi$ be the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ with dom $\partial\varphi \neq \phi$, $g(H) \cap dom(\partial\varphi) \neq \phi$.

Consider the problem of finding $u \in H$ such that

$$< T(u) + A(u), v - g(u) > + \varphi(v) - \varphi(g(u)) \geq 0 \text{ for all } v \in H. \quad (1.6)$$

In order to solve this variational inequality problem, Noor introduces resolvent operator and constructs the resolvent equatons.

For given nonlinear operators $T, A, g : H \to H$, consider the problem of finding $u \in H$ such that

$$Tg^{-1}J_\varphi z + \rho^{-1} R_\varphi z = -A(g^{-1}J_\varphi z). \quad (1.7)$$

Where $R_\varphi = I - J_\varphi$, I is identity operator, $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is resolvent operator, $\rho > 0$ is a constant and g^{-1} is the inverse of the operator g.

(1.7) is called the generalized resolvent equations of (1.6) .It has been shown in [17] that (1.6)and (1.7) are equivalent and Noor gived the algorithms of solving the problem (1.6).

In order to solve the problem (1.1), we will also use the resolvent operator and construct the generalized resolvent equatons of the problem (1.1).

2.Definitions and assumptions

Definition 2.1 The set-valued mapping $T : H \to C(H)$ is said to be $g-$strongly monotone with module δ, if there exists a constant $\delta > 0$ such that

$$< v_1 - v_2, g(u_1) - g(u_2) > \geq \delta \|u_1 - u_2\|^2, \text{ for all } u_1, u_2 \in H, \ v_1 \in T(u_1), v_2 \in T(u_2).$$

Definition 2.2 [1] The set-valued mapping $T : K \subset H \to 2^H$ is said to be $Lipschitz$ continuous with module L on the subset U of the set K, if there exists a constant $L > 0$ such that

$$H(T(u_1), T(u_2)) \leq L\|u_1 - u_2\|, \ \forall u_1, u_2 \in U$$

where $H(.,.)$ is the $Hausdorff$ metric on $C(H)$. i.e.

$$H(T(u_1),T(u_2)) = \max\{ \sup_{x\in T(u_1)} \inf_{y\in T(u_2)} \|x-y\|, \sup_{y\in T(u_2)} \inf_{x\in T(u_1)} \|x-y\|\}, \forall u_1,u_2 \in U$$

Definition 2.3 [1] The mapping $g: H \to H$ is said to be $Lipschitz$ continuous with module t on H, if there exists a constant $t > 0$ such that

$$\|g(u_1)-g(u_2)\| \le t\|u_1-u_2\|, \forall u_1,u_2 \in H$$

Definition 2.4 [1] The function $f: H \to R$ is said to be strongly convex uniformly, if there exists a constant $d > 0$ such that

$$f(v_1) - f(v_2) \ge < \nabla f(v_2), v_1 - v_2 > + d\|v_1 - v_2\|^2, \forall v_1, v_2 \in H.$$

Remark 2.1 If the function f is strongly convex uniformly on the convex set $S \subseteq H$, then f has a unique minimizer on the convex set S.

Lemma 2.1 [7] If the function ϕ satisfies $C.1 - C.4$. Then $\nabla \phi_2(u,v) = 0$ if and only if $u = v$.

Lemma 2.2 [14] If the function ϕ satisfies $C.3$, then we have $v_1, v_2 \in H$

$$< \nabla_2\phi(u,v_1) - \nabla_2\phi(u,v_2), v_1 - v_2 > \ge 2\lambda\|v_1 - v_2\|^2 \text{ for all } v_1, v_2 \in H.$$

Sometimes the function ϕ need to satisfy the following condition:

$C.5$ $\nabla_2\phi(u,.)$ is $Lipschitz$ continuous uniformly. i.e. there exists a constant $k > 0$ such that for any $u \in H$ we have

$$\|\nabla_2\phi(u,v_1) - \nabla_2\phi(u,v_2)\| \le k\|v_1 - v_2\|, \text{ for all } v_1, v_2 \in H.$$

Lemma 2.3 [8] If the function ϕ satisfies $C.1 - C.5$, then we have

$$\lambda\|u-v\|^2 \le \phi(u,v) \le (k-\lambda)\|u-v\|^2, \text{ for any } u, v \in H.$$

Definition 2.5 [15] If T is a maximal monotone operator from H into 2^H, then the resolvent operator associated with T is difined by $J_T(u) = (I + \rho T)^{-1}(u)$, for all $u \in H$. Where $\rho > 0$ is a constant and I is the identity operator. [15]

Lemma 2.4 For a given $u \in H$, $z \in H$ satisfies the inequality $(u - z, v - u) + \rho\varphi(v) - \rho\varphi(u) \ge 0$ for all $v \in H$, if and only if

$$u = J_\varphi(z),$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant, I is the identity operator.

Definition 2.6 [16] J_φ is a nonexpansive operator, that is,

$$\|J_\varphi(v) - J_\varphi(u)\| \le \|v - u\| \text{ for all } u \in H, \ v \in H.$$

Definition 2.7 [16] The operator $T: H \to C(H)$ is said to be

(1) $\alpha-$strongly monotone, if there exists a constant $\alpha > 0$ such that

$$< w_1 - w_2, u_1 - u_2 > \ge \alpha\|u_1 - u_2\|^2, \text{ for all } u_1, u_2 \in H, \ w_1 \in T(u_1), w_2 \in T(u_2).$$

(2) $H - \beta - Lipschitz$ continuous, if there exists a constant $\beta > 0$ such that

$$H(T(u_1),T(u_2)) \le \beta \|u_1 - u_2\|, \text{ for all } u_1, u_2 \in H,$$

where $H(.,.)$ is the $Hausdorff$ metric on $C(H)$.

Definition 2.8 The operator $T: H \to C(H)$ is said to be $\alpha - g$ strongly monotone, if there exists a constant $\alpha > 0$ such that

$$< w_1 - w_2, g(u_1) - g(u_2) > \ge \alpha \|u_1 - u_2\|^2, \ \forall u_1, u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2)$$

Definition 2.9 [17] The mapping $g: H \to H$ is said to be

(i) $\delta - Lipschitz$ continuous, if there exists a constant $\delta > 0$ such that

$$\|g(u) - g(v)\| \le \delta \|u - v\|, \text{ for all } u, v \in H.$$

(ii) σ strongly monotone, if there exists a constant $\sigma > 0$ such that

$$< g(u) - g(v), u - v > \ge \sigma \|u - v\|^2, \text{ for all } u, v \in H.$$

3 The gap function and error bounds of $GSMVIP(T, g, \varphi)$

We construct the gap function as follows:

$$G_\alpha(u) = \inf_{w_u \in T(u)} \sup_{v \in H} \psi_\alpha(w_u, u, v)$$
$$= \inf_{w_u \in T(u)} \sup_{v \in H} \{< w_u, g(u) - v > + \varphi(g(u)) - \varphi(v) - \alpha \phi(g(u), v)\}$$

for all $u \in H$.

Where $\alpha > 0$ is a constant, for given $u \in H$, any w_u is choosed from the set $T(u)$, the function ψ_α is given as follows:

$$\psi_\alpha(w_u, u, v) = < w_u, g(u) - v > + \varphi(g(u)) - \varphi(v) - \alpha \phi(g(u), v)$$

Lemma 3.1 If the function $\phi(u,.)$ is uniformly convex, for given $w_u \in T(u)$, then $-\psi_\alpha(w_u, u, .)$ is uniformly convex on H.

Proof :

Since $\phi(u,.)$ is uniformly convex, so, for any $u \in H$, there exists $\lambda > 0$ such that

$$\phi(g(u), v_1) - \phi(g(u), v_2) \ge < \nabla_2 \phi(g(u), v_2), v_1 - v_2 > + \lambda \|v_1 - v_2\|^2, \forall v_1, v_2 \in H.$$

Let

$$F(v) = -\psi_\alpha(w_u, u, v) = < w_u, v - g(u) > + \varphi(v) - \varphi(g(u)) + \alpha \phi(g(u), v).$$

We have

$$\begin{aligned}F(v_1)-F(v_2) &=< w_u, v_1-g(u)> +\varphi(v_1)-\varphi(g(u))+\alpha\phi(g(u),v_1)\\ &\quad -< w_u, v_2-g(u)> -\varphi(v_2)+\varphi(g(u))-\alpha\phi(g(u),v_2)\\ &=< w_u, v_1-v_2> +\varphi(v_1)-\varphi(v_2)+\alpha\{\phi(g(u),v_1)-\phi(g(u),v_2)\}\\ &\geq < w_u, v_1-v_2> +\varphi(v_1)-\varphi(v_2)+\alpha\{<\nabla_2\phi(g(u),v_2), v_1-v_2> +\lambda\|v_1-v_2\|^2\}\\ &=< w_u+\alpha\nabla_2\phi(g(u),v_2), v_1-v_2> +\varphi(v_1)-\varphi(v_2)+\alpha\lambda\|v_1-v_2\|^2\\ &\geq < w_u+\alpha\nabla_2\phi(g(u),v_2), v_1-v_2> + < x, v_1-v_2> +\alpha\lambda\|v_1-v_2\|^2\\ &=< w_u+x+\alpha\nabla_2\phi(g(u),v_2), v_1-v_2> +\alpha\lambda\|v_1-v_2\|^2\\ &=<\nabla F(v_2), v_1-v_2> +\alpha\lambda\|v_1-v_2\|^2\end{aligned}$$

Where $\alpha\lambda > 0$ and $x \in \partial\varphi(v_2)$. Hence , $-\psi_\alpha(w_u,u,.)$ is uniformly convex on H .

By Lemma 2.4， $-\psi_\alpha(w_u,u,.)$ is uniformly convex on H . Hence， for given $u \in H$ and $w_u \in T(u)$ ， $-\psi_\alpha(w_u,u,.)$ has a unique minimizer $v_\alpha(u,w_u)$.

For given $u \in H$ ， while $w_u = w^*$, $v = v_\alpha(u,w^*)$ ， the function G_α get the function value，then $G_\alpha(u)$ can be denoted as follows:

$$G_\alpha(u) =< w^*, g(u)-v_\alpha(u,w^*)> +\varphi(g(u))-\varphi(v_\alpha(u,w^*))-\alpha\phi(g(u),v_\alpha(u,w^*)) \quad (3.1)$$ where $v_\alpha(u,w^*)$ is the minimizer of $-\psi_\alpha(w^*,u,.)$ on H after givn $u \in H$ and while $w_u = w^*$.

Remark 3.1 Ifthere is not other explanations, throughout this paper, w^* and $v_\alpha(u,w^*)$ denote the above meaning.

Lemma 3.2 If the function ϕ satisfies $C.1-C.4$ .Then $u \in H$ and $w_u \in T(u)$ is the solution of $GSMVIP(T,g,\varphi)$ if and only if $g(u) = v_\alpha(u,w_u)$ for any constant $\alpha > 0$.

Proof :

$(\Leftarrow)$:For any $u \in H$ ， $w_u \in T(u)$ .Suppose that $v_\alpha(u,w_u)$ is the minimizer of $-\psi_\alpha(w_u,u,.)$ on H . We have

$$0 \in \partial(-\psi_\alpha(w_u,u,v_\alpha(u,w_u))) = w_u + \partial\varphi(v_\alpha(u,w_u)) + \alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)) .$$

Hence

$$-w_u - \alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)) \in \partial\varphi(v_\alpha(u,w_u)) .$$

By the definition of subgradient, we have

$$\varphi(v) \geq \varphi(v_\alpha(u,w_u)) + < -w_u - \alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)), v - v_\alpha(u,w_u) >$$

for any $v \in H$.

By above inequality, we have

$$< w_u, v-v_\alpha(u,w_u)> +\varphi(v)-\varphi(v_\alpha(u,w_u)) \geq -< \alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)), v-v_\alpha(u,w_u)> \quad (3.2)$$ for any $v \in H$.

If $g(u) = v_\alpha(u,w_u)$ ，by（3.2）and Lemma 2.1, we have

$$< w_u, v-g(u)> +\varphi(v)-\varphi(g(u)) \geq 0, \forall v \in H .$$

Hence， $u \in H, w_u \in T(u)$ is the solution of $GSMVIP(T,g,\varphi)$.

$(\Rightarrow)$:Suppose that $u \in H, w_u \in T(u)$ is the solution of $GSMVIP(T,g,\varphi)$, we have

$$< w_u, v - g(u) > + \varphi(v) - \varphi(g(u)) \geq 0, \forall v \in H .$$

Set $v = v_\alpha(u, w_u)$, we have

$$< w_u, v_\alpha(u, w_u) - g(u) > + \varphi(v_\alpha(u, w_u)) - \varphi(g(u)) \geq 0, \forall v \in H . \quad (3.3)$$

Since $g(u) \in H$, let $v = g(u)$, by (3.2), we have

$$< w_u, g(u) - v_\alpha(u, w_u) > + \varphi(g(u)) - \varphi(v_\alpha(u, w_u)) \geq - < \alpha \nabla_2 \phi(g(u), v_\alpha(u, w_u)), g(u) - v_\alpha(u, w_u) > . \quad (3.4)$$

By (3.3) and (3.4), we have

$$< \alpha \nabla_2 \phi(g(u), v_\alpha(u, w_u)), g(u) - v_\alpha(u, w_u) > \geq 0 .$$

Since $\alpha > 0$, so

$$< \nabla_2 \phi(g(u), v_\alpha(u, w_u)), g(u) - v_\alpha(u, w_u) > \geq 0 .$$

Since

$$\lambda \| g(u) - v_\alpha(u, w_u) \|^2 \geq 0 .$$

Hence

$$< \nabla_2 \phi(g(u), v_\alpha(u, w_u)), g(u) - v_\alpha(u, w_u) > + \lambda \| g(u) - v_\alpha(u, w_u) \|^2 \geq 0 .$$

Hence, by $C.2$, $C.4$ and the strongly convex of $\phi(u,.)$, we have

$$\begin{aligned} 0 &\leq < \nabla_2 \phi(g(u), v_\alpha(u, w_u)), g(u) - v_\alpha(u, w_u) > + \lambda \| g(u) - v_\alpha(u, w_u) \|^2 \\ &\leq \phi(g(u), g(u)) - \phi(g(u), v_\alpha(u, w_u)) \\ &\leq 0 \end{aligned}$$

i.e.

$$\phi(g(u), v_\alpha(u, w_u) = 0 .$$

Hence, by $C.4$, we have

$$g(u) = v_\alpha(u, w_u)$$

Theorem 3.1 If the function ϕ satisfies $C.1 - C.5$, and $\overset{=}{u} \in H, \overset{=}{w} \in T(\overset{=}{u})$ is the solution of $GSMVIP(T, g, \varphi)$, T is $g-$strongly monotone with module δ on H, $\delta > 0$, T, g are $Lipschitz$ continuous on H with respective module $L, t > 0$. Then for any $u \in H$, there exists $w_u \in T(u)$ such that

$$\left\| w_u - \overset{=}{w} \right\| \leq H(T(u), T(\overset{=}{u}))$$

and, while $\left\| w_u - \overset{=}{w} \right\| \leq H(T(u), T(\overset{=}{u}))$, we have

$$\left\| u - \overset{=}{u} \right\| \leq \frac{L + \alpha k t}{\delta} \| g(u) - v_\alpha(u, w_u) \|, \quad \forall u \in H, \quad \alpha > 0$$

$$\left\| w_u - \overset{=}{w} \right\| \leq \frac{L(L + \alpha k t)}{\delta} \| g(u) - v_\alpha(u, w_u) \|, \quad \forall u \in H, \quad \alpha > 0$$

Proof :

Suppose that $\overset{=}{u}\in H,\overset{=}{w}\in T(\overset{=}{u})$ is the solution of $GSMVIP(T,g,\varphi)$, then for any $u\in H$, by the difinition of $Hausdorff$ metric, there exists $w_u\in T(u)$ such that

$$\left\|w_u-\overset{=}{w}\right\|\le H(T(u),T(\overset{=}{u})). \quad (3.5)$$

If $\overset{=}{u}\in H,\overset{=}{w}\in T(\overset{=}{u})$ is the solution of $GSMVIP(T,g,\varphi)$, then we have

$$<\overset{=}{w},v-g(\overset{=}{u})>+\varphi(v)-\varphi(g(\overset{=}{u}))\ge 0,\forall v\in H. \quad (3.6)$$

For any $u\in H$, $w_u\in T(u)$, set $v=v_\alpha(u,w_u)\in H$, by (3.6), we have

$$<\overset{=}{w},v_\alpha(u,w_u)-g(\overset{=}{u})>+\varphi(v_\alpha(u,w_u))-\varphi(g(\overset{=}{u}))\ge 0,\ \forall u\in H,w_u\in T(u). \quad (3.7)$$

Set $v=g(\overset{=}{u})$, by (3.2), we have

$$<w_u,g(\overset{=}{u})-v_\alpha(u,w_u)>+\varphi(g(\overset{=}{u}))-\varphi(v_\alpha(u,w_u))\ge -<\alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)),g(\overset{=}{u})-v_\alpha(u,w_u)>. \quad (3.8)$$

By (3.7) and (3.8), we have

$$<w_u-\overset{=}{w},v_\alpha(u,w_u)-g(\overset{=}{u})>\le<\alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)),g(\overset{=}{u})-v_\alpha(u,w_u)> \quad (3.9)$$

$$\begin{aligned}
&<\alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)),g(\overset{=}{u})-v_\alpha(u,w_u)>\\
&=\alpha<\nabla_2\phi(g(u),v_\alpha(u,w_u)),g(\overset{=}{u})-g(u)>\\
&\quad+\alpha<\nabla_2\phi(g(u),v_\alpha(u,w_u)),g(u)-v_\alpha(u,w_u)>\\
&=\alpha<\nabla_2\phi(g(u),v_\alpha(u,w_u))-\nabla_2\phi(g(u),g(u)),g(\overset{=}{u})-g(u)>\\
&\quad-\alpha<\nabla_2\phi(g(u),g(u))-\nabla_2\phi(g(u),v_\alpha(u,w_u)),g(u)-v_\alpha(u,w_u)>\\
&\le\alpha\left\|\nabla_2\phi(g(u),v_\alpha(u,w_u))-\nabla_2\phi(g(u),g(u))\right\|\left\|g(\overset{=}{u})-g(u)\right\|\\
&\quad-2\alpha\lambda\left\|g(u)-v_\alpha(u,w_u)\right\|^2\\
&\le\alpha k\left\|g(\overset{=}{u})-g(u)\right\|\left\|g(u)-v_\alpha(u,w_u)\right\|-2\alpha\lambda\left\|g(u)-v_\alpha(u,w_u)\right\|^2\\
&\le\alpha kt\left\|\overset{=}{u}-u\right\|\left\|g(u)-v_\alpha(u,w_u)\right\|-2\alpha\lambda\left\|g(u)-v_\alpha(u,w_u)\right\|^2\\
&\le\alpha kt\left\|\overset{=}{u}-u\right\|\left\|g(u)-v_\alpha(u,w_u)\right\|.
\end{aligned} \quad (3.10)$$

By (3.9) and (3.10), we have

$$<w_u-\overset{=}{w},v_\alpha(u,w_u)-g(\overset{=}{u})>\le\alpha kt\left\|\overset{=}{u}-u\right\|\left\|g(u)-v_\alpha(u,w_u)\right\|. \quad (3.11)$$

Since T is $Lipschitz$ continuous on H, we have

$$H(T(u),T(\bar{\bar{u}})) \le L\left\|u-\bar{\bar{u}}\right\|. \qquad (3.12)$$

By (3.5) and (3.12), we have

$$\left\|w_u-\bar{\bar{w}}\right\| \le L\left\|u-\bar{\bar{u}}\right\|. \qquad (3.13)$$

By (3.11)、(3.13) and T is $g-$strongly monotone with module δ on H, we have

$$\begin{aligned}
\delta\left\|\bar{\bar{u}}-u\right\|^2 &\le < w_u-\bar{\bar{w}}, g(u)-g(\bar{\bar{u}})> \\
&\le < w_u-\bar{\bar{w}}, g(u)-v_\alpha(u,w_u)> + < w_u-\bar{\bar{w}}, v_\alpha(u,w_u)-g(\bar{\bar{u}})> \\
&\le \left\|w_u-\bar{\bar{w}}\right\|\left\|g(u)-v_\alpha(u,w_u)\right\| + < w_u-\bar{\bar{w}}, v_\alpha(u,w_u)-g(\bar{\bar{u}})> \\
&\le L\left\|\bar{\bar{u}}-u\right\| \ \left\|g(u)-v_\alpha(u,w_u)\right\| + \alpha kt\left\|\bar{\bar{u}}-u\right\| \ \left\|g(u)-v_\alpha(u,w_u)\right\| \\
&= (L+\alpha kt)\left\|\bar{\bar{u}}-u\right\| \ \left\|g(u)-v_\alpha(u,w_u)\right\|
\end{aligned}$$

i.e.

$$\left\|u-\bar{\bar{u}}\right\| \le \frac{L+\alpha kt}{\delta}\left\|g(u)-v_\alpha(u,w_u)\right\|, \quad \forall u\in H, \quad \alpha>0. \qquad (3.14)$$

By (3.13) and (3.14), we have

$$\left\|w_u-\bar{\bar{w}}\right\| \le \frac{L(L+\alpha kt)}{\delta}\left\|g(u)-v_\alpha(u,w_u)\right\|, \quad \forall u\in H, \quad \alpha>0.$$

Lemma 3.3 If the function ϕ satisfy $C.1-C.4$, for any $u\in H, \alpha>0$, while $w_u = w^* \in T(u)$, we have

$$G_\alpha(u) \ge \alpha\lambda\left\|g(u)-v_\alpha(u,w^*)\right\|^2 \ge 0.$$

If $G_\alpha(u)=0$, if and only if $u\in H, w^*\in T(u)$ is the solution of $GSMVIP(T,g,\varphi)$.

Proof :

For any $u\in H, w_u\in T(u), \alpha>0$, set $v=g(u)$, by (3.2), we have

$$< w_u, g(u)-v_\alpha(u,w_u)> + \varphi(g(u)) - \varphi(v_\alpha(u,w_u)) \ge -<\alpha\nabla_2\phi(g(u),v_\alpha(u,w_u)), g(u)-v_\alpha(u,w_u)>. \qquad (3.15)$$

While $w_u = w^* \in T(u)$, by (3.1) and (3.15), we have

$$\begin{aligned}
G_\alpha(u) &= < w^*, g(u)-v_\alpha(u,w^*)> + \varphi(g(u)) - \varphi(v_\alpha(u,w^*)) - \alpha\phi(g(u),v_\alpha(u,w^*)) \\
&\ge -<\alpha\nabla_2\phi(g(u),v_\alpha(u,w^*)), g(u)-v_\alpha(u,w^*)> - \alpha\phi(g(u),v_\alpha(u,w^*)) \\
&= -\{<\alpha\nabla_2\phi(g(u),v_\alpha(u,w^*)), g(u)-v_\alpha(u,w^*)> + \alpha\phi(g(u),v_\alpha(u,w^*))\} \\
&\ge \alpha\{-\phi(g(u),g(u)) + \lambda\left\|g(u)-v_\alpha(u,w^*)\right\|^2\} \\
&= \alpha\lambda\left\|g(u)-v_\alpha(u,w^*)\right\|^2.
\end{aligned}$$

Since $\alpha > 0$, $\lambda > 0$, we have

$$\alpha\lambda > 0.$$

Hence

$$G_\alpha(u) \geq \alpha\lambda \left\| g(u) - v_\alpha(u, w^*) \right\|^2 \geq 0.$$

Next we will prove the last part of Lemma 3.3.

$(\Rightarrow)$: If $G_\alpha(u) = 0$, we have $g(u) = v_\alpha(u, w^*)$. By Lemma 3.2, we have $u \in H, w^* \in T(u)$ is the solution of $GSMVIP(T, g, \varphi)$.

$(\Leftarrow)$: If $u \in H, w^* \in T(u)$ is the solution of $GSMVIP(T, g, \varphi)$, by Lemma 3.2, we have

$$g(u) = v_\alpha(u, w^*). \qquad (3.16)$$

By (3.1) and (3.16), we have

$$G_\alpha(u) = < w^*, g(u) - v_\alpha(u, w^*) > + \varphi(g(u)) - \varphi(v_\alpha(u, w^*)) - \alpha\phi(g(u), v_\alpha(u, w^*)) = 0.$$

Theorem 3.2 If the function ϕ satisfies $C.1 - C.5$, $\bar{\bar{u}} \in H, \bar{\bar{w}} \in T(\bar{\bar{u}})$ is the solution of $GSMVIP(T, g, \varphi)$, T is $g-$strongly monotone with module δ on H, $\delta > 0$, T, g are $Lipschitz$ continuous on H with respective module $L, t > 0$. Then for any $u \in H$, there exists $w_u \in T(u)$ such that

$$\left\| w_u - \bar{\bar{w}} \right\| \leq H(T(u), T(\bar{\bar{u}}))$$

especially, while $\left\| w_u - \bar{\bar{w}} \right\| \leq H(T(u), T(\bar{\bar{u}}))$ and $w_u = w^*$, we have

$$\left\| u - \bar{\bar{u}} \right\| \leq \frac{L + \alpha kt}{\delta\sqrt{\alpha\lambda}} \sqrt{G_\alpha(u)}, \quad \forall u \in H, \quad \alpha > 0$$

$$\left\| w_u - \bar{\bar{w}} \right\| \leq \frac{L(L + \alpha kt)}{\delta\sqrt{\alpha\lambda}} \sqrt{G_\alpha(u)}, \quad \forall u \in H, \quad \alpha > 0$$

Proof :

By theorem 3.1, for any $u \in H$, there exists $w_u \in T(u)$ such that

$$\left\| w_u - \bar{\bar{w}} \right\| \leq H(T(u), T(\bar{\bar{u}})).$$

While $\left\| w_u - \bar{\bar{w}} \right\| \leq H(T(u), T(\bar{\bar{u}}))$, we have

$$\left\| u - \bar{\bar{u}} \right\| \leq \frac{L + \alpha kt}{\delta} \left\| g(u) - v_\alpha(u, w_u) \right\|, \quad \forall u \in H, \quad \alpha > 0, \qquad (3.17)$$

$$\left\| w_u - \bar{\bar{w}} \right\| \leq \frac{L(L + \alpha kt)}{\delta} \left\| g(u) - v_\alpha(u, w_u) \right\|, \quad \forall u \in H, \quad \alpha > 0. \qquad (3.18)$$

While $w_u = w^*$, by (3.17) and (3.18), we have

$$\left\|u-\overset{=}{u}\right\| \le \frac{L+\alpha kt}{\delta}\left\|g(u)-v_\alpha(u,w^*)\right\|, \quad \forall u\in H, \ \alpha>0, \qquad (3.19)$$

$$\left\|w^*-\overset{=}{w}\right\|=\left\|w_u-\overset{=}{w}\right\|\le \frac{L(L+\alpha kt)}{\delta}\left\|g(u)-v_\alpha(u,w^*)\right\|, \quad \forall u\in H, \ \alpha>0. \qquad (3.20)$$

By lemma 3.3, for any $u\in H,\alpha>0$, while $w_u=w^*\in T(u)$, we have

$$G_\alpha(u)\ge \alpha\lambda\left\|g(u)-v_\alpha(u,w^*)\right\|^2 \qquad (3.21)$$

By（3.19）and （3.21）, we have

$$\left\|u-\overset{=}{u}\right\| \le \frac{L+\alpha kt}{\delta\sqrt{\alpha\lambda}}\sqrt{G_\alpha(u)}, \quad \forall u\in H, \ \alpha>0.$$

By（3.20）and （3.21）, we have

$$\left\|w_u-\overset{=}{w}\right\|\le \frac{L(L+\alpha kt)}{\delta\sqrt{\alpha\lambda}}\sqrt{G_\alpha(u)}, \quad \forall u\in H, \ \alpha>0.$$

Theorem 3.3 If the function ϕ satisfies $C.1-C.5$, $\overset{=}{u}\in H,\overset{=}{w}\in T(\overset{=}{u})$ is the solution of $GSMVIP(T,g,\varphi)$, T is $g-$strongly monotone with module δ on H, $\delta>0$, g is $Lipschitz$ continuous on H with module $t>0$. Then we have

$$\left\|u-\overset{=}{u}\right\|\le \frac{1}{\sqrt{\delta+\alpha(\lambda-k)t^2}}\sqrt{G_\alpha(u)}, \forall u\in H, \quad 0<\alpha<\frac{\delta}{(k-\lambda)t^2}.$$

Proof :

For given $u\in H$, choose $w_u\in T(u)$. Suppose that $\overset{=}{u}\in H,\overset{=}{w}\in T(\overset{=}{u})$ is the solution of $GSMVIP(T,g,\varphi)$, we have

$$<\overset{=}{w}, v-g(\overset{=}{u})>+\varphi(v)-\varphi(g(\overset{=}{u}))\ge 0, \forall v\in H.$$

Set $v=g(u)$, we have

$$<\overset{=}{w}, g(u)-g(\overset{=}{u})>+\varphi(g(u))-\varphi(g(\overset{=}{u}))\ge 0 \qquad (3.22)$$

$$\begin{aligned}
G_\alpha(u) &= \inf_{w_u\in T(u)}\sup_{v\in H}\psi_\alpha(w_u,u,v)\\
&= \inf_{w_u\in T(u)}\sup_{v\in H}\{<w_u, g(u)-v>+\varphi(g(u))-\varphi(v)-\alpha\phi(g(u),v)\}\\
&=<w^*, g(u)-v_\alpha(u,w^*)>+\varphi(g(u))-\varphi(v_\alpha(u,w^*))-\alpha\phi(g(u),v_\alpha(u,w^*))\\
&\ge <w^*, g(u)-g(\overset{=}{u})>+\varphi(g(u))-\varphi(g(\overset{=}{u}))-\alpha\phi(g(u),g(\overset{=}{u}))\\
&\ge <\overset{=}{w}, g(u)-g(\overset{=}{u})>+\delta\left\|u-\overset{=}{u}\right\|^2+\varphi(g(u))-\varphi(g(\overset{=}{u}))-\alpha\phi(g(u),g(\overset{=}{u}))\\
&=\{\delta\left\|u-\overset{=}{u}\right\|^2-\alpha\phi(g(u),g(\overset{=}{u}))\}+\{<\overset{=}{w}, g(u)-g(\overset{=}{u})>+\varphi(g(u))-\varphi(g(\overset{=}{u}))\}. \qquad (3.23)
\end{aligned}$$

By (3.22) and (3.23), we have

$$G_\alpha(u) \geq \delta \left\| u - \overset{=}{u} \right\|^2 - \alpha\phi(g(u), g(\overset{=}{u})). \quad (3.24)$$

By lemma 2.2, if the function ϕ satisfies $C.3$ and $C.5$, we have

$$2\lambda \|v_1 - v_2\|^2 \leq k \|v_1 - v_2\|^2,$$

i.e.

$$2\lambda \leq k.$$

Hence

$$k - \lambda > 0.$$

By lemma 2.3, we have

$$\lambda \left\| g(u) - g(\overset{=}{u}) \right\|^2 \leq \phi(g(u), g(\overset{=}{u})) \leq (k-\lambda) \left\| g(u) - g(\overset{=}{u}) \right\|^2. \quad (3.25)$$

By (3.25), we have

$$-\alpha\phi(g(u), g(\overset{=}{u})) \geq \alpha(\lambda - k) \left\| g(u) - g(\overset{=}{u}) \right\|^2$$

$$\geq \alpha(\lambda - k)t^2 \left\| u - \overset{=}{u} \right\|^2. \quad (3.26)$$

By (3.24) and (3.26), we have

$$G_\alpha(u) \geq [\delta + \alpha(\lambda - k)t^2] \left\| u - \overset{=}{u} \right\|^2,$$

i.e.

$$\left\| u - \overset{=}{u} \right\| \leq \frac{1}{\sqrt{\delta + \alpha(\lambda - k)t^2}} \sqrt{G_\alpha(u)}, \forall u \in H.$$

Since $\sqrt{\delta + \alpha(\lambda - k)t^2} > 0$, we have

$$0 < \alpha < \frac{\delta}{(k-\lambda)t^2}$$

4.The $D-$gap function and error bounds of $GSMVIP(T, g, \varphi)$

We define the $D-$gap function of $GSMVIP(T, g, \varphi)$ as follows:

$$\begin{aligned} D_{\alpha\beta}(u) &= G_\alpha(u) - G_\beta(u) \\ &= \inf_{w_u \in T(u)} \sup_{v \in H} \psi_\alpha(w_u, u, v) - \inf_{w_u \in T(u)} \sup_{v \in H} \psi_\beta(w_u, u, v) \\ &= \inf_{w_u \in T(u)} \sup_{v \in H} \{< w_u, g(u) - v > + \varphi(g(u)) - \varphi(v) - \alpha\phi(g(u), v)\} \\ &\quad - \inf_{w_u \in T(u)} \sup_{v \in H} \{< w_u, g(u) - v > + \varphi(g(u)) - \varphi(v) - \beta\phi(g(u), v)\} \end{aligned}$$

$$=< w_\alpha^*, g(u)-v_\alpha(u,w_\alpha^*) > +\varphi(g(u))-\varphi(v_\alpha(u,w_\alpha^*))-\alpha\phi(g(u),v_\alpha(u,w_\alpha^*))$$
$$-< w_\beta^*, g(u)-v_\beta(u,w_\beta^*) > -\varphi(g(u))+\varphi(v_\beta(u,w_\beta^*))+\beta\phi(g(u),v_\beta(u,w_\beta^*))$$
$$=< w_\alpha^*, g(u)-v_\alpha(u,w_\alpha^*) > - < w_\beta^*, g(u)-v_\beta(u,w_\beta^*) > +\varphi(v_\beta(u,w_\beta^*))-\varphi(v_\alpha(u,w_\alpha^*))$$
$$+\beta\phi(g(u),v_\beta(u,w_\beta^*))-\alpha\phi(g(u),v_\alpha(u,w_\alpha^*))$$

where $\alpha<\beta$, for given $u\in H$, while $w_u=w_\alpha^*$, $G_\alpha(u)$ gains the function value, and while $w_u=w_\beta^*$, $G_\beta(u)$ gains the function value . $v_\alpha(u,w_\alpha^*)$ denotes the minimizer of $-\psi_\alpha(w_\alpha^*,u,.)$, $v_\beta(u,w_\beta^*)$ denotes the minimizer of $-\psi_\beta(w_\beta^*,u,.)$ on H.

Lemma 4.1 If the function ϕ satisfies $C.3$, for any $u\in H$, we have

$$(\beta-\alpha)\phi(g(u),v_\beta(u,w_\alpha^*))\le D_{\alpha\beta}(u)\ \le(\beta-\alpha)\phi(g(u),v_\alpha(u,w_\beta^*))$$

Proof :

$$D_{\alpha\beta}(u)=G_\alpha(u)-G_\beta(u)$$
$$=\inf_{w_u\in T(u)}\sup_{v\in H}\psi_\alpha(w_u,u,v)-\inf_{w_u\in T(u)}\sup_{v\in H}\psi_\beta(w_u,u,v)$$
$$=\inf_{w_u\in T(u)}\sup_{v\in H}\{< w_u, g(u)-v > +\varphi(g(u))-\varphi(v)-\alpha\phi(g(u),v)\}$$
$$-\inf_{w_u\in T(u)}\sup_{v\in H}\{< w_u, g(u)-v > +\varphi(g(u))-\varphi(v)-\beta\phi(g(u),v)\}$$
$$=< w_\alpha^*, g(u)-v_\alpha(u,w_\alpha^*) > +\varphi(g(u))-\varphi(v_\alpha(u,w_\alpha^*))-\alpha\phi(g(u),v_\alpha(u,w_\alpha^*))$$
$$-\{< w_\beta^*, g(u)-v_\beta(u,w_\beta^*) > +\varphi(g(u))-\varphi(v_\beta(u,w_\beta^*))-\beta\phi(g(u),v_\beta(u,w_\beta^*))\}$$
$$\le< w_\beta^*, g(u)-v_\alpha(u,w_\beta^*) > +\varphi(g(u))-\varphi(v_\alpha(u,w_\beta^*))-\alpha\phi(g(u),v_\alpha(u,w_\beta^*))$$
$$-\{< w_\beta^*, g(u)-v_\beta(u,w_\beta^*) > +\varphi(g(u))-\varphi(v_\beta(u,w_\beta^*))-\beta\phi(g(u),v_\beta(u,w_\beta^*))\}$$
$$\le< w_\beta^*, g(u)-v_\alpha(u,w_\beta^*) > +\varphi(g(u))-\varphi(v_\alpha(u,w_\beta^*))-\alpha\phi(g(u),v_\alpha(u,w_\beta^*))$$
$$-\{< w_\beta^*, g(u)-v_\alpha(u,w_\beta^*) > +\varphi(g(u))-\varphi(v_\alpha(u,w_\beta^*))-\beta\phi(g(u),v_\alpha(u,w_\beta^*))\}$$
$$\le(\beta-\alpha)\phi(g(u),v_\alpha(u,w_\beta^*))$$

$$D_{\alpha\beta}(u)=G_\alpha(u)-G_\beta(u)$$
$$=\inf_{w_u\in T(u)}\sup_{v\in H}\psi_\alpha(w_u,u,v)-\inf_{w_u\in T(u)}\sup_{v\in H}\psi_\beta(w_u,u,v)$$
$$=\inf_{w_u\in T(u)}\sup_{v\in H}\{< w_u, g(u)-v > +\varphi(g(u))-\varphi(v)-\alpha\phi(g(u),v)\}$$
$$-\inf_{w_u\in T(u)}\sup_{v\in H}\{< w_u, g(u)-v > +\varphi(g(u))-\varphi(v)-\beta\phi(g(u),v)\}$$
$$=< w_\alpha^*, g(u)-v_\alpha(u,w_\alpha^*) > +\varphi(g(u))-\varphi(v_\alpha(u,w_\alpha^*))-\alpha\phi(g(u),v_\alpha(u,w_\alpha^*))$$
$$-\{< w_\beta^*, g(u)-v_\beta(u,w_\beta^*) > +\varphi(g(u))-\varphi(v_\beta(u,w_\beta^*))-\beta\phi(g(u),v_\beta(u,w_\beta^*))\}$$
$$\ge< w_\alpha^*, g(u)-v_\alpha(u,w_\alpha^*) > +\varphi(g(u))-\varphi(v_\alpha(u,w_\alpha^*))-\alpha\phi(g(u),v_\alpha(u,w_\alpha^*))$$
$$-\{< w_\alpha^*, g(u)-v_\beta(u,w_\alpha^*) > +\varphi(g(u))-\varphi(v_\beta(u,w_\alpha^*))-\beta\phi(g(u),v_\beta(u,w_\alpha^*))\}$$
$$\ge< w_\alpha^*, g(u)-v_\beta(u,w_\alpha^*) > +\varphi(g(u))-\varphi(v_\beta(u,w_\alpha^*))-\alpha\phi(g(u),v_\beta(u,w_\alpha^*))$$
$$-\{< w_\alpha^*, g(u)-v_\beta(u,w_\alpha^*) > +\varphi(g(u))-\varphi(v_\beta(u,w_\alpha^*))-\beta\phi(g(u),v_\beta(u,w_\alpha^*))\}$$
$$\ge(\beta-\alpha)\phi(g(u),v_\beta(u,w_\alpha^*))$$

Theorem 4.1 If the function ϕ satisfies $C.1-C.4$. Then the function $D_{\alpha\beta}$ is nonnegative on H. If $w_\alpha^* = w_\beta^* = w^*$, then $D_{\alpha\beta}(u) = 0$ if and only if $u \in H, w^* \in T(u)$ is the solution of $GSMVIP(T, g, \varphi)$.

Proof :

By lemma 4.1、$C.2$ and $\beta > \alpha$, we have

$$D_{\alpha\beta}(u) \geq (\beta - \alpha)\phi(g(u), v_\beta(u, w_\alpha^*)) \geq 0.$$

Hence, the function $D_{\alpha\beta}$ is nonnegative on H.

Next we will prove the last part of theorem 4.1.

$(\Leftarrow)$: If $u \in H, w^* \in T(u)$ is the solution of $GSMVIP(T, g, \varphi)$, by lemma 3.2, we have

$$g(u) = v_\alpha(u, w^*) . \tag{4.1}$$

If $w^* = w_\beta^*$, by (4.1), we have

$$g(u) = v_\alpha(u, w_\beta^*).$$

By $C.4$, we have

$$\phi(g(u), v_\alpha(u, w_\beta^*)) = 0 \tag{4.2}$$

By lemma 4.1, we have

$$D_{\alpha\beta}(u) \leq (\beta - \alpha)\phi(g(u), v_\alpha(u, w_\beta^*)). \tag{4.3}$$

By (4.2) and (4.3), we have

$$D_{\alpha\beta}(u) \leq 0. \tag{4.4}$$

Since $D_{\alpha\beta}$ is nonnegative on H, i.e.

$$D_{\alpha\beta}(u) \geq 0. \tag{4.5}$$

By (4.4) and (4.5), we have

$$D_{\alpha\beta}(u) = 0.$$

$(\Rightarrow)$: If $D_{\alpha\beta}(u) = 0$, by lemma 4.1, we have

$$(\beta - \alpha)\phi(g(u), v_\beta(u, w_\alpha^*)) \leq D_{\alpha\beta}(u) = 0. \tag{4.6}$$

Since

$$\beta - \alpha > 0. \tag{4.7}$$

Hence, by (4.7)and C.2, we have

$$(\beta - \alpha)\phi(g(u), v_\beta(u, w_\alpha^*)) \geq 0. \tag{4.8}$$

By (4.6) and (4.8), we have

$$(\beta - \alpha)\phi(g(u), v_\beta(u, w_\alpha^*)) = 0. \tag{4.9}$$

By (4.7) and (4.9), we have

$$\phi(g(u), v_\beta(u, w_\alpha^*)) = 0.$$

By C.4, we have

$$g(u)=v_{\beta}(u,w_{\alpha}^{*}). \qquad (4.10)$$

If $w_{\alpha}^{*}=w^{*}$, by (4.10), we have

$$g(u)=v_{\beta}(u,w^{*}).$$

By lemma 4.1, we have $u\in H, w^{*}\in T(u)$ is the solution of $GSMVIP(T,g,\varphi)$.

Theorem 4.2 Suppose that the function ϕ satisfies $C.1-C.5$. $\bar{\bar{u}}\in H, \bar{\bar{w}}\in T(\bar{\bar{u}})$ is the solution of $GSMVIP(T,g,\varphi)$, T is $g-$strongly monotone with module δ on H, $\delta>0$, T,g are $Lipschitz$ continuous on H with respective module $L,t>0$. Then for any $u\in H$, there exists $w_u\in T(u)$ such that

$$\left\|w_u-\bar{\bar{w}}\right\|\le H(T(u),T(\bar{\bar{u}}))$$

especially, while $\left\|w_u-\bar{\bar{w}}\right\|\le H(T(u),T(\bar{\bar{u}}))$ and $w_u=w_{\alpha}^{*}$, $\sqrt{D_{\alpha\beta}(u)}$ will give the global error bounds of $GSMVIP(T,g,\varphi)$, i.e.

$$\left\|\bar{\bar{u}}-u\right\|\le\frac{L+\beta kt}{\delta\sqrt{(\beta-\alpha)\lambda}}\sqrt{D_{\alpha\beta}(u)}, \forall u\in H, \beta>\alpha,$$

$$\left\|w_u-\bar{\bar{w}}\right\|\le\le\frac{L(L+\beta kt)}{\delta\sqrt{(\beta-\alpha)\lambda}}\sqrt{D_{\alpha\beta}(u)}, \forall u\in H, \beta>\alpha.$$

Proof :

By theorem 3.1, we have for any $u\in H$, there exists $w_u\in T(u)$ such that

$$\left\|w_u-\bar{\bar{w}}\right\|\le H(T(u),T(\bar{\bar{u}})).$$

And while $\left\|w_u-\bar{\bar{w}}\right\|\le H(T(u),T(\bar{\bar{u}}))$, we have

$$\left\|u-\bar{\bar{u}}\right\|\le\frac{L+\beta kt}{\delta}\left\|g(u)-v_{\beta}(u,w_u)\right\|, \quad \forall u\in H, \quad \beta>0, \qquad (4.11)$$

$$\left\|w_u-\bar{\bar{w}}\right\|\le\frac{L(L+\beta kt)}{\delta}\left\|g(u)-v_{\beta}(u,w_u)\right\|, \quad \forall u\in H, \quad \beta>0. \qquad (4.12)$$

While $w_u=w_{\alpha}^{*}$, by (4.11) and (4.12), we have

$$\left\|u-\bar{\bar{u}}\right\|\le\frac{L+\beta kt}{\delta}\left\|g(u)-v_{\beta}(u,w_{\alpha}^{*})\right\|, \quad \forall u\in H, \quad \beta>0, \qquad (4.13)$$

$$\left\|w_{\alpha}^{*}-\bar{\bar{w}}\right\|=\left\|w_u-\bar{\bar{w}}\right\|\le\frac{L(L+\beta kt)}{\delta}\left\|g(u)-v_{\beta}(u,w_{\alpha}^{*})\right\|, \quad \forall u\in H, \quad \beta>0. \qquad (4.14)$$

By lemma 4.1, we have

$$D_{\alpha\beta}(u) \ \geq (\beta-\alpha)\phi(g(u),v_\beta(u,w_\alpha{}^*)) . \qquad (4.15)$$

By lemma 2.3, we have

$$\phi(g(u),v_\beta(u,w_\alpha{}^*)) \ \geq \lambda\left\|g(u)-v_\beta(u,w_\alpha{}^*)\right\|^2 . \qquad (4.16)$$

By（4.13）、（4.14）、（4.15）and（4.16）, we have

$$\left\|\bar{\bar{u}}-u\right\| \ \leq \frac{L+\beta kt}{\delta\sqrt{(\beta-\alpha)\lambda}}\sqrt{D_{\alpha\beta}(u)}, \forall u\in H, \beta>\alpha ,$$

$$\left\|w_u-\bar{\bar{w}}\right\| \leq \frac{L(L+\beta kt)}{\delta\sqrt{(\beta-\alpha)\lambda}}\sqrt{D_{\alpha\beta}(u)}, \forall u\in H, \beta>\alpha .$$

5. The algorithms of $GSMVIP(T,g,\varphi)$ and convergence of algorethms

5.1 Equivalence and Iterative Algorithms

In order to solve the problem（1.1）， we consider the following resolvent equations:

Let $R_\varphi = I - J_\varphi$, where I is identity operator， $J_\varphi = (I+\rho\partial\varphi)^{-1}$ is resolvent equations. $T: H \to C(H)$ is a nonliner operator, we consider the problem of finding $z,u\in H$， $w\in T(u)$ such that

$$w+\rho^{-1}R_\varphi z = 0 . \qquad (5.1)$$

Where $\rho>0$ is a constant, we call (5.1) the generalized resolvent equations of the problem（1.1）.

If T： $H\to H$ is a single-valued operator，then the problem (5.1)is equivalent to finding $z\in H$ such that

$$TJ_\varphi z+\rho^{-1}R_\varphi z = 0 , \qquad (5.2)$$

we call (5.2) the resolvent equations. The resolvent equations is introduced by Noor，see reference[35,36].

Theorem 5.1 If $u\in H$， $w\in T(u)$ is the solution of $GSMVIP(T,g,\varphi)$ if and only if $u\in H$， $w\in T(u)$ satisfy the following condition:

$$g(u) = J_\varphi(g(u)-\rho w) ,$$

where $\rho>0$ is a constant， $J_\varphi = (I+\rho\partial\varphi)^{-1}$ is resolvent operator.

Proof :

$(\Rightarrow)$: Suppose that $u\in H$， $w\in T(u)$ is the solution of $GSMVIP(T,g,\varphi)$， we have

$$<w,v-g(u)>+\varphi(v)-\varphi(g(u))\geq 0$$

for all $v\in H$.

i.e.

$$<g(u)-(g(u)-\rho w),v-g(u)>+\rho\varphi(v)-\rho\varphi(g(u))\geq 0 ,$$

where $\rho>0$ is a constant.

Hence，by lemma 2.4, we have

$$g(u) = J_\varphi(g(u)-\rho w) .$$

$(\Leftarrow)$: While $u \in H$, $w \in T(u)$, we have

$$g(u) = J_\varphi(g(u) - \rho w).$$

By lemma 2.4, we have

$$< g(u) - (g(u) - \rho w), v - g(u) > + \rho\varphi(v) - \rho\varphi(g(u)) \geq 0,$$

for all $v \in H$.

i.e.

$$< w, v - g(u) > + \varphi(v) - \varphi(g(u)) \geq 0$$

for all $v \in H$.

Hence, $u \in H$, $w \in T(u)$ is the solution of $GSMVIP(T, g, \varphi)$.

Next we will give the iterative algorithms of the problem（5.1）.

Algorithm 5.1

From theorem 5.1, the problem （1.1）is equivalent to the fixed point problem of the type

$$u = (1-\lambda)u + \lambda\{u - g(u) + J_\varphi(g(u) - \rho w)\},$$

Where $0 < \lambda < 1$ is a parameter.

This fixed point formulation enables us to suggest the following algorithm:

Let T: $H \to C(H)$ be a multivalued operator, for given $u_0 \in H$, take $w_0 \in T(u_0)$ such that

$$u_1 = (1-\lambda)u_0 + \lambda\{u_0 - g(u_0) + J_\varphi(g(u_0) - \rho w_0)\}.$$

Since $w_0 \in T(u_0)$, there exists $w_1 \in T(u_1)$ such that

$$\|w_0 - w_1\| \leq H(T(u_0), T(u_1)),$$

where $H(.,.)$ is Hausdorff metric on $C(H)$.

Let

$$u_2 = (1-\lambda)u_1 + \lambda\{u_1 - g(u_1) + J_\varphi(g(u_1) - \rho w_1)\}.$$

Continuing this way, we can obtain te sequence $\{u_n\}$ and $\{w_n\}$ by the iterative algorithm

$$w_n \in T(u_n) : \|w_{n+1} - w_n\| \leq H(T(u_{n+1}), T(u_n)),$$

$$u_{n+1} = (1-\lambda)u_n + \lambda\{u_n - g(u_n) + J_\varphi(g(u_n) - \rho w_n)\},\quad n = 0,1,2,...$$

By theorem 5.1, we can establish the equivalence between the problem（1.1）and the problem（5.1）.

Theorem 5.2 If $u \in H$, $w \in T(u)$ is the solution of the problem（1.1）if and only if $z, u \in H$, $w \in T(u)$ is the solution of the generalized resolvent equations（5.1）, where

$$g(u) = J_\varphi(z) \tag{5.3}$$

$$z = g(u) - \rho w, \tag{5.4}$$

where $\rho > 0$ is a constant.

Proof :

$(\Rightarrow)$: If $u \in H$, $w \in T(u)$ is the solution of the problem (1.1), by theorem 3.1, we have

$$g(u) = J_\varphi(g(u) - \rho w) \ . \qquad (5.5)$$

By (5.5) and $R_\varphi = I - J_\varphi$, we have

$$\begin{aligned} R_\varphi(g(u) - \rho w) &= (I - J_\varphi)(g(u) - \rho w) \\ &= I(g(u) - \rho w) - J_\varphi(g(u) - \rho w) \\ &= g(u) - \rho w - J_\varphi(g(u) - \rho w) \\ &= g(u) - \rho w - g(u) \\ &= -\rho w. \end{aligned} \qquad (5.6)$$

Let $z = g(u) - \rho w$, by (5.5) and (5.6), we have

$$g(u) = J_\varphi(z),$$

$$R_\varphi(z) = -\rho w. \qquad (5.7)$$

By (5.7), we have

$$w + \rho^{-1} R_\varphi z = 0 .$$

$(\Leftarrow)$: If $z, u \in H$, $w \in T(u)$ is the solution of the generalized resolvent equations (5.1), we have

$$w + \rho^{-1} R_\varphi(z) = 0 . \qquad (5.8)$$

By (5.4) and (5.8), we have

$$\begin{aligned} \rho w &= -R_\varphi(z) \\ &= -(I - J_\varphi)(z) \\ &= J_\varphi(z) - z \\ &= J_\varphi(g(u) - \rho w) - g(u) + \rho w. \end{aligned}$$

i.e.

$$g(u) = J_\varphi(g(u) - \rho w) = J_\varphi(z) .$$

Hence, by theorem 5.1, $u \in H$, $w \in T(u)$ is the solution of the problem (1.1).

Algorithm 5.2

From theorem 5.2, we know that the problem (1.1) and the problem (5.1) is equivalent, hence, we can give the following iterative algorithm by the equivalence:

The generalized resolvent equations (5.1) can be written as

$$R_\varphi(z) = -\rho w . \qquad (5.9)$$

Using $R_\varphi(z) = (I - J_\varphi)(z)$ and (5.9), we have

$$z = J_\varphi(z) - \rho w . \qquad (5.10)$$

By (5.3) and (5.8), we have

$$z = g(u) - \rho w .$$

Hence，we can obtain the following iterative algorithm：

For given $z_0, u_0 \in H, w_0 \in T(u_0)$，compute the sequences $\{z_n\}$，$\{w_n\}$ and $\{u_n\}$ by iterative schemes:

$$g(u_n) = J_\varphi(z_n) \quad (5.11)$$

$$w_n \in T(u_n): \|w_{n+1} - w_n\| \le H(T(u_{n+1}), T(u_n)) \quad (5.12)$$

$$z_{n+1} = g(u_n) - \rho w_n, \quad n = 0, 1, 2, \ldots \quad (5.13)$$

Algorithm 5.3

The generalized resolvent equations（5.1）can be written as

$$0 = -\rho^{-1} R_\varphi(z) - w,$$

i.e.

$$R_\varphi(z) = (1 - \rho^{-1}) R_\varphi(z) - w. \quad (5.14)$$

By $R_\varphi(z) = (I - J_\varphi)(z)$ and（5.14）, we have

$$z = J_\varphi(z) + (1 - \rho^{-1}) R_\varphi(z) - w. \quad (5.15)$$

By（5.3）and（5.15）, we have

$$z = g(u) - w + (1 - \rho^{-1}) R_\varphi(z).$$

Hence, we can obtain another iterative algorithm:

For given $z_0, u_0 \in H, w_0 \in T(u_0)$，compute the sequences $\{z_n\}$，$\{w_n\}$ and $\{u_n\}$ by the iterative schemes:

$$g(u_n) = J_\varphi(z_n)$$

$$w_n \in T(u_n): \|w_{n+1} - w_n\| \le H(T(u_{n+1}), T(u_n))$$

$$z_{n+1} = g(u_n) - w_n + (1 - \rho^{-1}) R_\varphi(z_n), \quad n = 0, 1, 2, \ldots$$

5.2 Convergence Analysis of Iterative Algorithm

Next we will study those conditions under which the approximate solution obtained from Algorithm 5.2 converges to the exact solution of the generalized resolvent equations (5.1). In a similar way , one can study the convergence analysis of Algorithm 5.1 and 5.3.

Theorem 5.3 let the operator $T: H \to C(H)$ be $\alpha-$strongly monotone，$H - \beta - Lipschitz$ continuous. $g: H \to H$ is strongly monotone with module σ and $\delta - Lipschitz$ continuous，J_φ is nonexpansive mapping , and

$$\delta^2 < 2\sigma \quad (5.16)$$

$$\delta^2 > 2\sigma - 1 \quad (5.17)$$

$$\rho^2 \beta^2 > 2\alpha\rho - 1 \quad (5.18)$$

$$2\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2} < 1 \quad (5.19)$$

where $\sigma, \delta, \alpha, \rho, \beta > 0$ are constants.

Then there exists $z \in H$, which satisfies（5.1）and（5.4）and the sequences $\{z_n\}$ generated by Algorithm 5.2 converges to z strongly in H .

Proof :

Let $\{z_n\}$ be sequence generated by Algorithm 5.2, $z = g(u) - \rho w$ satisfies the generalized resolvent equations（5.1）and （5.4）, we have

$$\begin{aligned}\|z_{n+1} - z\| &= \|g(u_n) - g(u) - \rho(w_n - w)\| \\ &= \|g(u_n) - g(u) - (u_n - u) + (u_n - u) - \rho(w_n - w)\| \\ &\le \|g(u_n) - g(u) - (u_n - u)\| + \|(u_n - u) - \rho(w_n - w)\| \qquad (5.20)\end{aligned}$$

$$\begin{aligned}\|(u_n - u) - \rho(w_n - w)\|^2 &= \|u_n - u\|^2 - 2\rho < w_n - w, u_n - u > + \rho^2 \|w_n - w\|^2 \\ &\le \|u_n - u\|^2 - 2\alpha\rho \|u_n - u\|^2 + \rho^2 \beta^2 \|u_n - u\|^2 \\ &\le (1 - 2\alpha\rho + \rho^2\beta^2) \|u_n - u\|^2 \qquad (5.21)\end{aligned}$$

$$\begin{aligned}\|g(u_n) - g(u) - (u_n - u)\|^2 &= \|u_n - u\|^2 - 2 < u_n - u, g(u_n) - g(u) > + \|g(u_n) - g(u)\|^2 \\ &\le \|u_n - u\|^2 - 2\sigma \|u_n - u\|^2 + \delta^2 \|u_n - u\|^2 \\ &\le (1 - 2\sigma + \delta^2) \|u_n - u\|^2 \qquad (5.22)\end{aligned}$$

By（5.20）、（5.21）and（5.22）, we have

$$\begin{aligned}\|z_{n+1} - z\| &\le \|g(u_n) - g(u) - (u_n - u)\| + \|(u_n - u) - \rho(w_n - w)\| \\ &\le (\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}) \|u_n - u\| . \qquad (5.23)\end{aligned}$$

By（5.3）、（5.11）and（5.22）, we have

$$\begin{aligned}\|u_n - u\| &= \|u_n - u - (g(u_n) - g(u)) + J_\varphi(z_n) - J_\varphi(z)\| \\ &\le \|u_n - u - (g(u_n) - g(u))\| + \|J_\varphi(z_n) - J_\varphi(z)\| \\ &\le \sqrt{1 - 2\sigma + \delta^2} \|u_n - u\| + \|z_n - z\| \\ &\le x\|u_n - u\| + \|z_n - z\|, \qquad (5.24)\end{aligned}$$

where

$$x = \sqrt{1 - 2\sigma + \delta^2} .$$

By（5.24）, we have

$$\|u_n - u\| \le \frac{1}{1 - x} \|z_n - z\| . \qquad (5.25)$$

By（5.23）and（5.25）, we have

$$\begin{aligned}\|z_{n+1} - z\| &\le (\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}) \|u_n - u\| \\ &\le \frac{\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}}{1 - x} \|z_n - z\| \\ &\le \gamma \|z_n - z\| , \qquad (5.26)\end{aligned}$$

where

$$\gamma = \frac{\sqrt{1-2\sigma+\delta^2}+\sqrt{1-2\alpha\rho+\rho^2\beta^2}}{1-x}.$$

By（5.16）、（5.17）、（5.18）and（5.19）, we have

$$0<\gamma<1.$$

By（5.26）, we have

$$\begin{aligned}\|z_{n+1}-z\| &\le \gamma\|z_n-z\| \\ &\le \gamma^2\|z_{n-1}-z\| \\ &\le \gamma^3\|z_{n-2}-z\| \\ &\le \ldots \\ &\le \gamma^n\|z_1-z\| \\ &\le \tau\gamma^n, \end{aligned} \tag{5.27}$$

where

$$\tau=\|z_1-z\|.$$

Let $n\to+\infty$, by（5.27）, we have $z_{n+1}\to z$.

Theorem 5.4 Let the operator $T:H\to C(H)$ be $\alpha-g$ strongly monotone and $H-\beta-Lipschitz$ continuous. g is $L_1-Lipschitz$ continuous， g^{-1} exists and is $L_2-Lipschitz$ continuous，J_φ is nonexpansive mapping，and

$$\xi=L_1^2-2\rho\alpha+\rho^2\beta^2>0, \tag{5.28}$$

$$L_2\sqrt{\xi}<1, \tag{5.29}$$

where $L_1,L_2,\rho,\alpha,\beta>0$ are constants.

Then $z_n\to z, w_n\to w, u_n\to u$ strongly in H，where z_n，w_n，u_n are the approximate solutions from Algorithm 5.2，and $z,u\in H, w\in T(u)$ is the exact solution of the generalized resolvent equations（5.1）.

Proof :

By the Algorithm 5.2, we have

$$\begin{aligned}\|z_{n+1}-z_n\|^2 &= \|g(u_n)-g(u_{n-1})-\rho(w_n-w_{n-1})\|^2 \\ &= \|g(u_n)-g(u_{n-1})\|^2-2\rho< w_n-w_{n-1}, g(u_n)-g(u_{n-1})> \\ &\quad +\rho^2\|w_n-w_{n-1}\|^2 \\ &\le L_1^2\|u_n-u_{n-1}\|^2-2\rho\alpha\|u_n-u_{n-1}\|^2+\rho^2\{H(T(u_n),T(u_{n-1}))\}^2 \\ &\le L_1^2\|u_n-u_{n-1}\|^2-2\rho\alpha\|u_n-u_{n-1}\|^2+\rho^2\beta^2\|u_n-u_{n-1}\|^2 \\ &\le (L_1^2-2\rho\alpha+\rho^2\beta^2)\|u_n-u_{n-1}\|^2 \\ &\le \xi\|u_n-u_{n-1}\|^2, \end{aligned} \tag{5.30}$$

where

$$\xi=L_1^2-2\rho\alpha+\rho^2\beta^2.$$

By（5.11）, we have

$$u_n = g^{-1}(J_\varphi(z_n)). \qquad (5.31)$$

By (5.31), we have

$$\begin{aligned}\|u_n - u_{n-1}\|^2 &= \|g^{-1}(J_\varphi(z_n)) - g^{-1}(J_\varphi(z_{n-1}))\|^2 \\ &\le L_2{}^2 \|J_\varphi(z_n) - J_\varphi(z_{n-1})\|^2 \\ &\le L_2{}^2 \|z_n - z_{n-1}\|^2. \qquad (5.32)\end{aligned}$$

By (5.30) and (5.32), we have

$$\begin{aligned}\|z_{n+1} - z_n\|^2 &\le \xi \|u_n - u_{n-1}\|^2 \\ &\le \xi \|g^{-1}(J_\varphi(z_n)) - g^{-1}(J_\varphi(z_{n-1}))\|^2 \\ &\le \xi L_2{}^2 \|J_\varphi(z_n) - J_\varphi(z_{n-1})\|^2 \\ &\le \xi L_2{}^2 \|z_n - z_{n-1}\|^2 \\ &\le \gamma \|z_n - z_{n-1}\|^2, \qquad (5.33)\end{aligned}$$

where

$$\gamma = \xi L_2{}^2.$$

By (5.33), we have

$$\|z_{n+1} - z_n\| \le \sqrt{\gamma}\, \|z_n - z_{n-1}\| . \qquad (5.34)$$

By (5.28) and (5.29), we have

$$\sqrt{\gamma} < 1. \qquad (5.35)$$

By (5.34) and (5.35), $\{z_n\}$ is a Cauchy sequence in H, hence, there exists $z \in H$, $z_n \to z \in H$, as $n \to \infty$.

By (5.12) and T is $H-\beta-Lipschitz$ continuous, we have

$$\|w_{n+1} - w_n\| \le H(T(u_{n+1}), T(u_n)) \le \beta \|u_{n+1} - u_n\|. \qquad (5.36)$$

By (5.32), we have

$$\|u_{n+1} - u_n\| \le L_2 \|z_{n+1} - z_n\| . \qquad (5.37)$$

Hence, by (5.37) and $\{z_n\}$ is a Cauchy sequence in H, we have $\{u_n\}$ is also a Cauchy sequence in H. Hence, there exists $u \in H$, $u_n \to u \in H$ as $n \to \infty$.

By (5.36) and (5.37), we have

$$\|w_{n+1} - w_n\| \le H(T(u_{n+1}), T(u_n)) \le \beta \|u_{n+1} - u_n\| \le \beta L_2 \|z_{n+1} - z_n\|. \qquad (5.38)$$

Hence, by (5.38) and $\{z_n\}$ is a Cauchy sequence in H, we have that $\{w_n\}$ is a Cauchy in H, hence, there exists $w \in H$, $w_n \to w$ as $n \to \infty$.

Since T, g, J_φ are continuous and theorem 5.1 and 5.2, by (5.11) and (5.13), as $n \to \infty$, we have

$$g(u) = J_{\varphi}(z),$$

$$z = g(u) - \rho w = J_{\varphi}(z) - \rho w.$$

Next we will prove $w \in T(u)$. In fact

$$\begin{aligned} d(w,T(u)) &\le d(w,w_n) + d(w_n,T(u)) \\ &\le \|w_n - w\| + H(T(u_n),T(u)) \\ &\le \|w_n - w\| + \beta \|u_n - u\|, \end{aligned}$$

where $d(w,T(u)) = \inf\{\|w - v\|, v \in T(u)\}$.

Since $\{w_n\}$, $\{u_n\}$ are Cauchy sequences in H, hence, $d(w,T(u)) = 0$ as $n \to \infty$. Since $T(u) \in C(H)$, so $w \in T(u)$, using theorem 5.2, we see that $z,u \in H, w \in T(u)$ is the solution of the generalized resolvent equations (5.1).

From theorem 5.2, we see that the generalized resolvent equations (5.1) and the problem (1.1 are equivalent, hence the sequences $\{u_n\}$ and $\{w_n\}$ generated by Algorithm 5.2 converge to the exact solution of the problem (1.1).

References

[1] CJ.Zhang, Analysis of set-value mappings and its applications to economics ,Science(2004)Beijing(in chinese).

[2] SS.Zhang, Variational inequalities and related problems, Chongqing(2008)Chongqing(in chinese).

[3] YH. Hu, W.Song, Generalized gap functions and error bounds for generalizedvariational inequalities[J]. Applied Mathematics and Mechanics , 2009,30(3):301-308.

[4] G.Stampacchia. Forms bilineaires coercitivies surn les ensembles convexes,C.R.Acad.Sci.Pairs,1964,258:4413-4416.

[5] Fukushima M. Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems[J]. Mathematical Programming,1992,53(1): 99-110.

[6] WuJH, Florian M,Marcotte P.A general descent framework for the monotone variational inequality problem[J].Mathematical Programming,1993,61(3):281-300.

[7] Yamashita N,Taji K,Fukushima M.Unconstrained optimization reformulations of variational inequality problems[J].Journal of Optimization Theory and Applications,1997,92:439-456.

[8] Huang L R, Ng K F. Equivalent optimization formulations and error bounds for variational inequality problem[J].Journal of Optimization Theory and Applications, 2005,125(2):299-314.

[9] Solodov M V. Merit functions and error bounds for generalized variational inequalities[J].Journal of Mathematical Analysis and Applications, 2003,287(2):405-414.

[10] Noor M A. Merit functions for general variational inequalities[J] . Journal of Mathematical Analysis and Applications, 2006,316(2):736-752.

[11] A.Bensoussan.On the theory of option pricing[J]. Acta Applicandae Mathem-aticae,1984,2:139-158.

[12] J.-S.Pang.Newton's method for B-differentiable equations[J]. Mathematics of Operations Research 15,1990,166:311-341.

[13] S.P.Dirkse and M.C.Ferris. A nonmonotone stabilization scheme for mixed complementarity problems[J].Optimization Methods and Software, 1995,5:123-156.

[14] Qu B，Wang C Y, Zhang J Z. Convergence and error bound of a method for solving variational inequality problems via the generalized D-gap function[J].Journal of Optimization Theory and Applications,2003,l19：535-552.

[15] H. Brezis.Operateurs Maximaux Monotone et Semigroupes de Contractions dans les Espaces de Hilbert, North-Holland, Amsterdam 1973.

[16] Noor M A.Generalized Mixed Variational Inequalities and Resolvent Equations[M].Netherlands: Kluwer Academic Pub- lishers,1997:145-154.

[17] Noor M A.Resolvent equations technique for variational inequalities[J].Appl.Math,1997,4:347-358.

[18] L.Qi.Convergence analysis of some algorithms for solving nonsmooth equations[J].Mathematics of Operations Research ,1993,18:227-244.

[19] B.Xiao,P.T.Harker.A nonsmooth Newton method for variational inequalities theory[J]. Mathematical Programming, 1994,65:151-194.

[18] L.Qi and J.Sun.A nonsmooth version of Newton's method[J]. Mathematical Programming ,1993,58:353-367.

[19] S.M.Robinson.Strongly regular generalized equations[J]. Mathematics of Operations Research ,1980,5:43-62.

[20] A.B.Bakusinskji and B.T.Polyak.On the Solution of Variational Inequalities[J]. Soviet Mathematics Doklady,1974,15: 1705-1710.

[21] G.M.Korpelevich.The extragradient method for finding saddle points and other problems[J].Matecon,1976,12:747-756.

[22] Verma R.U. General Verma R U.General convergence analysis for two-step projection methods and applications to variational problems[J].Appl Math Lerr,2005,18(11):1286-1292.

[23] Chang S S, Josephlee H W,Chan C K. Generalized system for relaxed cocoercive variational inequalities in Hilbert spaces[J]. Appl Math Lett,2007,20:329-334.

[24] Verma R U .Projection methods ,algorithms and a new system of nonlinear variational inequalities[J].Computer＆Mathematics with Applications,2001,41:1025-1031.

[25] Farouq N E L. Pseudomonotone variational inequalities convergence of the auxiliary problem method[J].J Optim Theory Appl,2001,111(2):305-326.

[26] Verma R U. A class of quasivariational inequalities involving cocoercive mappings[J].Advances in Nonlinear Variational Inequalities,1999,2(2):1-12.

[27] Verma R U. Generalized class of partial relaxed monotonicity and its Connections[J].Adv Nonlinear Var Inequal, 2004,7(2):155-164.

[28] Verma R U. Nonlinear implicit variational inequalities involving partially relaxed pseudomonotone mappings[J].Comput Math Appl,2003,46:1703-1709.

[29] Verma R U.A new class of iterative algorithms for approximation-solvability of nonlinear variational inequalities[J]. Comput Math Appl,2001,41:505-512.

[30] Noor M A, Noor K I. Projection algorithms for solving a system of general variational inequalities[J].Nonlinear Analysis,2009,70:2700-2706.

[31] Noor M A. An implicit method for mixed variational inequalities [J]. Appl Math Lett,1998,11(4):109-113.

[32] Luo X P,Huang N J.A new class of variational inclusions with B-monoton operators in Banach spaces[J].J Comput Appl Math,2010,233:1888-1896.

[33] LUO Hong-lin, LUO Hui-lin. General Convergence Analysis for Three-step Projection Methods and Applications to Variational Problems[J].Chin. Quart. J. of Math,2008,24(2):239-243.

[34] YANG Jun, WU Zhong-lin. A General Projection Method for a System of Relaxed Coercive Variational Inequalities in Hilbert Spaces [J].Chin. Quart. J. of Math,2009,24(3):426-431.

[35] Noor, M.Theory of Variational Inequalities, Lecture Notes, Mathematics Department,King Saud University, Riyadh, Saudi Arabia, 1996.

[36] Noor, M.Some recent advances in variational inequalities[J]. NewZealand J. Math,1997,26:53–80.